Principal Component Analysis of the Volatility Smiles and Skews

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Motivation

- Implied volatilities are derived from market prices. So you may ask, if the underlying price changes how will the implied volatilities change?
- This is a very interesting question for option traders, because the answer will give us the *volatility sensitivity to price* term, $\partial\sigma / \partial S$, that is an important determinant of the option delta
- An answer to this question will also:
  - Tell us how to construct scenarios for prices and fixed strike (or fixed delta) implied volatilities
  - Indicate how to correlate the two diffusion processes in a two factor model for option pricing with stochastic volatility.

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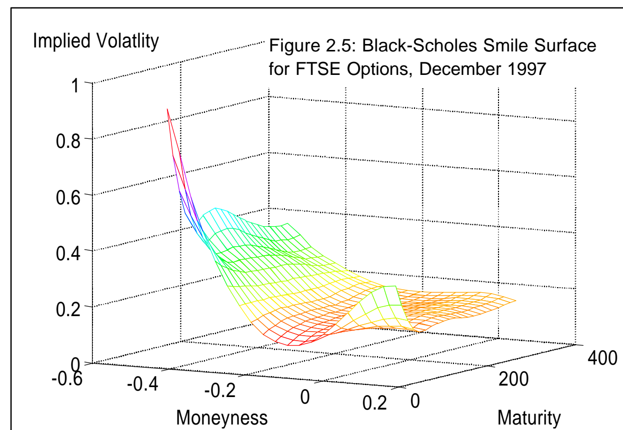
Application to Delta Hedging

- With non-constant volatility the delta of an option $f(S, \sigma)$ is:

$$\Delta(S, \sigma) = \partial f / \partial S + [\partial f / \partial \sigma][\partial \sigma / \partial S]$$
$$= \Delta_{BS} + \text{vega}\, [\partial \sigma / \partial S]$$

- Traders often approximate $\partial\sigma/\partial S$ by $\partial\sigma/\partial K$
- This paper shows how to calculate $\partial\sigma/\partial S$ using principal component analysis of the volatility smile

Application to Smile Scenarios



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Principal Component Analysis

- Various attempts to model volatility smiles and skew with principal component analysis have almost invariably used daily changes in implied volatilities, by strike or by moneyness, as the input to PCA.
- Derman and Kamal (1997) analyze S&P500 and Nikkei 225 index options where the volatility surface is specified by delta and maturity.
- Skiadopoulos, Hodges and Clewlow (1998) apply PCA to log differences of implied volatilities for fixed maturity buckets.
- Fengler et.al. (2000) employ a common PCA that allows options on equities in the DAX of different maturities to be analyzed simultaneously.

Fixed Strike Deviations

- There is an important difference between the research just cited and the approach taken in this paper.
- Instead of applying PCA to daily changes in implied volatilities, a PCA is applied to daily changes in the deviations of fixed strike volatilities from at-the-money volatility.
- The advantages of this approach are both empirical and theoretical.

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Empirical Advantages

- Time series data on fixed strike or fixed delta volatilities often display much negative autocorrelation, possibly because markets over-react.
- But the daily variations in fixed strike deviations from ATM volatility are much less noisy than the daily changes in fixed strike (or fixed delta) volatilities.
- Consequently the application of PCA to fixed strike deviations from ATM volatility yields more robust results.

Theoretical Advantages

- It will be shown that the models of the skew in equity markets that were introduced by Derman (1999) can be expressed in a form where fixed strike volatility deviations from ATM volatility always have the same relationship with the underlying index.
- The particular market regime is determined only by a different behaviour in ATM volatility.
- Thus the stability of PCA on daily changes in fixed strike deviations is implied by Derman's models.

