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# Oxford University Diploma in Mathematical Finance

Module 3, May 2000

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## Part 1: Factor Models



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## 1.1 The Capital Asset Pricing Model

- The capital asset pricing model (CAPM) is

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

where Y is the return to the stock, X is the return to the market index and $\varepsilon$ is the stock specific return

- The original derivation of the model was based on the mean-variance analysis of Markovitz (1959)
- If risk free lending and borrowing is available to investors the returns X and Y are excess over some risk free rate (Sharpe, 1964 and Lintner, 1965)

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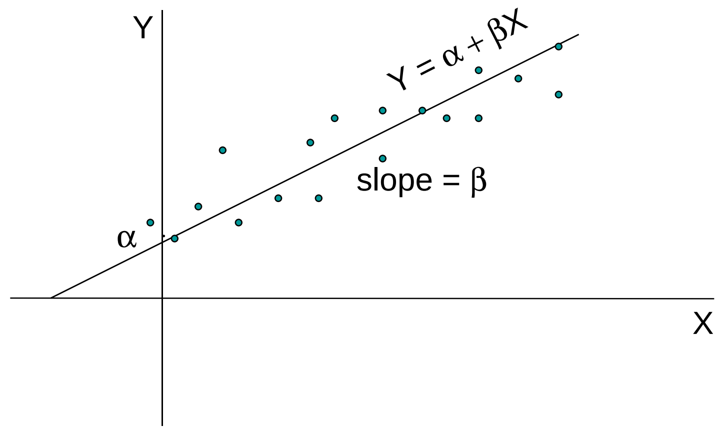
## Geometric Interpretation



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## Alpha and Beta

- The parameters of the basic CAPM are
  - the mis-pricing of the stock relative to the market, $\alpha$
  - the stock sensitivity to the market risk factor, $\beta$
- Active portfolio managers seek to gain incremental returns with a positive alpha
- But if markets are efficient and the CAPM is the correct model, alpha should be zero

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## Stock Sensitivity, Beta

- The beta provides a measure of how the stock responds to changes in the index
- If $\beta$ is insignificantly different from zero the index has no statistical effect on the stock
- If $\beta$ is insignificantly different from 1 then changes in the index are matched exactly by changes in the stock
- Stocks that have a beta significantly less than 1 are regarded as 'low risk' investments, those with beta significantly greater than 1 are categorised as 'high risk'

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## The Risk-Free Rate

- Short-term treasury bills are often taken as the 'risk free' rate. This ignores uncertainties about inflation and assumes that the bill is held to maturity
- However a risk free rate is not necessary
- If the covariance of the stock total return with the risk free return is the same as the covariance of the market return with the risk free return, then the beta may equally well be measured in terms of total returns
- This is so if the risk free rate really does have zero variance

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## Market and Specific Returns

- Changes in regulations have added to the incentive to quantify specific risk using internal models
- Factor models such as the CAPM allow the effect on returns due to the market to be stripped out (‘Index Stripping’) and so specific risk can be modelled separately
- But note that although the market portfolio should include all risky assets, many indices only contain a subset. So, for example, betas with respect to the Dow Jones 30 can be very different from betas with respect to the S&P 500

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## Estimating the Stock Alpha and Beta

- Estimates of alpha and beta will depend on
  - the data frequency
  - the historical time period used to estimate the model
  - the method used to calculate the estimates:
    - OLS (ordinary least squares regression)
    - EWMA (exponentially weighted moving averages)
    - Bayesian methods

