# Risk Decomposition for Portfolio Simulations

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**Abstract**

We describe a method to compute the decomposition of portfolio risk in additive asset components suitable for numerical simulations. The standard results in the covariance framework provide risk components computed from the derivative of the risk function with respect to the asset exposure. These results are generalized to a generic positively-homogeneous risk measure, but cannot be easily applied to value at risk because of the resulting instabilities. We show how introducing a new risk measure, the unbiased average value at risk, it is possible to split exactly the portfolio risk into stable additive components. The results obtained in this paper are general, stable, and can be used in portfolios containing products belonging any asset class. The risk-decomposition results are generalized to the computation of risk components at segment levels. Finally, we show a real-world application of the described method with a numerical example.

## 1 Introduction

One of the important results of modern portfolio theory is that portfolio risk is smaller than the sum of the risk of its constituent assets. This well-known fact is welcome because allows to lower risk but poses a challenge to the risk manager trying to identify the risk sources. Indeed, if a portfolio of assets smooths out the risk coming from all its assets, what shall we do to further reduce portfolio risk?

The purpose of risk decomposition is to determine how portfolio risk depends from its constituting assets. Given an asset in a portfolio one may ask two questions:

1. How much portfolio risk can I attribute to this asset?
2. How much would the portfolio risk change if I were to buy, or sell, a small quantity of this asset?

In the first instance we are looking for a global risk quantity: the contribution of an asset to the total portfolio risk, the answer is given by the asset *risk component*. In the second query we are looking for a small risk change: the marginal risk resulting from buying, or selling, a small position in an asset. It goes without saying that the answer to this second question is usually denoted as *marginal risk*. Answering to both questions will allow us to modify the portfolio risk profile and possibly reduce portfolio risk.

The bibliography on risk decomposition is quite limited. In the seminal paper of Garman, see reference [3], value at risk, sometimes shortened as *VaR* in this paper, is decomposed in the

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variance/covariance framework using the partial derivative of VaR. In reference [4] Jorion describes the same concepts at length and links risk component to marginal risk (see, later, formula (17)). Unfortunately this method cannot be directly applied to computer simulations of VaR because of its instability. A method that can be used for both Monte Carlo and historical simulations is described by Epperlein and and Smillie in reference [2]. However, such method feels a bit artificial in its use of kernels introduces unnecessary parameters. The method described here has the advantage of being simple enough, to retain the financial intuition, and robust so that it can be used daily by practitioners. All the parameters introduced have a straightforward statistical meaning.

In order to understand this paper it is not necessary to have a strong mathematical background, one only needs to be familiar with basic calculus and the computation of partial derivatives. It is necessary, however, to pay close attention to the notation. In order to keep the paper as simple as possible, we limit the discussions only to a single level of decomposition. Since risk components are additive, the generalization to multi-level decomposition is straightforward.

Risk decomposition can also be performed relative to a benchmark, however, the details of relative risk decomposition need a separate paper and are described in reference [5] together with risk allocation.

After introducing the paper notation in section 2, the standard risk decomposition in the covariance framework is described in section 3. In section 4 risk decomposition is defined for value at risk in a generic simulation framework, i.e. a framework that can be used both for Monte Carlo and historical simulations. The unstable components of value at risk can be replaced by more stable ones, as shown in section 5, by introducing the unbiased average value at risk. In section 6 the covariance and simulation methods are extended to a general risk decomposition formulation. The computation of risk components and marginal risk at sector level are shown in section 7. Finally, in section 8, we show an example of computation of risk decomposition for a simple, but realistic, portfolio.

# 2 Basic definitions and notation

In this section we describe the building blocks of a portfolio and the computation of its current and future value. It is also an opportunity to introduce the basic notation used in the paper.