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Derman's Models

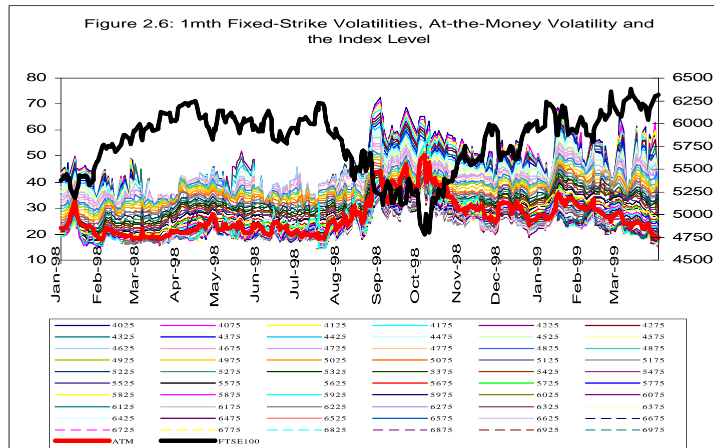

Figure 2.6: 1mth Fixed-Strike Volatilities, At-the-Money Volatility and the Index Level

Equity Index Volatility Regimes

- Derman (1999) formulated three different types of market regime and defined a different linear parameterization of the volatility skew in each regime.
- These are known as 'sticky' models, because each parameterization implies a different type of 'stickiness' for the local volatility in a binomial tree.
- For a fixed maturity t denote by σ_K the implied volatility of an option with strike K, and by σ_{ATM} the volatility of the t-maturity ATM option. Let S the current value of the index and σ_0 and S_0 the initial implied volatility and price used to calibrate the tree:

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Sticky Strike

In a **range bounded market** skews should be parameterized as

$$\sigma_K = \sigma_0 - b\,(K-S_0)$$

So fixed strike volatility σ_K is independent of the index level. Since

$$\sigma_{ATM} = \sigma_0 - b\,(S-S_0)$$

σ_{ATM} will decrease as the index increases

Local volatilities will be constant with respect to strike. That is, each option has its own binomial tree, with a constant volatility that is determined by the strike of the option. To see the effect of a change in the index all that happens is that the root of the tree is moved to the new level of the index. The same tree is still used to price the option.

Sticky Delta

In a **stable trending market** skews are parameterized as:

$$\sigma_K = \sigma_0 - b\,(K-S)$$

So fixed strike volatility σ_K will increase with the index level.
But σ_{ATM} will be independent of the index:

$$\sigma_{ATM} = \sigma_0$$

ATM volatility will remain constant as the index moves.

Local volatilities are constant with respect to delta. That is, it is the delta (or moneyness) of the option that determines the local volatility in the tree. For a change in the index we move to a different tree, the one corresponding to the new option delta.

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Sticky Tree

In **jumpy markets** skews should be parameterized as:

$$\sigma_K = \sigma_0 - b\ (K+S) + 2bS_0$$

So fixed strike volatility σ_K will decreases when the index goes up, and increase when the index falls. Since

$$\sigma_{ATM} = \sigma_0 - 2b\ (S-S_0)$$

ATM volatility will increase as the index falls, and twice as fast as the fixed strike volatilities do.

Local volatilities are no longer constant in the tree, but there is one unique tree that can be used to price all options, that is determined by the current skew.

Fixed Strike Deviations

In all of Derman's 'sticky' models there is a linear relationship between the deviation of a fixed strike volatility from ATM volatility and the underlying price:

$$\sigma_K - \sigma_{ATM} = -\ b\ (K-S)$$

Derman’s models imply that for any given maturity, the deviations of all fixed strike volatilities from ATM volatility will change by the same amount b as the index level changes.