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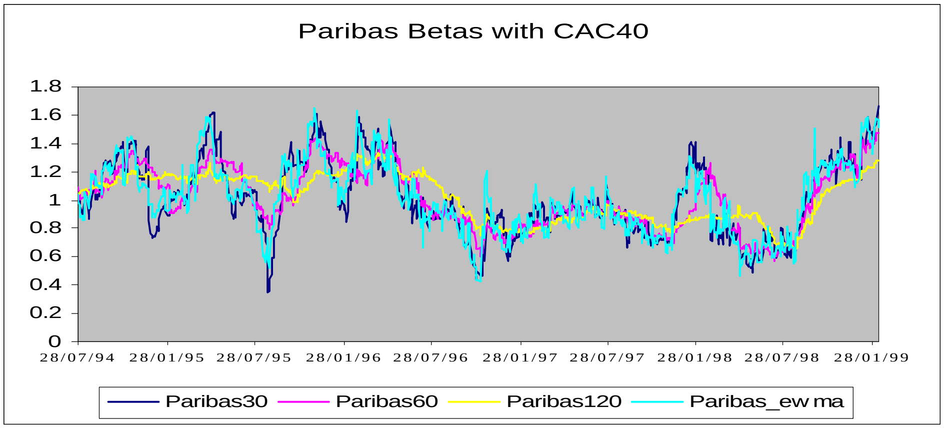
## Which is the Right Beta?



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## Estimating a CAPM by Regression

- A simple linear regression may be used to estimate the parameters of the CAPM for each stock in the portfolio.

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- The error term $\varepsilon_t$ is often assumed to be N(0, $\sigma^2$ **I** ), so stock specific returns are independent and identically distributed (i.i.d).
- In particular they are assumed to be homoscedastic and not auto-correlated, and we need a 'residual analysis' to know whether this assumption is valid

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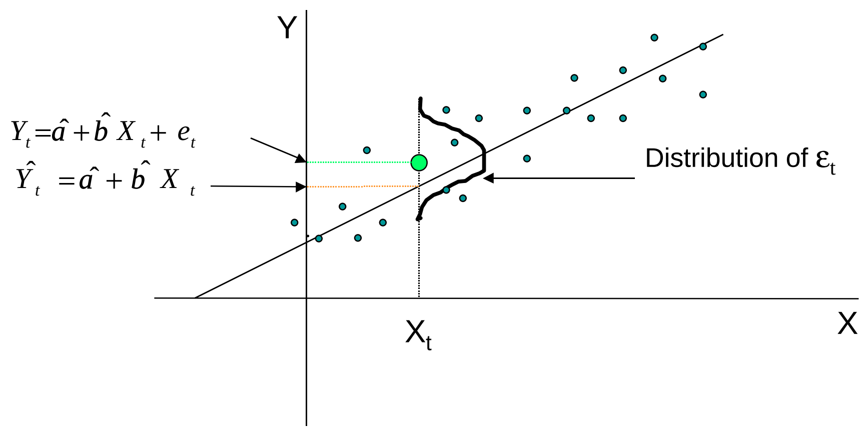
## Measuring the Specific Returns



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## Using OLS to Estimate the CAPM

- If the errors are i.i.d then ordinary least squares (OLS) may be used to estimate the parameters of the CAPM
- The OLS estimate of beta is COV(X,Y)/V(X)
- Thus beta = correlation*relative volatility
- Note that ignoring specific risk ⇒ correlation = 1, and then beta is just the relative volatility
- The OLS estimate of alpha is

  Mean stock return - beta*Mean market return

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## Example

- Estimating the daily CAPM for Electrobras with the Brazilian index Ibovespa using the regression tool in Excel gives the following output for the period 01/08/94 to 30/12/97:

| | *Coefficient* | *Standard Error* | *t Ratio* |
|---|---|---|---|
| Intercept | -0.00025 | 0.000608 | -0.41369 |
| Ibo Index | 1.211073 | 0.021586 | 56.1039 |

- But how do we know whether these parameter estimates are an accurate representation of reality?



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## Data Considerations

- Conditions in a firm or an industry can change quite abruptly.
- Correlations may be extremely variable and so betas may be unstable and difficult to predict.
- Many asset managers simply take an average beta over a long period of time, and typically 5 years of monthly data will produce robust results when fitting a CAPM.
- But such long-term averages may not be suitable for anything other than 'buy and hold' strategies.
- When interim rebalancing is required to take profits or cut losses, more sensitive beta estimates may be necessary.