## 2.1 Current and future portfolio values

Consider a market of $n$ assets available to be bought or sold in order to build a portfolio. Suppose that each asset[1] $i$, for $i = 1, \ldots, n$, has an outstanding value of $W_i$, so that the initial portfolio value is given by,

$$W = W_1 + \ldots + W_n \,. \tag{1}$$

Since not all assets in the universe might be present in the portfolio, we set $W_i = 0$ for the assets that are not part of the portfolio. Considering a number of assets bigger than the number of asset in our portfolio is useful, for example, when considering the risk consequences of adding a new asset to an existing portfolio.

In order to simplify the following discussions, in this paper we consider only assets for which the exposure to risk of an asset is the same as its outstanding value. Therefore, assets that have an outstanding value different from their risk exposure, such as futures or swaps, need to be split into two or more legs each of which is counted as a separate asset.

Consider now a one-period evolution of the portfolio: after a certain period of time, usually chosen to be one day, each asset will have a value, denoted by $u_i W_i$, different from the initial

[1] From now on we will often avoid to mention the possible values for $i$ which will always be understood to run from 1 to $n$.

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| **Country** |
|---|
| UK |
| France |
| Germany |
| USA |
| Mexico |
| Japan |
| China |

Table 1: List of possible segments for an example portfolio grouped according to the attribute *country*

value. We also assume that there is no re-balancing of the portfolio during the evolution period, hence the portfolio value after the evaluation period will be,

$$P = u_1\, W_1 + \,\ldots\, + u_n\, W_n\,. \tag{2}$$

Notice that whilst the values of the $W_i$'s are well known before and after the period, the quantities $u_i$'s are random variable for which we assume to be able to compute the expectation values and their moments. The choice of the probability space for the $u_i$'s is not trivial, however, we will not discuss in the current paper all the different possibilities and assume this choice to have been made.

## 2.2 Portfolio segments and asset attributes

In the analysis of a portfolio it is helpful to group the portfolio assets in subsets. We assume that any portfolio can be split into a number of sectors so that in each sub-portfolio assets are homogeneous. We will call *segments* the different components that grouped up make the portfolio and call *attribute* the particular property that each asset has in common with the others in a particular segment.

For example, in an equity portfolio one may wish to group the assets according to the country of origin. In this case the attribute is *country* and each stocks in the portfolio could be in one of the segments listed in table 1. The partitioning of a portfolio in segments can be accomplished using many different attributes. For example, one may decide that the portfolio assets should be identified by the following attributes: asset class, country, sector, and ISIN code. For instance, the value of these attributes for the Italian stock ENI are:

| | | |
|---|---|---|
| Asset class | ⇒ | Equity |
| Country | ⇒ | Italy |
| Sector | ⇒ | Energy |
| ISIN | ⇒ | IT0003132476 |

For simplicity, in the present paper, we consider only one attribute and one particular portfolio segmentation. In terms of notation we will denote with an alphabet letter each segment, or sector, so that the portfolio is split into sectors $a$, $b$, $c$, $\ldots$ . Furthermore we also assume that each asset belongs to one and only one sector.

The portfolio value, as given by expression (1), can be split as,

$$W = W_1 + \,\ldots\, + W_n = W_a + W_b + \,\ldots\,, \tag{3}$$

where for each segment, e.g. for segment $a$, the segment outstanding value $W_a$ is given by the sum of the asset outstanding values $W_i$'s for each asset in the sector. We use the notation

$$W_a = \sum_{i \in a} W_i\,, \tag{4}$$

where the symbol $i \in a$ means that the sum is performed over all assets in segment $a$. The future portfolio value, defined by expression (2), can also be decomposed as,

$$P = P_a + P_b + \ldots \,, \tag{5}$$

where for each segment, e.g. for segment $a$, we have

$$P_a = \sum_{i \in a} u_i \, W_i \,. \tag{6}$$

Now that we have defined the basic portfolio quantities, we can perform some statistical analysis on the portfolio future values and, in following sections, define risk in several different ways.

## 3 Covariance framework for risk

The first measure of portfolio risk ever used is probably volatility, i.e. the standard deviation of a portfolio value. This risk measure is also important because it is usually easy to compute and is widely used in portfolio optimization. In this section we briefly derive how to compute volatility in the covariance framework. The covariance framework will also give us a simple and analytical example on how to decompose risk into the sum of additive components.

