1. What is Implied Volatility?

- Implied volatility is:
  - the volatility of the underlying price process that is ‘implicit’ in the market price of the option, or put another way:
  - the forecast of the average volatility of the underlying over the life of the option that is implicit in investors expectations.
What Influences Implied Volatility?

- Implied volatility $\sigma$ depends on:
  - the price of the underlying $S$
  - the market price of the option $C$
  - the strike of the option $K$
  - the maturity of the option $t$
  - the interest rate $r$
  - and any other variables that influence the price of the option.

Black-Scholes Formula

If prices are governed by geometric Brownian motion (GBM) and there is perfect replication, then the current price of a call option $C$ has a closed form analytic solution:

$$ C = SN(x) - Ke^{-rt}N(x-\sigma\sqrt{t}) $$

where $x$ measures the ‘moneyness’ of the option:

- In-the-money (ITM) $x > 0$
- At-the-money (ATM) $x = 0$
- Out-of-the-money (OTM) $x < 0$

and $N(x)$ is the normal distribution function
Black-Scholes Implied Volatilities

- The values of S, K, r and t are all observable, so the volatility which is ‘implied’ in an observed market price C can be computed.

- No analytic form exists, but numerical methods (described in Chriss pp330-340) are used to approximate the value of the implicit function

\[ \sigma = f( C, S, K, r, t) \]

- Usually volatility is quoted as an annualized percentage:

\[ \text{Volatility} = 100 \sigma \sqrt{250} \%

Example

From the FT on June 15th '99: FTSE 100 Index options expiring on 18th June '99.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Calls</th>
<th>Puts</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Price</td>
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</tr>
<tr>
<td>6250</td>
<td>223</td>
<td>-</td>
</tr>
<tr>
<td>6300</td>
<td>176</td>
<td>-</td>
</tr>
<tr>
<td>6350</td>
<td>132</td>
<td>29.21</td>
</tr>
<tr>
<td>6400</td>
<td>92</td>
<td>26.32</td>
</tr>
<tr>
<td>6450</td>
<td>58.5</td>
<td>24.26</td>
</tr>
<tr>
<td>6500</td>
<td>33</td>
<td>22.51</td>
</tr>
<tr>
<td>6550</td>
<td>16</td>
<td>21.42</td>
</tr>
<tr>
<td>6600</td>
<td>5.5</td>
<td>19.53</td>
</tr>
<tr>
<td>6650</td>
<td>2</td>
<td>19.23</td>
</tr>
<tr>
<td>6700</td>
<td>0.5</td>
<td>18.59</td>
</tr>
</tbody>
</table>

The closing price on the FTSE was 6451.2.
Call and Put Volatility Skews

If call implied volatilities are significantly different from put implied volatilities it is because the evaluation model is inadequate. Probably it is a model based on spot price whereas the hedging instrument is a future.

Why the Differences between Call and Put Implied Volatilities?

- On June 15th 1999 the FTSE 100 future closed at 6486, but its theoretical fair value was 6453.92
- So the market price of a call was based on 6486 but the model price of the a call was based on 6453.92
- Market prices of call options therefore appear to be very expensive and the only way that the model can account for the high market price is to jack up the volatility.
- Similarly puts will appear less expensive than they should, so the implied volatility that is backed out of the model will be lower.
Differences between Implied and Statistical Volatility

• Implied volatilities and statistical volatilities are both forecasting the *same thing*: the volatility of the underlying asset over the life of the option.

• But the two types of volatility measure often differ.

• Because they use different data and different models:

- **Implied volatility**
  - Model is based on GBM:
    \[
    \frac{dS}{S} = \mu \, dt + \sigma \, dz
    \]
  - price increments are governed by a Wiener process (so they are independent and normal)
  - the volatility \( \sigma \) of the underlying asset \( S \) is constant.

- **Statistical volatility**
  - Return distributions are:
    - **Unconditional**
      - constant volatility
      - weighted averages
    - **Conditional**
      - stochastic volatility
      - GARCH or Diffusion

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Differences between Implied and Statistical Volatility

If statistical volatilities were correct, then differences between the implied and statistical measures of volatility would reflect a mis-pricing of the option. That is, the wrong option model is being used, or investors have irrational expectations.

If implied volatilities were correct (so the option pricing model is an accurate representation of reality, and investors expectations are correct so that there is no over- or under-pricing in the options market), then any observed differences between implied and statistical volatilities would reflect inaccuracies in the statistical forecast.

