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Multiple points of trajectories of multiparameter fractional Brownian motion

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Abstract. Consider $0 < \alpha < 1$ and the Gaussian process $Y(t)$ on $\mathbb{R}^N$ with covariance $E(Y(s)Y(t)) = |t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}$, where $|t|$ is the Euclidean norm of t. Consider independent copies $X^1, \ldots, X^d$ of Y and the process $X(t) = (X^1(t), \ldots, X^d(t))$ valued in $\mathbb{R}^d$. When $kN \leq (k-1)\alpha d$, we show that the trajectories of X do not have k-multiple points. If $N < \alpha d$ and $kN > (k-1)\alpha d$, the set of k-multiple points of the trajectories X is a countable union of sets of finite Hausdorff measure associated with the function $\varphi(\varepsilon) = \varepsilon^{kN/\alpha-(k-1)d}(\log\log(1/\varepsilon))^k$. If $N = \alpha d$, we show that the set of k-multiple points of the trajectories of X is a countable union of sets of finite Hausdorff measure associated with the function $\varphi(\varepsilon) = \varepsilon^d(\log(1/\varepsilon)\log\log\log 1/\varepsilon)^k$. (This includes the case $k = 1$.)

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1. Introduction

Classical results $[R], [T], [C-T]$, indicate that a portion R of the trajectory of $\mathbb{R}^d$ valued Brownian motion satisfies $0 < \mu_\varphi(R) < \infty$, where μ_φ is Hausdorff measure associated with the function φ given by

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$\varphi(\varepsilon) = \varepsilon^2 \log\log(1/\varepsilon)$ for $d \geq 3$ and $\varphi(\varepsilon) = \varepsilon^2 \log(1/\varepsilon) \log\log\log(1/\varepsilon)$ for $d = 2$. Classical results indicate [D-E-K-1] that no double points of the trajectories exist for $d \geq 4$, that double points, but not triple points exist for $d = 3$, [D-E-K-T] and that multiple points of all orders exist for $d = 2$ [D-E-K-2]. More recently, the size (with respect to Hausdorff measure) of the set of double points (for $d = 3$) or of k-multiple points ($d = 2$) was completely clarified [LG].

The basic process considered in this paper is the Gaussian process $X(t)$ from $\mathbb{R}^N$ to $\mathbb{R}^d$ such that

$$E(|X(t) - X(s)|^2) = d|t-s|^{2\alpha}$$

where $0 < \alpha < 1$, and where $|.|$ denotes the Euclidean norm. Following [K], we call this process the (N, d, α) Gaussian process. Brownian motion is the $(1, d, \frac{1}{2})$ Gaussian process; Lévy's multiparameter Brownian motion is the $(N, d, \frac{1}{2})$ process; Fractional Brownian motion is the $(1, d, \alpha)$ process. The components of the (N, d, α) Gaussian process are independent $(N, 1, \alpha)$ processes.

It would be hard to argue that Brownian motion is not the most important (N, d, α) Gaussian process. It is also extraordinary special. As soon as $N > 1$ or $d \neq \frac{1}{2}$, crucial properties (such as the Markov property) are lost, and the resemblance of the corresponding process with Brownian motion is only superficial. Our motivation for extending results classical for Brownian motion to the (N, d, α) Gaussian process is not the importance of this process, but rather that the case of Brownian motion suffers from an over abundance of special properties; and that moving away from these forces to find proofs that rely upon general principles, and arguably lie at a more fundamental level. Fractional Brownian motion might not be an object of central mathematical importance but abstract principles are.

In the transient case ($N < \alpha d$) it was shown in [T], following numerous previous results, that if L is a compact set of nonempty interior, then a.s., $0 < \mu_\varphi(L) < \infty$, where μ_φ denotes the Hausdorff measure associated to the function $\varphi(\varepsilon) = \varepsilon^{N/\alpha} \log\log(1/\varepsilon)$. The purpose of the present paper is to extend the *upper bound part* of this result to the case of multiple points and to the critical case $N = \alpha d$. The problem of lower bounds, that rely upon different techniques, and are possibly more difficult, remains open.

We say that a point x of $\mathbb{R}^d$ is a k-multiple point of the trajectory of X if we can find k points $t_1, \ldots, t_k$ of $\mathbb{R}^d$, all different, such that $x = X(t_\ell)$ for all $\ell \leq k$.

Theorem 1.1. *If $kN \leq (k-1)\alpha d$, a.s. there exist no k-multiple points.*

Comment. Take $N = 1, k = 2, \alpha = \frac{1}{2}, d = 4$ for a classical result. To obtain Theorem 1.1 in the equality case $kN = (k-1)\alpha d$ was part of the motivation behind the previous work [T2]. When trying to tackle the case of 4-dimensional Brownian motion the natural idea is first to control the Hausdorff measure of trajectories, and then, using independence, to show that an independent trajectory does not hit these sets; thus the results of [T2] were conceivably a first step towards Theorem 1.1. Unfortunately this approach, when attempted if either $N \neq 1$ or $\alpha \neq \frac{1}{2}$ runs into apparently untractable conditioning problems. Rather we will use a direct "global" approach, relying heavily upon the arguments of [T1]. This approach leads in fact to the following.

Theorem 1.2. *If* $kN < (k-1)\alpha d$, *given any* $1 > \eta > 0$, *the set of points* x *of* $\mathbb{R}^d$ *that can be written as* $x = X(t_1) = \cdots = X(t_k)$ *for*

$$\eta \leq |t_1|, \ldots, |t_k| \leq \eta^{-1}, \quad |t_i - t_j| \geq \eta \quad \text{if} \;\; i \neq j \tag{1.1}$$

is a.s. of finite measure for the Hausdorff measure associated with the function

$$\varphi(\varepsilon) = \varepsilon^{kN/\alpha-(k-1)d}(\log\log 1/\varepsilon)^k \;.$$

Next, we turn to the critical case $N = \alpha d$.

Theorem 1.3. *If* $N = \alpha d$, *a.s. for each compact* L *of* $\mathbb{R}^N$, *the set* $X(L)$ *is a.s. of finite measure for the Hausdorff measure associated to the function*

$$\varphi(\varepsilon) = \varepsilon^d \log(1/\varepsilon) \log\log\log(1/\varepsilon) \;.$$

This result seems to lie quite deeper than the corresponding result for $N < \alpha d$. The proof relies on a lower bound for a certain sojorn time. While this bound is not surprising, its proof contains the most creative arguments of the paper. We conjecture that if L has non empty interior, $X(L)$ has positive Hausdorff measure for the measure described in Theorem 1.3.