### 3.1 The covariance matrix

Consider the statistics of the portfolio possible future values, with a certain probability the assets in the portfolio can have a number of different values. We can compute the future portfolio average value as,

$$\bar{P} = \langle P \rangle = \bar{u}_1 W_1 + \ldots + \bar{u}_n W_n \,, \tag{7}$$

and its variance as,

$$\sigma^2 = \left\langle \left(P - \bar{P}\right)^2 \right\rangle = \sum_{i,j}^{n} \Gamma_{ij} \; W_i \, W_j \,, \tag{8}$$

where the symbol $\langle (.) \rangle$ denotes the average of (.). In expressions (7) and (8) we used equation (2), set $\bar{u}_i = \langle u_i \rangle$, and defined the covariance matrix $\Gamma_{ij}$ as,

$$\Gamma_{ij} = \langle (u_i - \bar{u}_i) \, (u_j - \bar{u}_j) \rangle \;. \tag{9}$$

Notice that the covariance matrix is symmetric, i.e, $\Gamma_{ij} = \Gamma_{ji}$.

In the *covariance framework* one assumes risk to be proportional to the portfolio standard deviation $\sigma$. Without losing any generality, in this framework, we will assume the proportionality constant between risk and the portfolio volatility to be one, so that the portfolio risk $\mathcal{R}^{\text{cov}}(P)$, is defined as,

$$\mathcal{R}^{\text{cov}}(P) = \sigma = \sqrt{\sum_{i,j}^{n} \Gamma_{ij} \; W_i \, W_j} \,. \tag{10}$$

### 3.2 Risk decomposition in the covariance framework

Given the covariance definition of portfolio risk it is easy to write $\mathcal{R}^{\text{cov}}(P)$ as the sum of components, i.e.,

$$\mathcal{R}^{\text{cov}}(P) = \frac{\sigma^2}{\sigma} = \frac{1}{\sigma} \sum_{i,j}^{n} \Gamma_{ij} \; W_i \, W_j \,, \tag{11}$$

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so that,

$$\mathcal{R}^{\text{cov}}(P) = \sum_{i=1}^{n} C_i \,, \tag{12}$$

where for each $i$,

$$C_i = W_i \sum_{j=1}^{n} \frac{\Gamma_{ij}}{\sigma} W_j \,, \tag{13}$$

is the $i$-th risk component. The sum on the right-hand side of equation (13) can be simplified noticing,

$$\frac{\partial \mathcal{R}^{\text{cov}}}{\partial W_i} = \frac{1}{2\,\sigma} \frac{\partial \sigma^2}{\partial W_i} = \frac{1}{2\,\sigma} \frac{\partial}{\partial W_i} \sum_{i,j}^{n} \Gamma_{ij}\; W_i\, W_j = \sum_{j=1}^{n} \frac{\Gamma_{ij}}{\sigma} W_j \,, \tag{14}$$

so that equation (13) becomes

$$C_i = W_i \, \frac{\partial \mathcal{R}^{\text{cov}}}{\partial W_i} \,. \tag{15}$$

In section 6 we will see that this decompositions of risk, obtained here in the covariance framework, is indeed more general and can be applied to any homogeneous risk function.

The terms in the sum on the right-hand side of equation (12) can be grouped by sector so that,

$$\mathcal{R}^{\text{cov}}(P) = C_a + C_b + \dots \;, \tag{16}$$

where for each sector, e.g. sector $a$, we have,

$$C_a = \sum_{i \in a} W_i \, \frac{\partial \mathcal{R}^{\text{cov}}}{\partial W_i} \,. \tag{17}$$

Equation (16) provides the risk decomposition of the portfolio by sector.