2. Smiles, Skews and Volatility Term Structures

- The smile effect in implied volatility refers to the fact that OTM options have higher implied volatilities than ATM options.
- Thus the plot of implied volatility vs moneyness (or strike) on a given day, for all options of a fixed maturity, will be ‘smile’ shaped
- The smile effect tends to increase as the option approaches expiry
Reasons for the Smile

- The volatility smile is a result of pricing model bias, and would not be found if options were priced using an appropriate model.
- Black-Scholes is based on the assumption of GBM. But Volatility is not constant, and neither are returns normally distributed.
- Thus OTM options have a greater chance of ending up ITM than the Black-Scholes formula allows.
- Consequently the Black-Scholes formula is biased to under-price OTM options.
- This under-pricing of the model compared to observed market behaviour yields higher implied volatilities for OTM options.

Reasons for the Skew

- The problem is compounded in equity markets because they often exhibit a leverage effect.
- That is volatility is often higher following market falls than it is following market rises of the same magnitude.
- So OTM puts require higher volatility to end up in-the-money than do OTM calls.
- This induces a pronounced negative ‘skew’ in the volatility smile.
Volatility Term Structures

- On any fixed date a plot of the fixed-strike implied volatilities of different maturities gives a term structure of volatilities.
- For example for the 6425 strike the implied volatility term structure on 15th June 1999 looked something like this:

Financial Times Prices of the FTSE 100 Index European Options on June 15th 1999

<table>
<thead>
<tr>
<th>Expiry end:</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strike</td>
<td>Call</td>
<td>Put</td>
<td>Call</td>
<td>Put</td>
<td>Call</td>
</tr>
<tr>
<td>6275</td>
<td>169</td>
<td>11</td>
<td>281</td>
<td>101</td>
<td>366.5</td>
</tr>
<tr>
<td>6325</td>
<td>126.5</td>
<td>19</td>
<td>245</td>
<td>115</td>
<td>332.5</td>
</tr>
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<td>89</td>
<td>31</td>
<td>214</td>
<td>133.5</td>
<td>300</td>
</tr>
<tr>
<td>6425</td>
<td>57</td>
<td>.49</td>
<td>184</td>
<td>154</td>
<td>269.5</td>
</tr>
<tr>
<td>6475</td>
<td>33</td>
<td>75</td>
<td>154.5</td>
<td>174</td>
<td>240</td>
</tr>
<tr>
<td>6525</td>
<td>16</td>
<td>108</td>
<td>128.5</td>
<td>197.5</td>
<td>213</td>
</tr>
<tr>
<td>6575</td>
<td>6.5</td>
<td>148.5</td>
<td>104</td>
<td>223</td>
<td>187</td>
</tr>
<tr>
<td>6625</td>
<td>2</td>
<td>194</td>
<td>83</td>
<td>252</td>
<td>163.5</td>
</tr>
</tbody>
</table>
Behaviour of Volatility Term Structures

- Long term volatilities will change much less than short term volatilities
- Volatility term structures mean revert to the long term average
- They may slope upwards or downwards, although they are not generally monotonic.

<table>
<thead>
<tr>
<th>Current Market Conditions</th>
<th>Slope of Term Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatile</td>
<td>Downwards</td>
</tr>
<tr>
<td>Tranquil</td>
<td>Upwards</td>
</tr>
</tbody>
</table>

The Smile Surface

- A smile surface is a surface plot of implied volatilities for different strikes (or moneyness) and maturities
- Slicing through this surface at a fixed strike or moneyness gives a volatility term structure
- Slicing through this surface at a fixed maturity gives a smile, which becomes more pronounced as maturity decreases
Fitting Smile Surfaces

- Reliable market data for all strikes and maturities are not available
- Data on OTM or very long term options is particularly unreliable since quotes may be left unchanged for days when trading is thin
- So smile surfaces must be interpolated using numerical methods such as cubic splines (see *Numerical recipes in C*).

Use of Smile Surfaces in Dynamic Delta Hedging

- The most basic dynamic hedge is to match a position in the underlying with an amount \( N(x) \) of an option
- This is hedge ratio is the option *delta*, and since
  \[
  x = \ln(S/Ke^{-r}) / (\sigma\sqrt{t} + \sigma\sqrt{t}/2)
  \]
  its value depends very much on implied volatility and maturity, as predicted by the current smile surface
- As the underlying moves over time, the position will need constant re-balancing to be *delta neutral*
- So, over a period of time, very large losses might be made if the wrong hedging volatility is used
3. Volatility Regimes

How should we model movements in implied volatility smile surfaces as the underlying price moves?

Derman’s ‘Sticky’ Models

1. Sticky Strike
   Bounded Market
   \[ \sigma_K = \sigma_0 - b(K - S_0) \]
   \( \sigma_K \) independent of \( S \)
   \[ \sigma_{ATM} = \sigma_0 - b(S - S_0) \]
   \( \sigma_{ATM} \) decreases as price increases

2. Sticky Delta
   Trending Market
   \[ \sigma_K = \sigma_0 - b(K - S) \]
   \( \sigma_K \) increases with \( S \)
   \[ \sigma_{ATM} = \sigma_0 \]
   \( \sigma_{ATM} \) independent of price

3. Sticky Tree
   Jumpy Market
   \[ \sigma_K = \sigma_0 - b(K + S) \]
   \( \sigma_K \) decreases with \( S \)
   \[ \sigma_{ATM} = \sigma_0 - 2bS \]
   \( \sigma_{ATM} \) moves twice as fast as the skew
Modelling the Relationship between ATM Volatility and Price

First question: How is ATM implied volatility likely to move as the underlying price changes?