Theorem 1.4. *Assume* $N = \alpha d$. *Then given* $1 > \eta > 0$, *the set of points* x *of* $\mathbb{R}^d$ *that can be written as* $x = X(t_1) = \cdots = X(t_k)$ *for* $t_1, \ldots, t_k$ *satisfying* (1.1) *is a.s. of finite measure for the Hausdorff measure associated to the function*

$$\varphi(\varepsilon) = \varepsilon^d(\log(1/\varepsilon)\log\log\log(1/\varepsilon))^k \;.$$

The paper is organized as follows. In Section 2, we list some of the abstract results we need. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we prove the key sojorn time estimate; and in Section 5, we prove Theorem 1.3 and sketch the proof of Theorem 1.4. While the proofs use many known ideas, they do require rather substantial technical inventiveness, and should provide ample reward for the motivated reader.

2. Preliminaries

Consider a set S and a Gaussian process $(Z(t))_{t\in S}$ valued in $\mathbb{R}^d (d \geq 1)$. We provide S with the canonical distance

$$d(s,t) = \|Z(s) - Z(t)\|_2 = (E|Z(t) - Z(s)|^2)^{\frac{1}{2}} \ .$$

We denote by $N(S,\varepsilon)$ the smallest number of open d-balls of radius ε needed to cover S. The proof of Theorems 1.1 and 1.2 will heavily rely upon the following;

Lemma 2.1. *Consider a function Ψ, such that for all $\varepsilon > 0$ and some $C > 0$ we have*

$$\Psi(2\varepsilon)/C \leq \Psi(\varepsilon) \leq C\Psi(\varepsilon/2) \ .$$

Assume that $N(S,\varepsilon) \leq \Psi(\varepsilon)$ for all $\varepsilon > 0$. Then we have

$$P\Big(\sup_{s,t\in S} |Z(t) - Z(s)| \leq u\Big) \geq \exp\left(-\frac{\Psi(u)}{K}\right) \ ,$$

where K depends upon C, d only.

In the case $d = 1$, this is proved in [T1]. The proof in the general case (via Sidak's lemma and chaining) is identical.

In order to work with the $(N,1,\alpha)$ process and to prove its existence, it is very useful to have a concrete representation of it. Such a representation is based upon the fact that if $0 < \alpha < 1$, there is a constant c depending upon α, N only such that for each t in $\mathbb{R}^N$ we have

$$|t|^{2\alpha} = c^2 \int_{\mathbb{R}^N} (1 - \cos\langle t,x\rangle) \frac{dx}{|x|^{2\alpha+N}} \ . \tag{2.2}$$

Consider two independent scattered Gaussian random measures m and m' on $\mathbb{R}^N$, with, for each $A \subset \mathbb{R}^N$

$$E(m(A)^2) = E(m'(A)^2) = \lambda(A) \ .$$

There, as well as in the rest of the paper, λ denotes Lebesgue measure. (We will not distinguish in the notation on which space λ lives; this should be clear from the context.) The process

$$Y(t) = c\int_{\mathbb{R}^N} (1-\cos\langle t,x\rangle)\frac{dm(x)}{|x|^{\alpha+\frac{N}{2}}} + \sin\langle t,x\rangle\frac{dm'(x)}{|x|^{\alpha+\frac{N}{2}}}$$

is a version of the $(N,1,\alpha)$ process, as is seen by checking through (2.2) that $E(Y(t)-Y(s))^2 = |t-s|^{2\alpha}$. (At this point we must apologize for the uncomplete formula (3.2) of [T2], a mistake that fortunately does not affect the rest of that paper.) To solve independence problems, we consider for $a,b \in [0,\infty)$ the process

$$Y(t,a,b) = c\int_{a\le|x|<b} (1-\cos\langle t,x\rangle)\frac{dm(x)}{|x|^{\alpha+\frac{N}{2}}} + \sin\langle t,x\rangle\frac{dm'(x)}{|x|^{\alpha+\frac{N}{2}}} \quad ,$$

We denote by $X(t,a,b)$ the $\mathbb{R}^d$-valued process consisting of d independent copies of $Y(t,a,b)$. If $b \le a'$, the processes $X(t,a,b)$, $X(t,a',b')$ are independent.

Only minor modifications to the argument of [T2], Corollary 3.3 are needed to obtain the following

Lemma 2.2. *Consider $r,d,b>0$ and $A = r^2a^{2-2\alpha}+b^{-2\alpha}$. If $A \le r^{2\alpha}$, for*

$$u \ge K\left(A\log\frac{2r^{2\alpha}}{A}\right)^{\frac{1}{2}} \quad ,$$

we have for all $t \in \mathbb{R}^N$

$$P\left(\sup_{|t'-t|\le r}|X(t)-X(t')-(X(t,a,b)-X(t',a,b))| \ge u\right) \le \exp\left(-\frac{u^2}{\lambda A}\right) \quad .$$

Here, as well as in the rest of the paper, K denotes a constant depending upon N,α,d, only, that may vary at each occurrence. (Specific constants are denoted by $K_1, K_2, \ldots$.)

The following standard estimate will also be used.

Lemma 2.3. *Given R, there is a number $K(R)$, depending only upon R,N,d,α such that, for $\varepsilon \le \frac{1}{2}$*

$$P\Big(\forall t,t' \in \mathbb{R}^N, |t|,|t'| \le R, |t-t'| \le \varepsilon$$
$$\Rightarrow |X(t)-X(t')| \le K(R)\varepsilon^\alpha\big(\log\tfrac{1}{\varepsilon}\big)^{\frac{1}{2}}\Big) \ge 1-\varepsilon \quad .$$

Of crucial importance will be the following lemma of L. Pitt. [P]

Lemma 2.4. *Consider $t \in \mathbb{R}^N, u>0$. The conditional variance of $Y(t)$ given all $Y(t')$ for $|t'-t| \ge u$ is at least $K^{-1}\min(u,|t|)^{2\alpha}$.*

3. Proofs of Theorems 1.1 and 1.2

Consider $\eta > 0$ and $t_1, \ldots, t_k$ in $\mathbb{R}^N$ with

$$\min_{i \le k} |t_i| \ge \eta; \quad \min_{i \ne j} |t_i - t_j| \ge \eta \ . \tag{3.1}$$

Given $\rho > 0$, we consider the random set

$$M(\rho) = \{x \in \mathbb{R}^d; \quad \forall i \le k, \exists u_i, |u_i - t_i| \le \rho, x = X(u_1) = \cdots = X(u_k)\} \ .$$

(Of course $M(\rho)$ depends also upon $t_1, \ldots, t_k$). We will show the following.

