Principal Component Analysis of the Volatility Smiles and Skews

Professor Carol Alexander
Chair of Risk Management
ISMA Centre
University of Reading
www.ismacentre.rdg.ac.uk

Motivation

- Implied volatilities are derived from market prices. So you may ask, if the underlying price changes how will the implied volatilities change?
- This is a very interesting question for option traders, because the answer will give us the volatility sensitivity to price term, \( \partial \sigma / \partial S \), that is an important determinant of the option delta.
- An answer to this question will also:
  - Tell us how to construct scenarios for prices and fixed strike (or fixed delta) implied volatilities
  - Indicate how to correlate the two diffusion processes in a two factor model for option pricing with stochastic volatility.
Application to Delta Hedging

- With non-constant volatility the delta of an option \( f(S, \sigma) \) is:

\[
\Delta(S, \sigma) = \frac{\partial f}{\partial S} + \left[ \frac{\partial f}{\partial \sigma} \right] \frac{\partial \sigma}{\partial S}
\]

\[
= \Delta_{BS} + \text{vega} \left[ \frac{\partial \sigma}{\partial S} \right]
\]

- Traders often approximate \( \frac{\partial \sigma}{\partial S} \) by \( \frac{\partial \sigma}{\partial K} \)

- This paper shows how to calculate \( \frac{\partial \sigma}{\partial S} \) using principal component analysis of the volatility smile

Application to Smile Scenarios

Figure 2.5: Black-Scholes Smile Surface for FTSE Options, December 1997
Principal Component Analysis

• Various attempts to model volatility smiles and skew with principal component analysis have almost invariably used daily changes in implied volatilities, by strike or by moneyness, as the input to PCA.
• Derman and Kamal (1997) analyze S&P500 and Nikkei 225 index options where the volatility surface is specified by delta and maturity.
• Skiadopoulos, Hodges and Clewlow (1998) apply PCA to log differences of implied volatilities for fixed maturity buckets.
• Fengler et.al. (2000) employ a common PCA that allows options on equities in the DAX of different maturities to be analyzed simultaneously.

Fixed Strike Deviations

• There is an important difference between the research just cited and the approach taken in this paper.
• Instead of applying PCA to daily changes in implied volatilities, a PCA is applied to daily changes in the deviations of fixed strike volatilities from at-the-money volatility.
• The advantages of this approach are both empirical and theoretical.
Empirical Advantages

• Time series data on fixed strike or fixed delta volatilities often display much negative autocorrelation, possibly because markets over-react.
• But the daily variations in fixed strike deviations from ATM volatility are much less noisy than the daily changes in fixed strike (or fixed delta) volatilities.
• Consequently the application of PCA to fixed strike deviations from ATM volatility yields more robust results.

Theoretical Advantages

• It will be shown that the models of the skew in equity markets that were introduced by Derman (1999) can be expressed in a form where fixed strike volatility deviations from ATM volatility always have the same relationship with the underlying index.
• The particular market regime is determined only by a different behaviour in ATM volatility.
• Thus the stability of PCA on daily changes in fixed strike deviations is implied by Derman's models.
Equity Index Volatility Regimes

- Derman (1999) formulated three different types of market regime and defined a different linear parameterization of the volatility skew in each regime.
- These are known as ‘sticky’ models, because each parameterization implies a different type of ‘stickiness’ for the local volatility in a binomial tree.
- For a fixed maturity $t$ denote by $\sigma_K$ the implied volatility of an option with strike $K$, and by $\sigma_{ATM}^t$ the volatility of the $t$-maturity ATM option. Let $S$ the current value of the index and $\sigma_0$ and $S_0$ the initial implied volatility and price used to calibrate the tree:
Sticky Strike

In a **range bounded market** skews should be parameterized as

\[ \sigma_K = \sigma_0 - b (K - S_0) \]

So fixed strike volatility \( \sigma_K \) is independent of the index level. Since

\[ \sigma_{ATM} = \sigma_0 - b (S - S_0) \]

\( \sigma_{ATM} \) will decrease as the index increases

---

Sticky Delta

In a **stable trending market** skews are parameterized as:

\[ \sigma_K = \sigma_0 - b (K - S) \]

So fixed strike volatility \( \sigma_K \) will increase with the index level. But \( \sigma_{ATM} \) will be independent of the index:

\[ \sigma_{ATM} = \sigma_0 \]

ATM volatility will remain constant as the index moves.

---

Local volatilities are constant with respect to delta. That is, it is the delta (or moneyness) of the option that determines the local volatility in the tree. For a change in the index we move to a different tree, the one corresponding to the new option delta.
Sticky Tree

In jumpy markets skews should be parameterized as:

\[ \sigma_K = \sigma_0 - b(K+S) + 2bS_0 \]

So fixed strike volatility \( \sigma_K \) will decreases when the index goes up, and increase when the index falls. Since

\[ \sigma_{ATM} = \sigma_0 - 2b(S-S_0) \]

ATM volatility will increase as the index falls, and twice as fast as the fixed strike volatilities do.