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## Bayesian Betas

- Some securities firms (for example Merrill Lynch) have published stock betas based on Bayesian methods.
- Bayesian regression will give beta estimates that are influenced by prior beliefs about the stock beta.
- Depending on the nature of these beliefs the Bayesian betas can be quite different from standard beta estimates.

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## Example

- Compare the OLS and Bayesian estimates of the beta for Electrobras in the Brazilian 'Ibovespa' index based on daily data for the period 01/08/94 to 30/12/97.

| | OLS | Bayesian with Prior $N(0.5,0.1^2)$ | Bayesian with Prior $N(0.5, 0.01^2)$ |
|---|---|---|---|
| Estimate of Market Beta | 1.211 | 1.179 | 0.6256 |
| Estimated Standard Error | 0.021586 | 0.0211 | 0.0091 |

Bayesian estimates have lower standard errors

Uncertain prior ⇒ Bayesian and OLS estimates are similar

Confident prior ⇒ Bayesian and OLS estimates are different

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## Bayesian Betas

- Factor Model: $\mathbf{y} = \mathbf{X}\beta + \varepsilon$ where $\varepsilon \sim N(0, \sigma^2\mathbf{I})$
- OLS estimates of market betas: $\mathbf{b} = (\mathbf{X'X})^{-1}\mathbf{X'y}$
- Assume the conjugate prior: $\beta \sim N(\beta_0, \Sigma_0)$.
- Then the Bayesian estimates of the market betas are:

$$\mathbf{b}^* = \Sigma^{*\,-1} ( \Sigma_0^{-1}\beta_0 + \Sigma_1^{-1}\mathbf{b})$$

with inverse covariance matrix $\Sigma^{*\,-1} = c\,(\Sigma_0^{-1} + \Sigma_1^{-1})$

where $\Sigma_1 = s^2(\mathbf{X'X})^{-1}$ and c = m/(m-2), m being the degrees of freedom in the model.

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## Time-Varying Betas

- Estimates based on constant parameter models can be very inaccurate
- Sensitivities vary over time, but OLS estimates don't fully reflect this variation
- Use exponentially weighted moving average (EWMA) or GARCH models of covariance and variance to estimate time-varying sensitivities for each stock in the portfolio
- The specific return for each stock are then given by the residual $Y_t - \beta_t X_t$
- Specific risks are calculated from the EWMA (or GARCH) covariance matrix of the residuals

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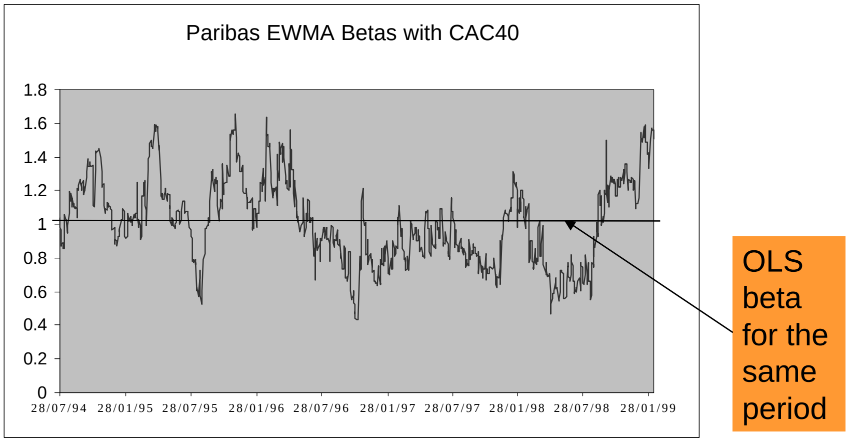
## Comparison of OLS and EWMA Betas



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## Portfolio Betas

- Portfolio betas may be measured directly by using returns data at the portfolio level.
- More usually the individual stock betas are weighted by the proportion of the portfolio Y invested in stock y, $w_y$.
- Then summation gives the net portfolio beta:

$$\beta_Y = \Sigma w_y \beta_y$$

## Portfolio Betas

- The portfolio beta provides a very simple framework for predicting portfolio returns and modelling portfolio risk.
- A 1% fall in the market is expected to be matched by a $\beta$% fall in the portfolio
- Portfolios with betas greater than 1 are considered more risky than those with betas less than 1.