Proposition 3.1. a) *If ρ is small enough, and if $kN = (k-1)\alpha d, M(\rho)$ is almost surely empty.* b) *If $kN < (k-1)\alpha d$, then for some constant $K(\eta)$, that depends upon η, α, N, d, but NOT upon ρ, we have*

$$E\mu_\varphi(M(\rho)) \le K(\eta)\rho^{kN}$$

where μ_φ is the Hausdorff measure associated to the function φ of Theorem 1.2.

It should be clear that this implies Theorems 1.1 and 1.2. For $i \le k$, we set

$$B_i = \{u \in \mathbb{R}^N; |u - t_i| \le \rho\}; \quad B_i' = \{u \in \mathbb{R}^N; |u - t_i| \le 2\rho\} \ .$$

We will determine ρ later on. We start the proof by a strange move; for each $i \le k$, we select a point t_i' with $|t_i - t_i'| = 3\rho$. The motivation for this is as follows.

Lemma 3.2. *For some number C_1 (depending possibly upon $t_1, \ldots, t_k$, ρ, η, N, α, d) we have, for all $i \le k$*

$$u_1, u_2 \in B_i' \Rightarrow |E((Y(u_1) - Y(u_2))Y(t_i'))| \le C_1|u_1 - u_2| \ .$$

Proof. This follows from the fact that

$$E(Y(u)X(t_i')) = |t_i'|^{2\alpha} + |u|^{2\alpha} + |t_i' - u|^{2\alpha}$$

and that we have taken care to ensure $|t_i' - u| \ge \rho$ (and $|u| \ge \eta - 2\rho$). □

The previous lemma would not work for $t_i' = t_i$.

We now denote by Σ_2 the σ-algebra generated by $(X(t_i'), i \le k)$. We set

$$X^2(t) = E(X(t)|\Sigma_2); \quad X^1(t) = X(t) - X^2(t) \ .$$

The processes $X^1(t)$ and $X^2(t)$ are independent. The main construction will depend only upon the process $X^1(t)$; given that process, we will then use an averaging argument upon $X^2(t)$.

The bad news is that the process $X^1(t)$ is somewhat more mysterious than the process $X(t)$, because it is not really clear what the conditioning does. The good news is that for our purposes $X^1(t)$ is a very small perturbation of $X(t)$, and we will be able to deduce all the information we need about $X^1(t)$ from the study of $X(t)$.

Lemma 3.3. *For some constant C_2 (possibly depending upon $t_1, \ldots, t_k, \eta, \rho, N, \alpha, d$) we have, for $i \le k$ and $u_1, u_2 \in B'_i$:*

$$|X^2(u_1) - X^2(u_2)| \le C_2 |u_1 - u_2| \max_{i \le k} |X(t'_i)| \ .$$

Proof. This follows from Lemma 3.2, since

$$X^2(u) = \sum_{i,j \le k} a_{ij} E(X(u)X(t'_i))X(t'_j)$$

for numbers a_{ij} depending only upon $t'_1, \ldots, t'_k$.

The main estimate is as follows.

Proposition 3.4. *There is a constant $\delta > 0$ with the following property. Given $r_0 \le \delta$, and for $i \le k$ a point u_i in $\mathbb{R}^N$, we have*

$$P\left(\exists r, r_0^2 \le r \le r_0, \sup_{i \le k} \sup_{|t-u_i| \le 2\sqrt{N} r} |X(t) - X(u_i)| \le K_1 r^\alpha \left(\log\log \frac{1}{r} \right)^{-\alpha/N} \right)$$
$$\ge 1 - \exp\left(- \left(\log \frac{1}{r_0} \right)^{\frac{1}{2}} \right) \ . \tag{3.2}$$

Comment. The term $2\sqrt{N}$, that plays no important role, is simply for convenience when replacing balls by cubes.

Proof. First we prove that given r,

$$P\left(\sup_{t \le k} \sup_{|t-u_i| \le 2\sqrt{N} r} |X(t) - X(u_i)| \le u \right) \ge \exp\left(- \frac{Kr}{u^{\frac{1}{\alpha}}} \right) \ . \tag{3.3}$$

To see this, we simply apply Lemma 2.1 to the $\mathbb{R}^{kd}$ valued process

$$Z(t_1, \ldots, t_k) = (X(t_i))_{i \le k}$$

defined for $|t_i - u_i| \le 2\sqrt{N} r$, keeping in mind that

$$E|Z(t_1, \ldots, t_k) - Z(t'_1, \ldots, t'_d)|^2 \le \sum_{i \le k} |t_i - t'_i|^{2\alpha}$$

to see that we can use $\Psi(\varepsilon) = Kr/\varepsilon^{\frac{1}{\alpha}}$.

Once (3.3) is obtained, the rest of the argument is very similar to the proof of Proposition 4.1 of [T2], and there seems to be no point in reproducing it. □

We now start the main construction. For $p \geq 1$, consider the set

$$R_p = \left\{ (u_1, \dots, u_k) \in B'_1 \times \dots \times B'_k, \exists r, \quad 2^{-2p} \leq r \leq 2^{-p}, \right.$$
$$\left. \sup_{i \leq k} \sup_{|t-u_i| \leq 2\sqrt{N}r} |X(t) - X(u_i)| \leq K_2 r^\alpha \left(\log\log \frac{1}{r} \right)^{-\alpha/N} \right\} .$$

It then follows from Fubini's theorem and Proposition 3.4 that we have $\sum_p P(\Omega^c_{p,1}) < \infty$, where $\Omega_{p,1}$ is the event

$$\lambda(R_p) \geq \lambda(B'_1 \times \dots \times B'_k) \left(1 - \exp\left(-\frac{\sqrt{p}}{4} \right) \right) .$$