Local volatilities are no longer constant in the tree, but there is one unique tree that can be used to price all options, that is determined by the current skew.

Fixed Strike Deviations

In all of Derman's 'sticky' models there is a linear relationship between the deviation of a fixed strike volatility from ATM volatility and the underlying price:

\[ \sigma_K - \sigma_{ATM} = -b(K-S) \]

Derman’s models imply that for any given maturity, the deviations of all fixed strike volatilities from ATM volatility will change by the same amount \( b \) as the index level changes.
Effect of Index Change

Figure 6.5a: Parallel Shift in Skew Deviations as Price Moves Up

\[ \sigma_{L}(t) - \sigma_{H}(t) \]

\[ d_{L} \]

\[ d_{H} \]

Strike

Figure 2b: Parallel Shifts in Fixed-Strike Volatilities as Price Moves Up

\[ \sigma_{L} = \sigma_{L} + b(t) \]

\[ \sigma_{L} = \sigma_{L} + b(t) \]

\[ \sigma_{L} = \sigma_{L} - b(t) \]

\[ \sigma_{L} = \sigma_{L} - b(t) \]

\[ \sigma_{L} = \sigma_{L} - d_{L} + b(t) \]

Trending

Range-bounded

Jumpy

Copyright 2001, Carol Alexander
Non-Parallel Shifts in Very Short Term Volatilities

- Derman’s models predict that all shifts in the skew are parallel.
- Three month implied volatilities on the FTSE100 and SP500 do appear to have fairly parallel shifts.
- However, the behaviour of two month and, more particularly, one month volatilities in the FTSE100 market appear to be highly non-linear.
- Often there is a range narrowing in the skew when the market moves up and a range widening when the market moves down.
1mth Deviations

Figure 3: Deviations of 1mth Fixed-Strike Volatility from At-the-Money Volatility

Principal Component Analysis

- Non-linear movements in the skew may be modelled using principal component analysis
- For a fixed volatility maturity $t$ and strike $K$ a three component principal component decomposition is used:

$$\Delta(\sigma_K - \sigma_{ATM}) = \omega_{K,1} P_1 + \omega_{K,2} P_2 + \omega_{K,3} P_3$$  \hspace{1cm} (1)

- Daily data on $\Delta(\sigma_K - \sigma_{ATM})$ are used to estimate the time series of principal components $P_1$, $P_2$ and $P_3$, and the constant factor weights $\omega_{K,1}$, $\omega_{K,2}$ and $\omega_{K,3}$. 
Factor Weights (3 mth)

<table>
<thead>
<tr>
<th>PC</th>
<th>Eigenvalue</th>
<th>Cumulative R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>13.3574</td>
<td>0.742078</td>
</tr>
<tr>
<td>P2</td>
<td>2.257596</td>
<td>0.8675</td>
</tr>
<tr>
<td>P3</td>
<td>0.691317</td>
<td>0.905906</td>
</tr>
</tbody>
</table>

| P1  | 0.53906 | 0.74624 | 0.26712 |
| P2  | 0.6436  | 0.7037  | 0.1862  |
| P3  | 0.67858 | 0.58105 | 0.035155|
| 4225| 0.8194  | 0.48822 | -0.03331|
| 4625| 0.84751 | 0.34675 | -0.19671|
| 4725| 0.86724 | 0.1287  | -0.41161|
| 4825| 0.86634 | 0.01742 | -0.43254|
| 4925| 0.80957 | -0.01649| -0.28777|
| 5025| 0.9408  | -0.18548| 0.068028|
| 5125| 0.92639 | -0.22766| 0.13049 |
| 5225| 0.92764 | -0.21065| 0.12154 |
| 5325| 0.93927 | -0.22396| 0.14343 |
| 5425| 0.93046 | -0.25167| 0.16246 |
| 5525| 0.90232 | -0.20613| 0.017523|
| 5625| 0.94478 | -0.2214 | 0.073863|
| 5725| 0.94202 | -0.22928| 0.073997|
| 5825| 0.93583 | -0.22818| 0.074602|
| 5925| 0.90699 | -0.22788| 0.068758|

Dynamics of Fixed Strike Volatilities with Respect to Price

Each component is assumed to have a linear relationship with daily changes $\Delta S$ in the underlying:

$$P_i \approx \gamma_i \Delta S \quad (2)$$

The skew will only shift parallel as the index moves if

$$\gamma_2 = \gamma_3 = 0.$$  

If $\gamma_2 < 0$ the range of the skew will narrow and if $\gamma_2 > 0$ the range of the skew will widen as the index moves up.