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Effect of Index Change

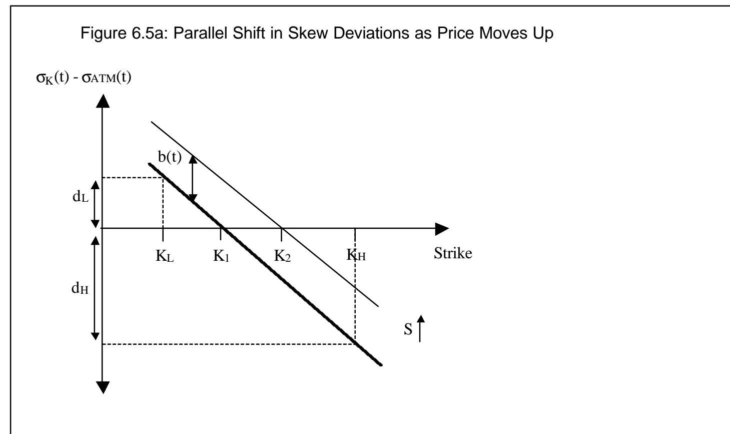

Figure 6.5a: Parallel Shift in Skew Deviations as Price Moves Up

Effect of Index Change

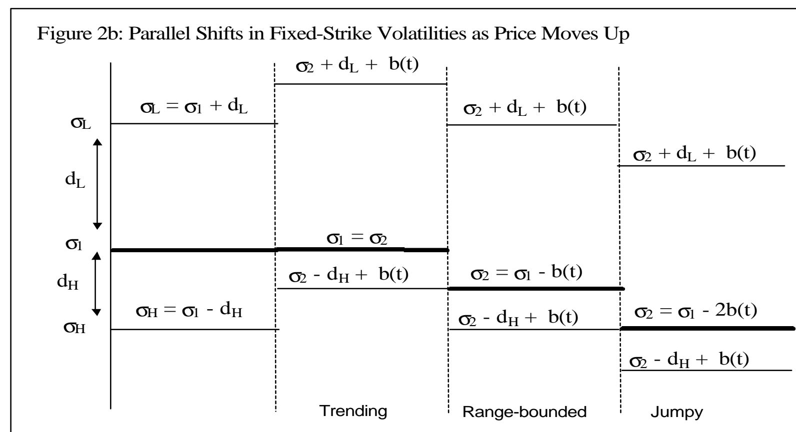

Figure 2b: Parallel Shifts in Fixed-Strike Volatilities as Price Moves Up

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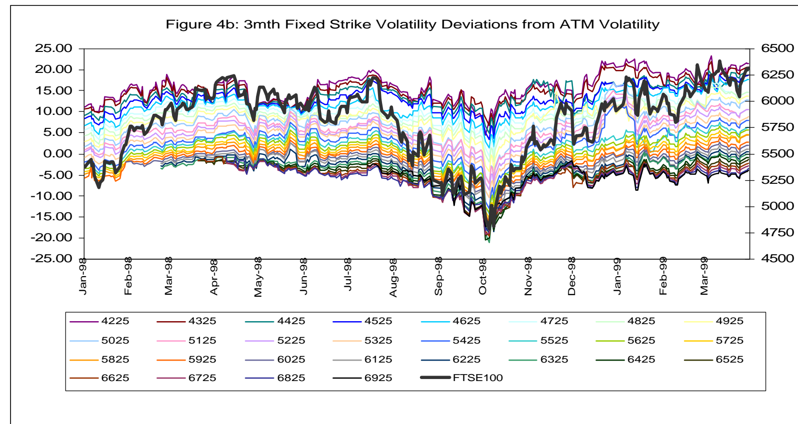
3mth Deviations



Non-Parallel Shifts in Very Short Term Volatilities

- Derman's models predict that all shifts in the skew are parallel.
- Three month implied volatilities on the FTSE100 and SP500 do appear to have fairly parallel shifts
- However the behaviour of two month and, more particularly of one month volatilities in the FTSE100 market appear to be highly non-linear
- Often there is a range narrowing in the skew when the market moves up and a range widening when the market moves down.