To apply Lemma 3.3, we consider $\beta > 0$ with $\alpha + \beta < 1$ and the event $\Omega_{p,2}$ given by

$$\max_{i \leq k} |X(t'_i)| \leq 2^{\beta p}$$

so that $\sum_p P(\Omega^c_{p,2}) < \infty$. It follows from Lemma 3.3 that there is p_0 such that if $p \geq p_0$, then on the event $\Omega_{p,3} = \Omega_{p,1} \cap \Omega_{p,2}$, we have

$$(u_1, \dots, u_k) \in R_p \Rightarrow \exists r, 2^{-2p} \leq r \leq 2^{-p};$$
$$\sup_{i \leq k} \sup_{|t-u_i| \leq 2\sqrt{N}r} |X^1(t) - X^1(u_i)| \leq K_3 r^\alpha \left(\log\log \frac{1}{r} \right)^{-\alpha/N} . \tag{3.4}$$

Let us recall that a dyadic cube of order ℓ is a product of intervals $[m2^{-\ell}, (m+1)2^{-\ell}[$. For u in $\mathbb{R}^N$, denote by $C_\ell(u)$ the dyadic cube of order ℓ that contains u. For $u_1, \dots, u_k$ in $\mathbb{R}^N$, denote by $C_\ell(u_1, \dots, u_k) = C_\ell(u_1) \times \dots \times C_\ell(u_k)$ the dyadic cube of order ℓ of $\mathbb{R}^{kN}$ that contains $(u_1, \dots, u_k) \in \mathbb{R}^{kN}$. We say that $C_\ell(u_1) \times \cdots \times C_\ell(u_k)$ is a *good* cube of order ℓ if it has the property that

$$\forall i \leq k, \qquad \sup_{s,t \in C_\ell(u_i) \cap B_i} |X^1(s) - X^1(t)| \leq d_\ell \ , \tag{3.5}$$

where $d_\ell = 8K_3 2^{-\ell\alpha} (\log\log 2^\ell)^{-\alpha/N}$. It follows from (3.4) that (if $p \geq p_0$), each point $(u_1, \dots, u_k)$ of R_p is contained in a good dyadic cube of order ℓ, with $p \leq \ell \leq 2p$. Thus, we can find a covering of R_p by a disjoint family $\mathscr{H}_1$ of good dyadic cubes. This family depends only upon the process $X^1(t)$. Consider the family $\mathscr{H}_2$ of dyadic cubes of order $2p$ of $\mathbb{R}^{kN}$, that meet $B_1 \times \cdots \times B_k$, but are not contained in any cube of $\mathscr{H}$. For p large enough, these cubes are contained in $B'_1 \times \cdots \times B'_k$, and hence in $B'_1 \times \cdots \times B'_k \backslash R_p$, so that (when $\Omega_{p,3}$, occurs) their number is at most

$$C_k 2^{2Nkp} \exp\left(-\frac{\sqrt{p}}{4}\right) \tag{3.6}$$

when C_k does not depend upon p.

We set $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$. This family of dyadic cubes covers $B_1 \times \cdots \times B_k$, and is always well defined (although we cannot say much about it unless $\Omega_{p,3}$ occurs).

Next, we proceed to the construction of a certain family of balls of $\mathbb{R}^d$. For each cube in A in $\mathscr{H}$ we pick a distinguished point v_A in A, say $v_A = (v_{A,1}, \ldots, v_{A,k}), v_{A,i} \in \mathbb{R}^N$ for $i \leq k$. We consider the ball B_A of $\mathbb{R}^d$ defined as follows:

- If $A \in \mathscr{H}_1$ is a dyadic cube of order ℓ, we take for B_A the ball of center $X(v_{A,1})$, of radius $r_A = 4d_\ell$
- If $A \in \mathscr{H}_2$ (this is a dyadic cube of order $2p$), B_A is the ball of center $X(v_{A,1})$ of radius $r_A = K_4 2^{-2\alpha p}\sqrt{\log p}$.

There we choose K_4 large enough that $\sum_{p\geq 1} P(\Omega_{p,4}) < \infty$ where $\Omega_{p,4}$ is the following event:

For each dyadic cube C of order 2^{-2p} of $\mathbb{R}^N$ that meets $\bigcup_{i\leq k} B_i$, we have

$$\sup_{t,u\in C} |X(t) - X(u)| \leq K_4 2^{-2\alpha p}\sqrt{p}\ .$$

This is possible by Lemma 2.3. We consider the event $\Omega_p = \Omega_{p,3} \cap \Omega_{p,4}$.

For each A in $\mathscr{H}$, we define the event Ω_A as

$$\forall i, 2 \leq i \leq k, \quad |X(v_{A,1}) - X(v_{A,i})| \leq r_A\ . \tag{3.7}$$

We consider the family $\mathscr{F}$ of balls $B_A (A \in \mathscr{H})$ for which Ω_A occurs.

Lemma 3.5. *On Ω_p, $\mathscr{F}$ covers $M(\rho)$.*

Proof. Consider x in $M(\rho)$. By definition, we can find for $i \leq k$ a point u_i in B_i such that $X(u_i) = x$. The point $(u_1, \ldots, u_k)$ belongs to a certain cube A of $\mathscr{H}$. We will show that B_A contains x, and that B_A belongs to $\mathscr{F}$. We will consider only the case $A \in \mathscr{H}_1$. (The similar case $A \in \mathscr{H}_2$ is left to the reader.) Consider the distinguished point $(v_{A,1}, \ldots, v_{A,k})$ of A. By (3.5) we have

$$\forall i \leq k, \quad |X^1(v_{A,i}) - X^1(u_i)| \leq d_\ell\ .$$

Thus (if $p \geq p_0$) by Lemma 3.3 we have

$$|X(v_{A,i}) - X(u_i)| \leq 2d_\ell.$$

Since $X(u_i) = x$, this implies that Ω_A occurs, and that $x \in B_A$. $\square$

Consider the function

$$f(x) = x^{kN/2}\left(\log\log\tfrac{1}{x}\right)^k.$$

Lemma 3.6. *If Ω_p occurs, and p is large enough, we have*

$$\sum_{A \in \mathscr{H}} f(r_A) \leq K\lambda(B_1 \times \cdots \times B_k)\ .$$

Proof. If $A \in \mathscr{H}_1$ is a dyadic cube of order ℓ, simple estimates show that

$$f(r_A) \leq K2^{-\ell kN}.$$

If $A \in \mathscr{H}_2$ (is thus a dyadic cube of order $2p$) we have

$$f(r_A) \leq K2^{-2pkN}p^{kN/2\alpha}(\log p)^k\ .$$

Recalling (3.6) yields the result. $\square$

We now denote by Σ_1 the σ-algebra generated by the process $(X^1(t))$. Thus $\mathscr{H}$ depends upon Σ_1 only. The basis of the averaging argument is as follows.