NB: The gamma are time-varying parameters representing the conditional correlations between the principal componants and the index. The unconditional correlations are zero, by definition.
Effect of Index Change: $\gamma_2 < 0$

Figure 5a: Non-Parallel Shift in Skew Deviations as Price Moves Up

![Graph showing non-parallel shift in skew deviations](image)

Figure 6a: Effect on Fixed-Strike Volatilities as Price Moves Up ($\gamma_2 < 0$)

![Graph showing effect on fixed-strike volatilities](image)

Range Narrowing

Normally $e_L$ is a little greater than $d_L$ unless $\gamma_2$ becomes very large and negative. The range of the skew will narrow, more so when $\gamma_2$ is very large and negative.

But $e_H$ will be less than $d_H$ so most of the movement in the skew will come from the low strike volatilities and there may be little movement in high strike volatilities.
Effect of Index Change: $\gamma_2 > 0$

Figure 5b: Non-Parallel Shift in Skew Deviations as Price Moves Up

Range Widening

Figure 6b: Effect on Fixed-Strike Volatilities as Price Moves Up ($\gamma_2 > 0$)

Normally $\sigma_1$ is a little less than $d_1$ and $\sigma_2$ is certainly greater than $d_1$.

Unless $\sigma_1$ is substantially less than $d_1$, more than usual movement in high strike volatilities will be observed. And low strike volatilities will move less than they do when $\gamma_2$ is negative.
Empirical Evidence (1mth)

Figure 7a: Gamma Estimates for 1mth Volatilities

Empirical Evidence (2mth)

Figure 7b: Gamma Estimates for 2mth Volatilities
Empirical Evidence (3mth)

Dynamics of ATM Volatility with Respect to Price

- Assume
  \[ \Delta \sigma_{\text{ATM}} = \beta \Delta S \]  

- Estimate the ATM volatility sensitivity \( \beta \) with an exponentially weighted moving average (again with \( \lambda = 0.94 \), as for the gamma coefficients).

- It is found that the sensitivity of ATM volatility will move with the level of the index. It will not jump unless the index jumps:
Fixed Strike Volatility Sensitivity: $\frac{\partial \sigma}{\partial S}$

The sensitivity of the fixed strike volatility $\sigma_k$ to the index is given by combining (1), (2) and (3):

$$\Delta \sigma_k = \beta_k \Delta S$$

where

$$\beta_k = \beta + \sum \omega_{k,i} \gamma_i$$
Scenarios for Fixed Strike Volatilities

Figure 9: Change in 1mth Fixed Strike Volatility per Unit Increase in Index

Low Strike Sensitivities

- Low strike volatilities are normally more sensitive to index changes than high strike volatilities
- The 4675 volatility gains about 1 or 2 basis points for every point fall in the FTSE index, but the sensitivity varies considerably during the period
- High sensitivity is associated with range narrowing of the skew as the index increases, and widening as the index increases, with most of the movement coming from low strike volatility
High Strike Sensitivities

- High strike volatilities are less sensitive, moving between about 0.5 and 1 basis points for every point change in the FTSE index.
- But these sensitivities showed a marked increase during the crash period: since the FTSE fell by 1500 points during the crash, the 5875 sensitivity of about 1.5 basis points indicates a 22.5% increase in 5875 volatility.
- At the height of the crash the 5875 sensitivity was an impressive -0.028, indicating a further 2.8 basis point increase in 5875 volatility would have occurred for every point off the FTSE at that time.

Conclusions

- This paper has presented a new principal component model of fixed strike volatility deviations from ATM volatility. It has been used to quantify the change that should be made to any given fixed strike volatility per unit change in the underlying.
- Empirical application of the model to the FTSE 100 index options has shown that 2mth and 3mth skews should normally be shifted parallel as the index moves, as predicted by Derman's models.
Conclusions

• But for very short term volatility the empirical analysis has revealed two distinct regimes of equity index volatilities.
• In stable markets the range of the 1mth skew narrows as the index moves up and widens as the index moves down. Most of the movement is in low strike volatilities.
• In jumpy markets the high strike volatilities move much more than usual. During the market crash and recovery of 1998 the 1mth skew range actually narrowed as the index fell and widened as the index moved up.

Summary

• Principal component analysis is a powerful analytical tool for the computation of movements in fixed strike implied volatilities as the underlying price moves.
• The model presented in this paper has extended Derman’s models of the skew in equity index markets.
• It will admit non-linear movements in the volatility smile as the underlying moves.
• It gives a formal model for computing $\frac{\partial \sigma}{\partial S}$ with applications to delta hedging.
• It has been applied to equity index markets but also has applications to currency and interest rate option markets; this is a subject of future research.
References

• Derman, E. and M. Kamal (1997) “The Patterns of Change in Implied Index Volatilities” Quantitative Strategies Research Notes, Goldman Sachs