## Portfolio Objectives

- A portfolio manager seeking only to track an index will diversify the portfolio to achieve
  - a beta of 1
  - an alpha of 0
  - residual returns as small as possible to minimise the tracking error.
- On the other hand active portfolio managers may have betas that are somewhat greater than 1 if they are willing to accept an increased risk for the incremental return above the index

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## Using the CAPM for Risk Attribution

- The CAPM attributes portfolio risk to three sources:
- Assuming $COV(X_t, \varepsilon_t) = 0$ for all t, taking variances gives a risk decomposition into market and specific components:

$$V(Y_t) = \beta^2 V(X_t) + V(\varepsilon_t)$$

- Thus portfolio risk, as measured by the variance of portfolio returns $V(Y_t)$, depends on
  - risk factor sensitivities (only one in the CAPM)
  - market risk (variance of the risk factor)
  - specific risk (residual variance)



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## Limitations of the CAPM

- Of course, the assumption $COV(X_t, \varepsilon_t) = 0$ depends very much on model specification.
- In particular if relevant risk factors are omitted from the model, the variation from these factors can only be attributed to the specific or residual risk.
- If these omitted factors are correlated with the included factors, the assumption $COV(X_t, \varepsilon_t) = 0$ will not hold.
- So it is important to include all possible risk factors in the model for a linear portfolio, and often there are too many sources of risk to capture with the simple CAPM alone.

## 1.2 APT Factor Models

- More general factor models based on the Arbitrage Pricing Theory (APT) represent the returns to each asset in a cash portfolio with many risk factors:

$$Y_{jt} = \beta_{1j} X_{1t} + \beta_{2j} X_{2t} + \ldots\ldots + \beta_{kj} X_{kt} + \varepsilon_{jt}$$

$Y_j$ = returns to jth asset in the portfolio ($j = 1,\ldots, n$)

$X_i$ = return to the ith factor ($i = 2, \ldots.\ K$, and $X_1 = 1$)

$\beta_{ij}$ = beta of the jth asset to the ith factor

$\varepsilon_j$ = residual of the jth asset



## The Multicollinearity Problem

- There is a danger that highly correlated factors will cause severe model bias
- A rule of thumb is not to use factors that have a correlation higher than the $R^2$ from the regression
- But this can place too many constraints on a model that is specified according to economic theory or fundamental market factors
- Since multicollinearity is not a problem in statistical factor models (the principal components are orthogonal by definition), more and more models are being based on this approach

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## Risk Decomposition

Just as in the CAPM, multi-factor models allow the portfolio risk to be attributed to risk factor sensitivities, factor risk and specific risk:

$$V_P = \mathbf{W'BV_XB'W} + \mathbf{W'VW}$$

$V_p$ = portfolio variance

**W** = nx1 vector of portfolio holdings

**B** = nxk sensitivity matrix

**W'B** = net sensitivities

$\mathbf{V}_x$ = kxk covariance matrix of factor returns

**V** = nxn covariance matrix of specific risks

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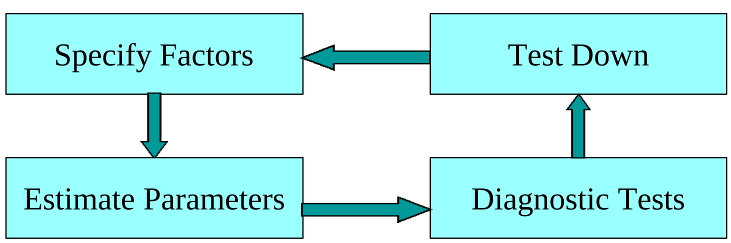
## 1.3 Model Specification

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## Types of Factor Models

- Macroeconomic factor models

  The effect of factors such as inflation, interest rates, and growth is commonly estimated by regression

- Fundamental factor models

  Equity indices, FX and interest rates, P/E ratios, dividend yields, market cap - again estimated by regression

- Statistical factor models

  Principal components or factor analysis - eigenvalue methods are described in part 2.