Lemma 3.7. *If ρ is small enough, for some constant $K(\eta)$ depending only upon η, N, d, α, we have*

$$P(\Omega_A|\Sigma_1) \leq K(\eta)r_A^{(k-1)d}\ .$$

Proof. Given $v_i \in B_i$ for $i \leq k$, it suffices to show that if r is small, for any choice of a_i in $\mathbb{R}^d$ we have

$$P(\forall i, 2 \leq i \leq k, \quad |X^2(v_1) - X^2(v_i) - a_i| \leq r) \leq K(\eta)r^{(k-1)d}\ .$$

With obvious notation, using independence, it suffices to prove that

$$P(\forall i, 2 \leq i \leq k, \quad |Y^2(v_1) - Y^2(v_i) - b_i| \leq r) \leq K(\eta)r^{k-1}\ .$$

Proceeding by induction over k, it suffices to show that the conditional variance of $Y^2(v_i)$ given $Y^2(v_1), \ldots, Y^2(v_{i-1})$ remains bounded below by a number depending only upon α, η. But this follows easily from the fact that $E|Y(v_i) - Y(t_i')|^2 \leq (2\rho)^{2\alpha}$ and Lemma 2.4. $\square$

Assume now $kN \leq (k-1)\alpha d$, and consider the function

$$\varphi(x) = x^{kN/\alpha-(k-1)d}\left(\log\log\tfrac{1}{x}\right)^k \ .$$

Lemma 3.8. *If ρ is small enough that Lemma 3.7 holds true, we have for p large enough*

$$E\left(1_{\Omega_p}\sum_{A\in\mathscr{F}}\varphi(r_A)\right) \le K(\eta)\lambda(B_1\times\cdots\times B_k) \ .$$

Proof. We compute the left-hand side by first taking expectation conditionally in Σ_1, and combining Lemmas 3.6 and 3.7. □

It is now routine to deduce Proposition 3.1 from Lemmas 3.5 and 3.8.

4. Sojourn time estimate

We now start the study of the critical case, $N = \alpha d$. The proof of Theorem 1.3 is based upon an estimate for the tails of the "sojourn time"

$$T_\varepsilon = \lambda(\{t\in\mathbb{R}^N;\quad |t|\le 1,\quad |X(t)|\le\varepsilon\}) \ .$$

Theorem 4.1. *For $1\le u\le\frac{1}{K}\log\frac{1}{\varepsilon}$, and $\varepsilon\le\frac{1}{2}$, we have*

$$P\left(T_\varepsilon\ge u\varepsilon^d\log\tfrac{1}{\varepsilon}\right)\ge e^{-Ku} \ . \tag{4.1}$$

The proof of Theorem 1.4 is based upon the following extension of Theorem 4.1.

Theorem 4.2. *Consider $t_1,\ldots,t_k\in\mathbb{R}^N$, and $\eta=\inf_{i\ne j}|t_i-t_j|$. For $a\le\eta/2$, $\varepsilon>0$ consider*

$$T=\lambda(\{(u_1,\ldots,u_k)\in\mathbb{R}^{kN};\forall i\le k,|u_i-t_i|\le a,|X(t_i)-X(u_i)|\le\varepsilon\}).$$

Then we have, for all $\varepsilon\le a^\alpha/2$, all $1\le u\le\frac{1}{K}\log\frac{a^\alpha}{\varepsilon}$ that

$$P(T\ge(u\varepsilon^d\log\tfrac{a^\alpha}{\varepsilon})^k)\ge \mathrm{e}^{-Ku} \ .$$

If the processes $(X(t))_{t\in B_i}$ $(B_i=\{t;|t-t_i|\le a\})$ were independent, Theorem 4.2 would follow from Theorem 4.1. (Since, however, Theorem 4.1 relies upon arguments for which independence is the worst case lack of independence is not an issue.)

The standard way to write down these results would consist of providing the proof of Theorem 4.2, arguing that Theorem 4.1 is a special case.

This would make the proof harder to read; which would be a pity, since it is the argument in the paper that seems to require the most imagination. Thus, we have decided rather to write down the proof of Theorem 4.1, and to leave the pretty straightforward extension required by Theorem 4.2 to the interested reader.

The starting point of our approach is the following simple result.

Lemma 4.3. *Consider a r.v.* $X \geq 0$, *and assume*

$$EX^n \geq n^n/K_1^n; \quad EX^{2n} \leq K_2^{2n}(2n)^{2n}$$

Then

$$P\left(X \geq \frac{n}{2K_1}\right) \geq \frac{1}{(16K_1K_2^2)^n} \ .$$

Comment. Thus $P(X \geq u) \geq e^{-Ku}$ for $u = \frac{n}{2K_1}$.

Proof. The (elementary) Paley-Zygmund inequality states that, for a r.v. $Y \geq 0$, we have

$$P\left(Y \geq \frac{EY}{2}\right) \geq \frac{1}{4}\frac{(EY)^2}{EY^2} \ . \tag{4.2}$$

We use this for $Y = X^n$. We get

$$P\left(X^n \geq \frac{n^n}{2K_1^n}\right) \geq \frac{1}{4}\frac{n^{2n}/K_1^n}{K_2^{2n}(2n)^{2n}} \geq \frac{1}{(16K_1K_2^2)^n} \ . \qquad \square$$

We will use Lemma 4.3 with $X = T_\varepsilon$; thus we have to get upper and lower bounds for ET_ε^n, that will be obtained respectively in (4.5) and (4.12) below. As is classical, we write

$$T_\varepsilon^n = \int_{C_n} \prod_{i \leq n} 1_{\{|X(t_i)| \leq \varepsilon\}} \, d\lambda(t_1, \ldots, t_n)$$

where

$$C_n = \{t_1, \ldots, t_n; \forall i \leq n, |t_i| \leq 1\} \ ,$$

and thus

$$ET_\varepsilon^n = \int_{C_n} P(\forall i \leq n, |X(t_i)| \leq \varepsilon) \, d\lambda(t_1, \ldots, t_n) \ . \tag{4.3}$$

To find an upper bound, we observe that by Lemma 2.4 we have

$$P(|X(t_n)| \le \varepsilon \,|\, \forall i \le n-1, |X(t_i)| \le \varepsilon) \le \frac{K\varepsilon^d}{\max(\varepsilon^{\frac{1}{\alpha}}, |t_n|, \min_{i<n}|t_n - t_i|)^{\alpha d}} \quad . \tag{4.4}$$