# 4 VaR decomposition in a simulation framework

So far we have seen that, when risk is defined as the standard deviation of the portfolio future values, it is possible to write portfolio risk as the sum of contributions coming from each asset as given by equation (15). In this section we consider few other risk measures, value at risk in primis, in the framework of portfolio simulations and show how it is also possible to decompose them into additive components. We will then generalize the results of section 3 and those of this section to a generic risk decomposition in section 6.

In the previous section we defined risk as the standard deviation of the portfolio value in a certain probability space. While that definition of risk is completely legitimate and has been used in risk management for a long time, it is often not detailed enough for portfolios containing corporate bonds, exotic options, credit derivatives, or other complex financial products. For example one important shortcoming of the standard deviation used as a risk measure is that it is symmetric in losses and gains. In order to improve on the standard deviation as a risk measure, at least two other frameworks have been used by practitioners: historical simulations, described for example in reference [7], and Monte Carlo simulations.

In this section we take a common view to both historical and Monte Carlo simulations considering a *generic simulation framework* to compute the additive components of risk. We show that, by its own nature, value at risk gives rise to unstable components. In order to overcome this problem, in the next section, we introduce a proxy risk measure, namely *average value at risk*, to replace value at risk. Finally we show how to use the extra parameters in average value at risk to obtain an unbiased risk decomposition of value at risk.

## 4.1 A generic simulation framework

In section 2 we discussed how the evaluate portfolio changes from an initial value $W$ to a future value $P$. Recall that while the initial value $W$ is well known, the final value $P$ is a random variable. The change in portfolio value is caused by the change in the intrinsic value of the underlying assets, as described by the coefficients $u_i$'s and does not arise because of rebalancing.

In the general simulation framework we assume that $N$ simulations are performed[2]. In each simulation $s$, each asset $i$ changes its value from $W_i$ to $u_i^s\, W_i$, so that the asset percentage loss is given by

$$\frac{W_i - u_i^s\, W_i}{W_i} = 1 - u_i^s\,. \tag{18}$$

The portfolio value $P^s$ in simulation $s$ is equal to

$$P^s = u_1^s\, W_1 + \,\ldots\, + u_n^s\, W_n\,. \tag{19}$$

For simplicity we assign the same weight to each simulation and leave the general case where each simulation has a different weight to the reader. Finally, we assume that $N$ is large so that all the financially relevant scenarios, especially those with large losses, are included in the simulations.

## 4.2 Decomposition of simulation-generated value at risk

Consider a large number of simulations $N$ and a percentile $c$, for example $c$=99%. Value at risk is defined as the loss in the scenario $\bar{s}$ so that 99% of the portfolio scenarios have an higher value than $P^{\bar{s}}$. In other words, we take all the simulated portfolios $P^s$, order them by value and choose the scenario that is between the worst 1% and the best 99%.

More formally, for a generic percentile $c$, value at risk is the loss incurred in the simulation $\bar{s}$ so that $c\,N$ scenarios (in a later paragraph we describe how to apply this method when $c\,N$ is not an integer) have a $P^s$ not higher than $P^{\bar{s}}$. The scenario $\bar{s}$ is chosen so that,

$$\text{Prob}(P^s \le P^{\bar{s}}) = 1 - c \qquad \text{and} \qquad \text{Prob}(P^s > P^{\bar{s}}) = c\,. \tag{20}$$

We write $\mathcal{R}^{\text{VaR}}(P)$ to denote the portfolio value at risk and, when the dependence from the percentile $c$ is to be made explicit, we write $\mathcal{R}_c^{\text{VaR}}(P)$. In the simulation framework, given a percentile $c$ and a corresponding scenario $\bar{s}$ satisfying equation (20), we define value at risk as the best of the $N(1-c)$ worst losses, i.e.,

$$\mathcal{R}^{\text{VaR}}(P) = \max(W - P^{\bar{s}}, 0)\,. \tag{21}$$

Notice that value at risk, representing a loss, cannot be negative. We will not consider the trivial case in which value at risk is zero and always assume risk to be positive. Hence, expression (21) can be written as

$$\mathcal{R}^{\text{VaR}}(P) = W - P^{\bar{s}} \;=\; W_1 - \bar{u}_1 W_1 + \ldots + W_n - \bar{u}_n W_n\,, \tag{22}$$

so that,

$$\mathcal{R}^{\text{VaR}}(P) = (1-\bar{u}_1)\, W_1 + \ldots + (1-\bar{u}_n)\, W_n\,, \tag{23}$$

where,

$$\bar{u}_1 = u_1^{\bar{s}}\,, \quad \ldots\,, \quad \bar{u}_n = u_n^{\bar{s}}\,. \tag{24}$$