Scatter Plots

Daily changes in FTSE and 1mth ATM vol

Daily changes in FTSE and 3mth ATM vol

Copyright ISMA centre, January 2000
Scatter Plots

Daily Change in Cable and 1M Imp Vol

Daily Change in Cable and 3M Imp Vol

Constructing a Joint Distribution of $\Delta S$ and $\Delta \sigma_{\text{ATM}}$

$$\text{prob}(\Delta \sigma_{\text{ATM}} \text{ and } \Delta S) = \text{prob}(\Delta \sigma_{\text{ATM}} | \Delta S) \times \text{prob}(\Delta S)$$
Estimating prob($\Delta\sigma_{ATM} \mid \Delta S$)

- To give conditional probabilities prob($\Delta\sigma_{ATM} \mid \Delta S$) one needs to model the relationship between implied volatility and the price.
- The linear model of ATM implied volatility and the price has been employed:

$$\Delta\sigma_{ATM} = \alpha + \beta \Delta S + \varepsilon$$

in which case

$$\Delta\sigma_{ATM} \mid \Delta S \sim N(\alpha + \beta \Delta S, \sigma_\varepsilon^2)$$

Daily Data on $\sigma_{ATM}$ and $S$

Does the FTSE100 index price have a negative relationship with 3 month ATM volatility?
Daily Data on $\sigma_{\text{ATM}}$ and $S$

Does the Cable rate have a negative relationship with 1 month ATM volatility?

$\Delta\sigma_{\text{ATM}} = \alpha + \beta \Delta\text{FTSE} + \varepsilon$

Coefficient on Daily Change in FTSE

Significance of Coefficient on Daily Change in FTSE

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We shall assume that prob(ΔS) is represented by a normal density

\[ ΔS \sim N(μ, σ^2) \]

The parameters could be obtained from statistical forecasts of the mean and variance.

Their values will depend very much on current market circumstances.

Assume the index was fairly stable, as reflected by a marginal density for one-day changes in FTSE100 of

\[ ΔS \sim N(0, 35^2) \]

The OLS estimate of a linear relationship between the one-day changes in 1 month ATM volatility \( Δσ_{ATM} \) and \( ΔS \) on 31.03.99 was:

\[
Δσ_{ATM} = -0.3 - 0.017 ΔS
\]

\( (-2.73) \quad (-10.01) \)

s.e. regression = 0.492

\[ ⇒ Δσ_{ATM} \mid ΔS \sim N(-0.3 - 0.017 ΔS, 0.492^2) \]
Prob($\Delta \sigma_{ATM}$ and $\Delta S$) in a Stable Market

$\Delta S \sim N(0, 35^2), \quad \Delta \sigma_{ATM} \mid \Delta S \sim N(-0.3-0.017 \Delta S, 0.492^2)$

Prob($\Delta \sigma_{ATM}$ and $\Delta S$) in a Jumpy Market

$\Delta S \sim N(-30, 60^2), \quad \Delta \sigma_{ATM} \mid \Delta S \sim N(-0.03 \Delta S, 1.25^2)$
4. Principal Component Models of the Smile

FTSE100 Index, 3 month ATM Volatility and the Skew: Jan ‘98 to Mar ‘99

Relationship between the Index and the Skew Deviations

Deviation of fixed strike volatility from ATM volatility

3 months

Prof. C.O. Alexander
Relationship between the Index and the Skew Deviations

Why Does $\sigma_K - \sigma_{ATM}$ Increase with $S$?

1. Sticky Strike
   - Bounded Market
   - $\sigma_K = \sigma_0 - b(K-S_0)$
   - $\sigma_K$ independent of $S$
   - $\sigma_{ATM} = \sigma_0 - b(S-S_0)$
   - $\sigma_{ATM}$ decreases as index increases

2. Sticky Delta
   - Trending Market
   - $\sigma_K = \sigma_0 - b(K-S)$
   - $\sigma_K$ increases with $S$
   - $\sigma_{ATM} = \sigma_0$
   - $\sigma_{ATM}$ independent of index

3. Sticky Tree
   - Jumpy Market
   - $\sigma_K = \sigma_0 - b(K+S)$
   - $\sigma_K$ decreases with $S$
   - $\sigma_{ATM} = \sigma_0 - 2bS$
   - $\sigma_{ATM}$ moves twice as fast as the skew
How Should the Skew be Modified as the Index Changes?