We observe that, setting $t_0 = 0$, we have

$$\max\left(\varepsilon^{\frac{1}{\alpha}}, |t_n|, \min_{i<n}|t_n - t_i|\right)^{-\alpha d} \le \sum_{0 \le j \le n} \max(\varepsilon^{\frac{1}{\alpha}}, |t_n - t_j|)^{-\alpha d} \quad .$$

Combining with (4.3), and since $N = \alpha d$, we get the induction relation

$$ET_\varepsilon^n \le Kn\varepsilon^d \log\tfrac{1}{\varepsilon} E(T_\varepsilon^{n-1})$$

from which it follows that

$$ET_\varepsilon^n \le \left(Kn\varepsilon^d \log\tfrac{1}{\varepsilon}\right)^n \quad . \tag{4.5}$$

Next, we turn to the more delicate task of finding a lower bound for ET_ε^n of the correct order. It suffices to consider the case $n = 2^p$. For $k < p$, we set $F_k = \{t_1, \ldots, t_{2^k}\}$.

Lemma 4.4. *Assume $|t_1|^\alpha \ge 2^{-p}\varepsilon$, and assume that $d(t_i, F_k)^\alpha \ge \varepsilon 2^{k-p}$ for $k < p,\ 2^k < i \le 2^{k+1}$. Then*

$$P(\forall i \le n, |X(t_i)| \le \varepsilon) \ge \frac{1}{|t_1|^{\alpha d}} \left(\frac{\varepsilon^d}{K}\right)^n \prod_{0 \le k < p} \prod_{2^k < i \le 2^{k+1}} \frac{1}{d(t_i, F_k)^{\alpha d}} \quad .$$

Proof. Step 1. For $k \ge 0$ and $2^k < i \le 2^{k+1}$, we consider $1 \le a(i) \le 2^k$ such that $|t_i - t_{a(i)}| = d(t_i, F_k)$. We observe that if

$$|X_{t_1}| \le \varepsilon 2^{-p} \tag{4.6}$$

$$\forall k < p,\ \forall i, 2^k < i \le 2^{k+1}, \quad |X_{t_i} - X_{t_{a(i)}}| \le \varepsilon 2^{k-p} \tag{4.7}$$

then

$$\forall i \le 2^p, \quad |X_{t_i}| \le 2\varepsilon \quad .$$

Step 2. We recall Sidak's theorem: for any family (Y_j) of jointly gaussian centered r.v., we have

$$P(\forall j, |Y_j| \le \varepsilon_j) \ge \prod_j P(|Y_j| \le \varepsilon_j) \quad . \tag{4.8}$$

We denote by $(Y^\ell(t))_{\ell \le d}$ the components of $X(t)$. Thus, by (4.8) the probability that (4.6) and (4.7) occur is at least

$$\prod_{\ell\le d}\left[P\left(|Y^\ell(t_1)|\le\frac{\varepsilon 2^{-p}}{\sqrt{d}}\right)\prod_{k<p}\prod_{2^k<i\le 2^{k+1}}P\left(|Y^\ell(t_i)-Y^\ell(t_{a_i})|\le\frac{\varepsilon 2^{k-p}}{\sqrt{d}}\right)\right] \ . \quad (4.9)$$

Step 3. We recall the elementary fact that

$$\forall\eta>0,\quad P(|Y^\ell(u)-Y^\ell(t)|\le\eta)\ge\frac{1}{K}\min\left(1,\frac{\eta}{|t-u|^\alpha}\right)\ ,$$

as follows simply from the fact that normal law has density of order 1 close to zero. Thus (4.9) is bounded below by

$$\frac{1}{K^n}\min\left(1,\frac{\varepsilon 2^{-p}}{\sqrt{d}|t_1|^\alpha}\right)^d\prod_{k<p}\prod_{2^k<i\le 2^{k+1}}\min\left(1,\frac{\varepsilon 2^{k-p}}{\sqrt{d}|t_i-t_{a(i)}|^\alpha}\right)^d$$

$$=\frac{1}{K^n}\left(\frac{2^{-p}\varepsilon}{|t_1|^\alpha}\right)^d\left(\prod_{k<p}\prod_{2^k<i\le 2^{k+1}}\frac{\varepsilon 2^{k-p}}{|t_i-t_{a(i)}|^\alpha}\right)^d\ .$$

To conclude, it suffices to observe that

$$2^{-p}\prod_{k<p}2^{(k-p)2^k}\ge K^{-n}\ . \qquad \square$$

Lemma 4.5. *Assume $n\le\frac{1}{K}\log\frac{1}{\varepsilon}$. Then there exists a subset D of C_{2^p} with the following properties*

Every $(t_1,\ldots,t_{2^p})$ in D satisfies the condition of Lemma 4.4. (4.10)

$$\int_{(t_1,\ldots,t_{2^p})\in D}\frac{1}{|t_1|^{\alpha d}}\prod_{k<p}\prod_{2^k<i\le 2^{k+1}}\frac{1}{d(t_i,F_k)^{\alpha d}}\,d\lambda(t_1,\ldots,t_{2^p}) \quad (4.11)$$

$$\ge\left(\frac{1}{K}\right)^n\prod_{k<p}\left(2^{-p+k}\log\frac{1}{\varepsilon}\right)^{2^k}\prod_{k<p}(2^k)!\ .$$

Before the reader tries to swallow this condition, it might be helpful to see why this finishes the proof of Theorem 4.1. Indeed, combining with Lemma 4.4, we see that

$$ET_\varepsilon^n\ge\left(\frac{\varepsilon^d}{K}\right)^n\left(\log\frac{1}{\varepsilon}\right)^n\prod_{k<p}2^{-(p-k)2^k}2^{\sum_{k<p}k2^k}\ge\left(\frac{n\varepsilon^d}{K}\log\frac{1}{\varepsilon}\right)^n\ . \quad (4.12)$$

Proof of Lemma 4.5. To construct D, we set

$$\varepsilon_k=\varepsilon^{(1+2^{-p+k})/2}\ .$$

Observe that ε_k decreases, and that

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \varepsilon^{2^{-p+k-1}}$$

Thus

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} \leq \varepsilon^{2^{-p-1}} \leq \frac{1}{8} \tag{4.13}$$

if $n = 2^p$ satisfies $n \leq \frac{1}{K} \log \frac{1}{\varepsilon}$. Define

$$H_k(t) = \{x \in \mathbb{R}^N; \quad \varepsilon_{k+1} \leq |x - t| \leq \varepsilon_k / 4\}.$$