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1mth Deviations

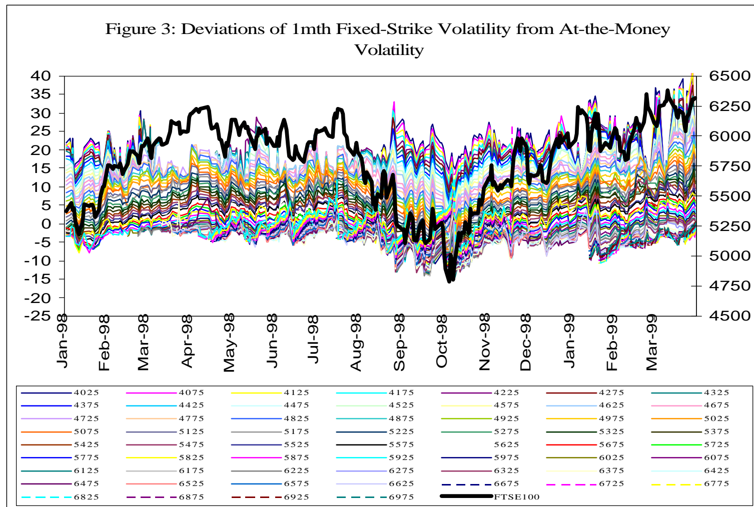

Figure 3: Deviations of 1mth Fixed-Strike Volatility from At-the-Money Volatility

Principal Component Analysis

- Non-linear movements in the skew may be modelled using principal component analysis
- For a fixed volatility maturity t and strike K a three component principal component decomposition is used:

$$\Delta(\sigma_K - \sigma_{ATM}) = \omega_{K,1}\ P_1 + \omega_{K,2}\ P_2 + \omega_{K,3} P_3 \qquad (1)$$

- Daily data on $\Delta(\sigma_K - \sigma_{ATM})$ are used to estimate the time series of principal components P_1, P_2 and P_3, and the constant factor weights $\omega_{K,1}$, $\omega_{K,2}$ and $\omega_{K,3}$.

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Factor Weights (3 mth)

	P1	P2	P3
4225	0.53906	0.74624	0.26712
4325	0.6436	0.7037	0.1862
4425	0.67858	0.58105	0.035155
4525	0.8194	0.48822	-0.03331
4625	0.84751	0.34675	-0.19671
4725	0.86724	0.1287	-0.41161
4825	0.86634	0.017412	-0.43254
4925	0.80957	-0.01649	-0.28777
5025	0.9408	-0.18548	0.068028
5125	0.92639	-0.22766	0.13049
5225	0.92764	-0.21065	0.12154
5325	0.93927	-0.22396	0.14343
5425	0.93046	-0.25167	0.16246
5525	0.90232	-0.20613	0.017523
5625	0.94478	-0.2214	0.073863
5725	0.94202	-0.22928	0.073997
5825	0.93583	-0.22818	0.074602
5925	0.90699	-0.22788	0.068758

PC	Eigenvalue	Cumulative R^2
P1	13.3574	0.742078
P2	2.257596	0.8675
P3	0.691317	0.905906

Dynamics of Fixed Strike Volatilities with Respect to Price

Each component is assumed to have a linear relationship with daily changes ΔS in the underlying:

$$P_i \approx \gamma_i \; \Delta S \qquad (2)$$

The skew will only shift parallel as the index moves if

$$\gamma_2 = \gamma_3 = 0.$$

If $\gamma_2 < 0$ the range of the skew will narrow and if $\gamma_2 > 0$ the range of the skew will widen as the index moves up.

NB The gamma are **time-varying** parameters representing the conditional correlations between the principal componants and the index. The unconditional correlations are zero, by definition.