- Step 1: Model the skew deviations from ATM volatility with a principal component analysis on $\Delta(\sigma_K(t) - \sigma_{ATM}(t))$.
- Step 2: Model the relationship between the Index and the skew deviations as:
  
  $$
  \text{ith principal component } (t) = \gamma_{0,i}(t) + \gamma_i(t) \Delta FTSE + \eta_i(t)
  $$

  
  $[t = \text{option maturity (1mth, 2mth or 3mth)}]$

---

PCA of the Skew Deviations

- A principal components analysis of the daily change in $\sigma_K - \sigma_{ATM}$ shows that typically 80-90% of the variation in $\sigma_K - \sigma_{ATM}$ can be explained by 3 principal components.
- The factor weights show that the principal components are capturing:
  - parallel shift (PC1)
  - tilt (PC2)
  - convexity (PC3)
## Variation Explained by Principal Components

<table>
<thead>
<tr>
<th></th>
<th>Eigenvalue</th>
<th>Cumulative $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>3 month data</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PC1</td>
<td>13.3574</td>
<td>0.742078</td>
</tr>
<tr>
<td>PC2</td>
<td>2.257596</td>
<td>0.8675</td>
</tr>
<tr>
<td>PC3</td>
<td>0.691317</td>
<td>0.905906</td>
</tr>
<tr>
<td><strong>2 month data</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PC1</td>
<td>19.68491</td>
<td>0.855866</td>
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<tr>
<td>PC2</td>
<td>0.866442</td>
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</tr>
<tr>
<td>PC3</td>
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<td>0.929831</td>
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<tr>
<td><strong>1 month data</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PC1</td>
<td>25.7177</td>
<td>0.476254</td>
</tr>
<tr>
<td>PC2</td>
<td>11.6942</td>
<td>0.692813</td>
</tr>
<tr>
<td>PC3</td>
<td>6.119627</td>
<td>0.806139</td>
</tr>
</tbody>
</table>

## Factor Weights in PCA of $\sigma_K - \sigma_{ATM}$

<table>
<thead>
<tr>
<th>Strike</th>
<th>PC1</th>
<th>PC2</th>
<th>PC3</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.039156</td>
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<td>5925</td>
<td>0.90699</td>
<td>-0.22788</td>
<td>0.068758</td>
</tr>
</tbody>
</table>

Prof. C.O. Alexander
\[ \text{ith PC} = \gamma_0 + \gamma_1 \Delta \text{FTSE} + \eta_i \]

**\( \gamma_1 \) (parallel shift):**
- Always positive and usually very highly significant.

**\( \gamma_2 \) (tilt):**
- Often negative (stable market) but sometimes positive (jumpy market) or zero (trending market).

**\( \gamma_3 \) (convexity):**
- Often has the opposite sign to \( \gamma_2 \).

---

**What Happens to \( \sigma_K - \sigma_{\text{ATM}} \) as the Index Increases in a Stable Market?**

\[ \sigma_K - \sigma_{\text{ATM}} \]

- \( \gamma_1 > 0 \) (a) \( \sigma_K - \sigma_{\text{ATM}} \) increases with the Index
- \( \gamma_1 < 0 \) (b) The range (or slope) of the skew decreases as the index increases
Stable Market Regime

- Low K vol
- ATM vol
- High K vol

Most of the movement in volatilities comes from the low strikes

As the index moves $\sigma_K - \sigma_{ATM}$ is relatively constant for low strike volatilities.

But for high strike volatilities $\sigma_K - \sigma_{ATM}$ decreases (increases) as the index increases (decreases).

What Happens to $\sigma_K - \sigma_{ATM}$ as the Index Increases in a Jumpy Market?

- $\gamma_1 > 0$

(a) $\sigma_K - \sigma_{ATM}$ increases with the Index
(b) The range (or slope) of the skew increases as the index increases

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Jumpy Market Regime

As the index moves $\sigma_K - \sigma_{ATM}$ is quite stable for high strike volatilities.

But for low strike volatilities $\sigma_K - \sigma_{ATM}$ increases (decreases) as the index increases (decreases).

What Happens to $\sigma_K - \sigma_{ATM}$ as the Index Increases in a Trending Market?

(a) $\sigma_K - \sigma_{ATM}$ increases with the Index

(b) The range of the skew does not much change as the index increases
Trending Market Regime

As the index increases $\sigma_K - \sigma_{ATM}$ also increases for all strikes.

So for some just OTM strikes $\sigma_K - \sigma_{ATM}$ can move from negative to positive as the option moves to ITM.

Sticky Models vs PCA

Sticky regimes models assume the skew is a linear function of the strike.

Principal component analysis includes linear and non-linear effects.
Reading

N.A. Chriss (1997) *Black-Scholes and Beyond* IRWIN Chapter 8