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## Specifying the Factors

- Determine potential factors by a consensus view from market analysts
- Certain factors such as P/E ratio, book-to-price ratio, debt/equity ratio and market capitalisation have emerged as standard 'fundamental' factors.
- Economic factor models explain stock returns with the equity index, inflation, interest rates and other macro-economic factors
- In statistical factor models one can specify how much of the variation is to be explained as an alternative to specifying the number of factors

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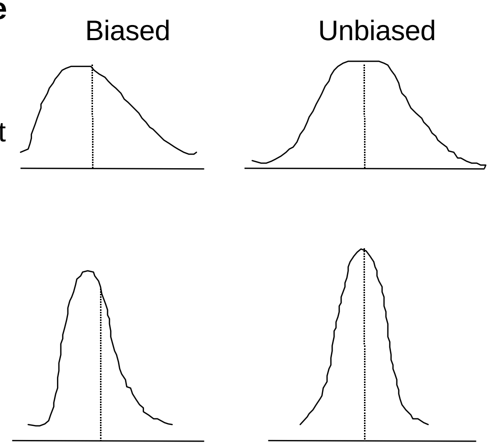
## Parameter Estimation

**OLS estimators are unbiased and efficient……**




**…if errors are i.i.d**



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## OLS

- Write the model in matrix form:

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon$$

where **y** is the vector of time series data on stock returns, $\beta$ is the risk factor sensitivity vector, $\varepsilon$ denotes the error process and the risk factor data is put into a Txk matrix

$$\mathbf{X} = \begin{pmatrix} | & | & & & | \\ X_1 & X_2 & . & . & X_k \\ | & | & & & | \end{pmatrix}$$

- The OLS estimate of $\beta$ is $(\mathbf{X'X})^{-1}\mathbf{X'y}$

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## Diagnostic Tests

- The significance and robustness of individual factor sensitivities is given by their t-ratios

t = coefficient estimate/(estimated s.e. of coefficient estimate)

- Standard errors of coefficients are estimated in $s^2 (\mathbf{X'X})^{-1}$ where **X** is the matrix of data on all explanatory variables
- $s^2$ is the OLS estimate of $\sigma^2$
- $s^2$ = sum of squared residuals /(T-k) where T is the number of observations and k is the number risk factors including a constant.

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## Example for Testing Down

| Variable | Coefficient | t-Ratio |
|---|---|---|
| Constant | 0.0002 | 1.2 |
| Factor 1 | 0.76 | 4.5 |
| Factor 2 | 0.34 | 1.5 |
| Factor 3 | 1.17 | 10.2 |
| Factor 4 | -0.88 | -2.3 |
| Factor 5 | 0.33 | 0.97 |

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## Specific Return Diagnostics

- Most methods will not give good parameter estimates unless the stock specific returns are i.i.d.
- So there are standard tests for:
  - Autocorrelation
  - Heteroscedasticity
  - Normality

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## And if Diagnostic Tests Fail…..?

- If the specific return diagnostics indicate that residuals are autocorrelated, or heteroscedastic then the standard OLS method for estimating alpha and beta is no longer best
- Instead generalized least squares (GLS) could be used

  The GLS estimate of $\beta$ is $(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y}$

  where $\boldsymbol{\Omega}$ is the covariance matrix of the residuals
- So the parameters are estimated in two stages: first use OLS to estimate $\boldsymbol{\Omega}$, then use this estimate in the GLS formula

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## Recommended Texts

J. Campbell, A. Lo and A. MacKinley 'The Econometrics of Financial Markets' Princeton UP (1997)

J.D. Hamilton 'Time Series Analysis' Princeton UP (1994)

W. Enders 'Applied Econometric Time Series' Wileys (1995)