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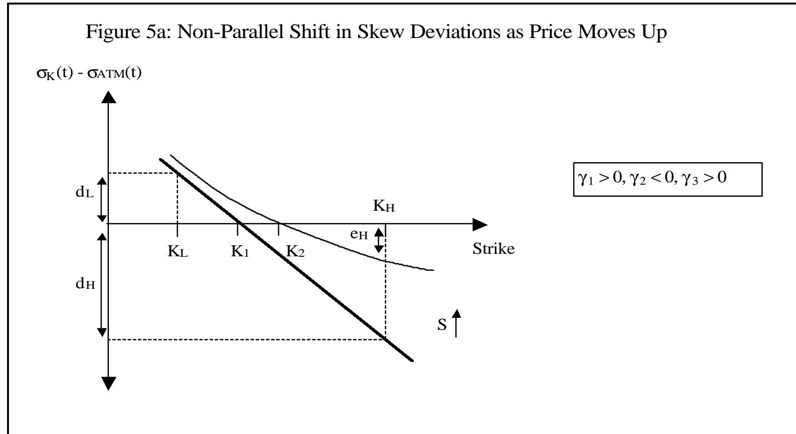
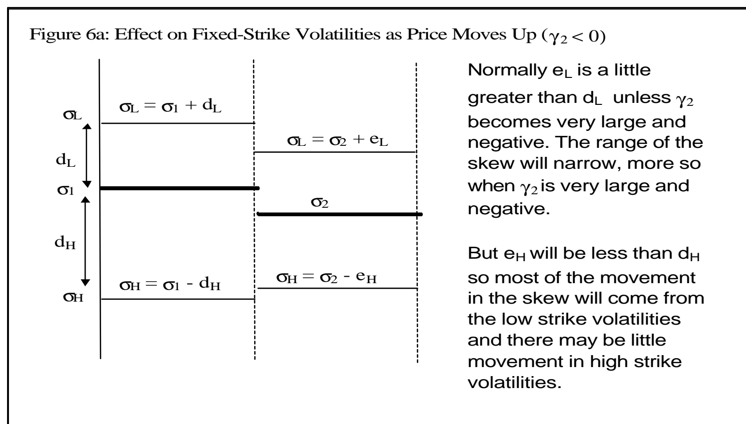
Effect of Index Change: $\gamma_2 < 0$



Range Narrowing



Normally e_L is a little greater than d_L unless γ_2 becomes very large and negative. The range of the skew will narrow, more so when γ_2 is very large and negative.

But e_H will be less than d_H so most of the movement in the skew will come from the low strike volatilities and there may be little movement in high strike volatilities.

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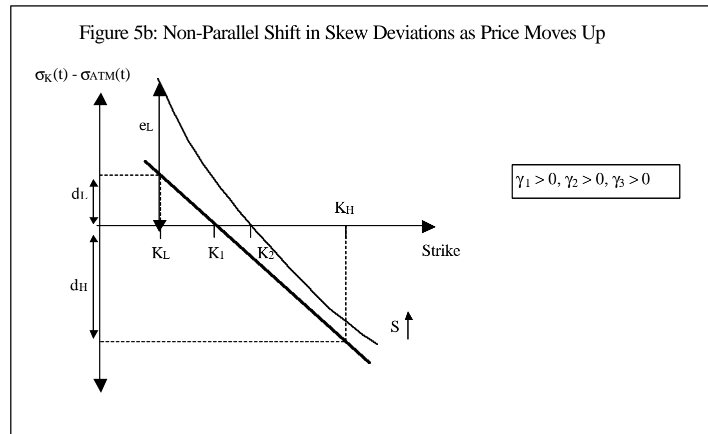
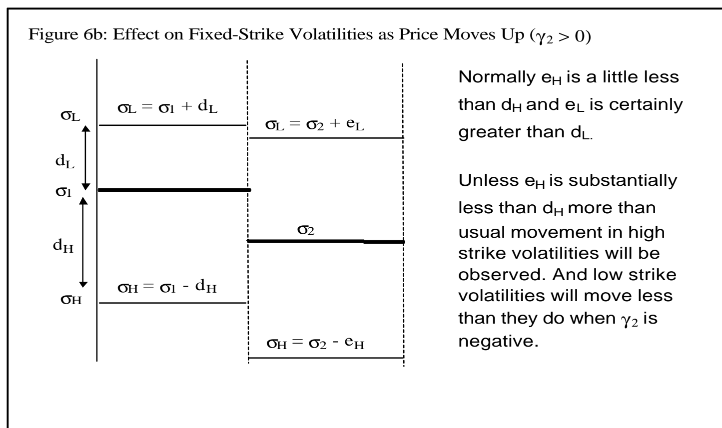
Effect of Index Change: $\gamma_2 > 0$



Range Widening



Normally e_H is a little less than d_H and e_L is certainly greater than d_L.

Unless e_H is substantially less than d_H more than usual movement in high strike volatilities will be observed. And low strike volatilities will move less than they do when γ_2 is negative.