We define D by the following conditions. We require that $t_1 \in H_0(0)$. Next, if $2^k < \ell \leq 2^{k+1}$ for $0 \leq k < p$, we require that there is an index $a(\ell) \leq 2^k$ such that $t_\ell \in H_k(t_{a(\ell)})$. Moreover, the map $\ell \to a(\ell)$ is one to one. It might be useful to think of D as being constructed recursively. Once $(t_\ell), \ell \leq 2^k$ has been constructed, one then throws a point exactly in each of the sets $H_k(t_\ell), \ell \leq 2^k$. If $2^k < \ell \leq 2^{k+1}$, we now show that $\varepsilon_{k+1} \leq d(t_\ell, F_k) \leq \varepsilon_k/4$. The right hand side inequality is obvious. To prove the left hand side inequality one simply observes by induction over k using (4.13), that $|t_\ell - t_{\ell'}| \geq \varepsilon_k$ if $\ell \neq \ell', \ell, \ell' \leq 2^k$. To prove the condition of Lemma 4.4, it suffices to observe that for $0 \leq k < p$, we have

$$\varepsilon_{k+1} = \varepsilon^{\frac{1}{2}(1+2^{-p+k+1})} \geq \varepsilon 2^{k-p} \quad .$$

Indeed, as k increases, ε_{k+1} decreases while 2^k increases; thereby, it suffices to check the above inequality for $k = p - 1$ where it becomes $\varepsilon \geq \varepsilon/2$. To check that $D \subset C_{2^p}$, we simply use the fact that (by induction on k) if $i \leq 2^{k+1}$, then $|t_i| \leq \sum_{0 \leq \ell \leq k} \varepsilon_\ell/4$, so that by (4.13) $|t_i| \leq 2\varepsilon_0 \leq 2\sqrt{\varepsilon}$.

Finally, it remains to prove (4.11). But this is an easy consequence of the fact that

$$\int_{t \in H_k(0)} \frac{1}{|t|^{\alpha d}} \, dt \geq \frac{1}{K} 2^{-p+k} \log \frac{1}{\varepsilon} \quad . \qquad \square$$

5. End of proofs in the critical case

Throughout this section, we set $R_p = 2^{-2^{2^p}}$. As a first step, we have to prove a result of the nature of Proposition 3.4. This is somewhat more delicate that what one would hope.

Proposition 5.1. *We can find $\beta, 1 < \beta < 1/\alpha, x_0 > 0$ and p_0 such that if $p \geq p_0$ and $x_0 \leq x \leq p$ the event*

$$\exists r, \quad R_{2p} \le r \le R_p, \lambda(\{t; |t| \le r^\beta, |X(t)| \le 3r\}) \ge \frac{xr^d}{K} \log \frac{1}{r} \tag{5.1}$$

has a probability at least $1 - 2^{-(1+d)2^{2p+x_0-x}}$.

Proof. Consider $\gamma > 1$ to be determined later (as well as β), and $r_\ell = 2^{-\gamma^\ell}$. Thus $R_{2p} \le r_\ell \le R_p$ for $2^{2^p} \le \gamma^\ell \le 2^{2^{2p}}$. It follows from Theorem 4.1 (and rescaling) that the event

$$\lambda(\{|t| \le r_\ell^\beta; |X(t)| \le r_\ell\}) \ge x(1-\alpha\beta)\frac{r_\ell^d}{K} \log \frac{1}{r_\ell}$$

has probability $\ge 2^{-x}$ if $x \ge x_0$. As ℓ varies, these events are not independent. To create independence, we replace the process $X(t)$ by the process $X(t, a_\ell, b_\ell)$ where a_ℓ, b_ℓ will be chosen later. Set $A_\ell = r_\ell^{2\beta} a_\ell^{2-2\alpha} + b_\ell^{-2\alpha}$. We see that if we arrange that $A_\ell \le 2r_\ell^{2\beta'}$ for some $\beta' > 1$, then Lemma 2.2 implies that for p large enough

$$P\left(\sup_{|t| \le r_\ell^\beta} |X(t) - X(t, a_\ell, b_\ell)| \ge r_\ell\right) \le \exp\left(-\frac{1}{Kr_\ell^{2\beta'-2}}\right) .$$

This suggests the choice $b_\ell = r_\ell^{-\beta'/\alpha}, a_\ell = r_\ell^{-(\beta'-\beta)/(1-\alpha)}$. Since $r_{\ell+1} = r_\ell^\gamma$, we see that given γ, we can choose $\beta' > 1$ and $\beta > 1/\alpha$ such that $b_\ell \le a_{\ell+1}$ for each ℓ. The events

$$\lambda(\{t; |t| \le r_\ell^\beta; |X(t, a_\ell, b_\ell)| \le 2r_\ell\}) \ge x(1-\alpha\beta)\frac{r_\ell^d}{K} \log \frac{1}{r_\ell}$$

are independent, and each has a probability $\ge 2^{-x-1}$ if p_0 is large enough. For $\gamma \le 2$, there are at least $2^{2p}/K \log \gamma$ such events. Thus the probability that one such event occurs is at least

$$\begin{aligned} 1 - (1 - 2^{-x-1})^{2^{2p}/K \log \gamma} &\ge 1 - \exp(-2^{2p-x}/K \log \gamma) \\ &\ge 1 - \tfrac{1}{2} 2^{-(1+d)2^{2p+x-x_0}} \end{aligned}$$

if γ has been chosen close enough to one. But then (if p_0 is large enough) the probability that

$$\exists \ell, r_\ell \le R_{2p}, \sup_{|t| \le r_\ell^\beta} |X(t) - X(t, a_\ell, b_\ell)| \ge r_\ell$$

is at most

$$\frac{\lambda}{\log \gamma} 2^{2p} \exp(-1/KR_p^{2\beta'-2}) \le \frac{1}{2} 2^{-(1+d)2^{2p+x-x_0}} .$$

(In summary, we choose γ, then β and β', then p_0!.) $\square$

Comment. Proposition 5.1 will be used mostly for $x = p$. A difficulty there is that for this choice of x it does not seem possible to guarantee that the probability that (5.1) does not happen is less than (say) $1/\log(R_{2p}^{-1}) \cong 2^{-2^{2p}}$. This makes it impossible to control (as we did in the transient case) the contribution of the points where (5.1) fails by the modules of continuity. (To circumvent the difficulty, we will use a two stage procedure).