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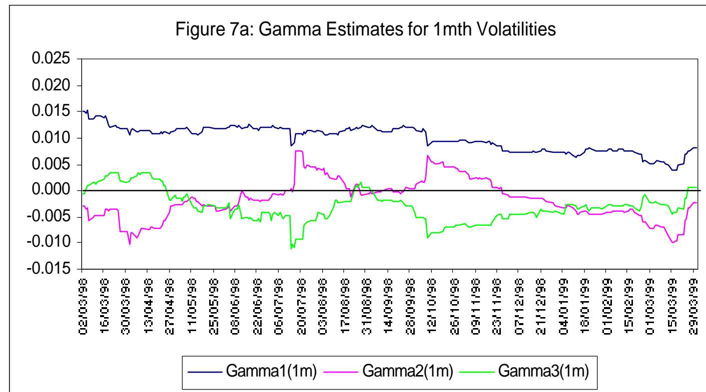
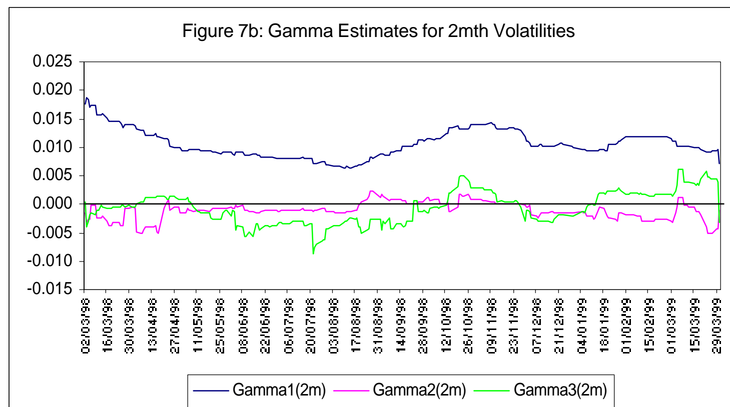
Empirical Evidence (1mth)



Empirical Evidence (2mth)



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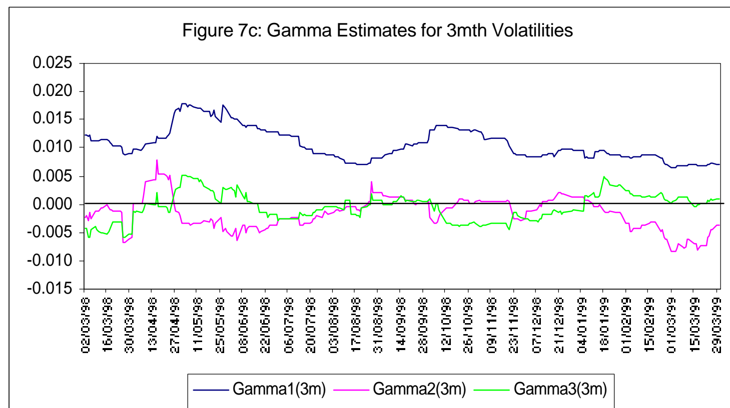
Empirical Evidence (3mth)



Dynamics of ATM Volatility with Respect to Price

- Assume

$$\Delta\sigma_{ATM} \approx \beta \, \Delta S \qquad (3)$$

- Estimate the ATM volatility sensitivity β with an exponentially weighted moving average (again with λ = 0.94, as for the gamma coefficients).
- It is found that the sensitivity of ATM volatility will move with the level of the index. It will not jump unless the index jumps:

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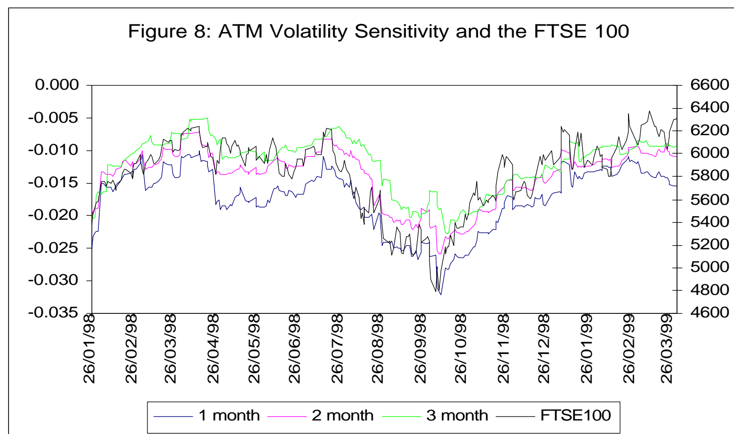
ATM Volatility Sensitivity



Fixed Strike Volatility Sensitivity: $\partial\sigma / \partial S$

The sensitivity of the fixed strike volatility σ_k to the index is given by combining (1), (2) and (3):

$$\Delta\sigma_k \approx \beta_K \, \Delta S$$

where $\beta_K = \beta + \Sigma \, \omega_{K,i} \, \gamma_i$

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Scenarios for Fixed Strike Volatilities

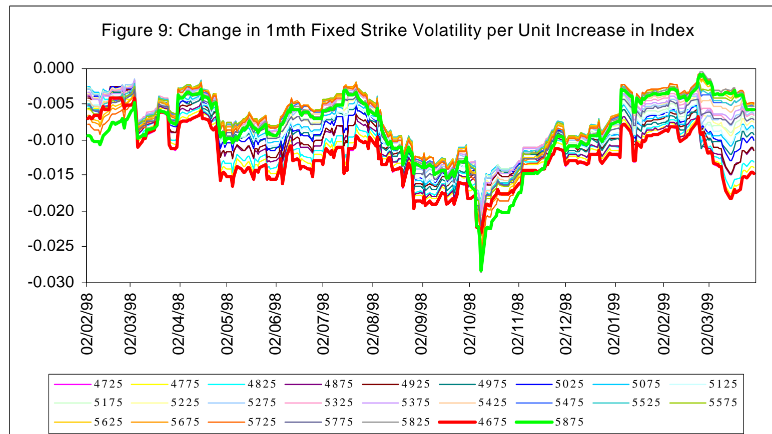

Figure 9: Change in 1mth Fixed Strike Volatility per Unit Increase in Index

Low Strike Sensitivities

- Low strike volatilities are normally more sensitive to index changes than high strike volatilities
- The 4675 volatility gains about 1 or 2 basis points for every point fall in the FTSE index, but the sensitivity varies considerably during the period
- High sensitivity is associated with range narrowing of the skew as the index increases, and widening as the index increases, with most of the movement coming from low strike volatility

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High Strike Sensitivities

- High strike volatilities are less sensitive, moving between about 0.5 and 1 basis points for every point change in the FTSE index
- But these sensitivities showed a marked increase during the crash period: since the FTSE fell by 1500 points during the crash, the 5875 sensitivity of about 1.5 basis points indicates a 22.5% increase in 5875 volatility
- At the height of the crash the 5875 sensitivity was an impressive -0.028, indicating a further 2.8 basis point increase in 5875 volatility would have occurred for every point off the FTSE at that time.

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Conclusions

- This paper has presented a new principal component model of fixed strike volatility deviations from ATM volatility. It has been used to quantify the change that should be made to any given fixed strike volatility per unit change in the underlying.

- Empirical application of the model to the FTSE 100 index options has shown that 2mth and 3mth skews should normally be shifted parallel as the index moves, as predicted by Derman's models.

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Conclusions

- But for very short term volatility the empirical analysis has revealed two distinct regimes of equity index volatilities
- In stable markets the range of the 1mth skew narrows as the index moves up and widens as the index moves down. Most of the movement is in low strike volatilities.
- In jumpy markets the high strike volatilities move much more than usual. During the market crash and recovery of 1998 the 1mth skew range actually narrowed as the index fell and widened as the index moved up.

Summary

- Principal component analysis is a powerful analytical tool for the computation of movements in fixed strike implied volatilities as the underlying price moves
- The model presented in this paper has extended Derman's models of the skew in equity index markets
- It will admit non-linear movements in the volatility smile as the underlying moves.
- It gives a formal model for computing $\partial\sigma / \partial S$ with applications to delta hedging
- It has been applied to equity index markets but also has applications to currency and interest rate option markets; this is a subject of future research.

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