We now complete the proof of Theorem 1.3. We set $B_1 = \{t; |t| \le 1\}$. Using Fubini's theorem, we see that $\sum_p P(\Omega_p^c) < \infty$ where Ω_p is the event defined by the following conditions

$$\lambda(U_p) \ge (1 - 2^{-2^p})\lambda(3B_1) \tag{5.2}$$

where

$$U_p = \left\{ t \in 3B_1;\ \exists R_{2p} \le r \le R_p; \lambda(\{u; |u-t| \le r, |X(t) - X(u)| \le 4r\}) \right.$$
$$\left. \ge p\frac{r^d}{K}\log\frac{1}{r} \ge \frac{1}{K}r^d \log\frac{1}{r}\log\log\log\frac{1}{r} \right\} .$$

$$\lambda(V_p) \ge (1 - 2^{-(1+d)2^{4p}})\lambda(2B_1) \tag{5.3}$$

where

$$V_p = \{t \in 2B_1, \exists r, R_{4p} \le r \le R_{2p};$$
$$\lambda(\{u; |u-t| \le r^\beta; |X(t) - X(u)| \le 4r\}) \ge \tfrac{1}{K} r^d \log\tfrac{1}{r} \} .$$

If C is a dyadic cube of order $\ell \ge 2^{2^{4p}}$ that meets B_1, its image under $X(\cdot)$ has diameter at most $K2^{-\ell\alpha}\sqrt{\ell}$. (5.4)

To ensure (5.2), we use Proposition 5.1 with $x = p$; and with $x = x_0$ (and $2p$ rather than p) to ensure (5.3). As for (5.4), this of course follows from Lemma 2.3.

Before we finish the proof, let us recall the following standard fact.

Lemma 5.2. *Given a family of balls $\mathscr{F}$ of bounded radius of $\mathbb{R}^d$, there is a disjoint subfamily $\mathscr{F}'$ such that if one enlarges the radius of the balls of $\mathscr{F}'$ by a factor 5 (without changing their centers) the resulting family $\mathscr{F}''$ covers $\mathscr{F}$.*

Proof. Think of Vitali's covering theorem. $\square$

We now continue the proof of Theorem 1.3. For a ball A of $\mathbb{R}^d$, we denote by r_A its radius. Consider the family $\mathscr{F}$ of balls A of $\mathbb{R}^d$, of radius $R_{2p} \le r_A \le R_p$ that satisfy

$$\lambda(\{u \in 3B_1; X(u) \in A\}) \ge \frac{r_A^d}{K}\log\frac{1}{r_A}\log\log\log\frac{1}{r_A} \ . \tag{5.5}$$

Consider the disjoint subfamily $\mathscr{F}'_1$ of $\mathscr{F}_1$ and the family $\mathscr{F}''_1$ given by Lemma 5.2. Consider the family $\mathscr{F}_2$ of balls A of $\mathbb{R}^d$, of radius $R_{4p} \le r_A \le R_{2p}$, that are disjoint from the balls of $\mathscr{F}''_1$ and that satisfy

$$\lambda(\{u \in 2B_1; X(u) \in A\}) \ge \frac{r_A^d}{K} \log \frac{1}{r_A} \ . \tag{5.6}$$

and the disjoint subfamily $\mathscr{F}'_2$ and the family $\mathscr{F}''_2$ given by Lemma 5.2.

First, we observe from (5.5) that

$$\sum_{A \in \mathscr{F}'_1} r_A^d \log \frac{1}{r_A} \log\log\log \frac{1}{r_A} \le K \ . \tag{5.7}$$

Next, we observe that if $X(u) \in A$ for $A \in \mathscr{F}_2$, then we must have $u \notin U_p$ (for otherwise there is $r, R_{2p} \le r \le R_p$, such that the ball of center $X(u)$ and radius r belongs to $\mathscr{F}_1$, and thus is contained in the union of the balls of $\mathscr{F}''_1$). Thus

$$\bigcup_{A \in \mathscr{F}'_2} \{u \in 2B_1; X(u) \in A\} \subset 3B_1 \backslash U_p$$

and combining with (5.6), (5.2),

$$\sum_{A \in \mathscr{F}'_2} r_A^d \log \frac{1}{r_A} \le K 2^{-2^p} \ .$$

Since $\log\log\log \frac{1}{r_A} \le Kp$ for $A \in \mathscr{F}'_2$, we get

$$\sum_{A \in \mathscr{F}'_2} r_A^d \log \frac{1}{r_A} \log\log\log \frac{1}{r_A} \le K \tag{5.8}$$

(with huge room to spare) Consider now the smallest integer ℓ such that $K 2^{-\ell\alpha} \sqrt{\ell} \le R_{4p}$, where K is the constant of (5.4). Thus $\ell \le K 2^{2^{4p}}$.

Consider the family G of balls obtained by taking each ball of $\mathscr{F}'_1, \mathscr{F}'_2$, and tripling its radius. If $u \in B_1$ is such that $X(u)$ does not belong to the union of G, the dyadic cube of order ℓ that contains it is entirely in $2B_1 \backslash V_p$. There are at most $M = K 2^{N\ell} 2^{-2^{4p}(1+d)}$ dyadic cubes of order ℓ contained in $2B_1 \backslash V_p$. The image of each of them has a diameter $\le K 2^{-\ell\alpha} \sqrt{\ell} := r_0$. Thus the part of $X(B_1)$ not covered by G can be covered by M balls of radius r_0, and

$$\begin{aligned} M r_0^d \log \frac{1}{r_0} \log\log\log \frac{1}{r_0} &\le K M 2^{-\ell N} \ell^{1+\frac{d}{2}} p \\ &\le K p 2^{-d 2^{4p-1}} \le K \ . \end{aligned}$$

The proof is complete. $\square$

As for the proof of Theorem 1.4, it is best described by saying that one combines the methods of Theorem 1.2 and Theorem 1.3; or, alternatively, that this proof is to Theorem 1.3 what Theorem 1.2 is to the results of [T2]. There is however one difficulty, namely that (with the notation of Theorem 1.2) one controls $|X^2(t) - X^2(s)|$ only by $K|t-s|$, while in the argument of Theorem 1.3, there are points possibly at distance of order one, the images of which are put together in one of the balls of our covering. This difficulty is solved by a refinement of the covering principle of Lemma 5.2 (using balls in $\mathbb{R}^{Nk+dk}$ rather than $\mathbb{R}^{dk}$). The details are better left to the interested reader.

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