Expression (23) gives a decomposition of value at risk in additive components similar to expression (12), however with the risk components $C_i$ given by,

$$C_1 = (1-\bar{u}_1)\, W_1\,, \qquad \ldots\,, \qquad C_n = (1-\bar{u}_n)\, W_n\,. \tag{25}$$

[2] $N$ is the number of simulations and should not be confused with the number of assets in the universe $n$.

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**Technical note on fractional scenarios** In practice unless $N(1-c)$ is an integer there is no scenario $\bar{s}$ satisfying equation (20). In order to deal with this problem, given a generic $N$ and a generic $c$, we define $c_1$ as the highest percentile $c_1 < c$ such that $N(1-c_1)$ is an integer and $c_2$ as the lowest percentile $c_2 > c$ such that $N(1-c_2)$ is also an integer (we obviously have $N(1-c_1)+1 = N(1-c_2)$). The coefficients $\bar{u}_i$'s can be obtained by searching for the scenarios $s_1$ and $s_2$ satisfying expression (20) with $c$ respectively substituted by $c_1$ and $c_2$. Hence define $\bar{u}_i$ as the linear interpolation of $u_i^{s_1}$ and $u_i^{s_2}$ with respect to the percentiles:

$$\bar{u}_i = \frac{c_2 - c}{c_2 - c_1} u_i^{s_1} + \frac{c - c_1}{c_2 - c_1} u_i^{s_2} \,. \tag{26}$$

We can use this synthetic scenario in the value at risk computation of expressions (23) and (25).

## 4.3 Volatility decomposition in the simulation framework

In section 3 we obtained a decomposition of volatility into additive components starting from the covariance matrix. Here we obtain an analogous result for the portfolio volatility computed in a generic simulation framework.

According to equation (8) the portfolio variance can be written in general as,

$$\begin{aligned} \sigma^2 &= \frac{N}{N-1} \left\langle \left(P - \bar{P}\right) \left(P - \bar{P}\right) \right\rangle \\ &= \frac{N}{N-1} \left\langle \left[\left(u_1 - \bar{u}_1\right) W_1 + \ldots + \left(u_n - \bar{u}_n\right) W_n\right] \left(P - \bar{P}\right) \right\rangle \\ &= \frac{N}{N-1} \left\langle \left(u_1 - \bar{u}_1\right) \left(P - \bar{P}\right) \right\rangle W_1 + \ldots + \left\langle \left(u_n - \bar{u}_n\right) \left(P - \bar{P}\right) \right\rangle W_n \,, \end{aligned} \tag{27}$$

where we used equations (2) and (7) to write the first term inside the average. This expression should be compared with equations (15) and (16) to provide the generic volatility decomposition,

$$\sigma = W_1 M_1^\sigma + \ldots + W_n M_n^\sigma \,, \tag{28}$$

where,

$$M_i^\sigma = \frac{N}{\sigma\,(N-1)} \left\langle \left(u_i - \bar{u}_i\right) \left(P - \bar{P}\right) \right\rangle \,. \tag{29}$$

When the portfolio distribution is known, this expression allows the computation of the marginal volatility $M_i^\sigma$ without computing the covariance matrix. Equation (29) can be used in the particular case of the generic simulation framework to give,

$$M_i^\sigma = \frac{1}{\sigma\,(N-1)} \sum_{s=1}^{N} \left(u_i^s - \bar{u}_i\right) \left(P^s - \bar{P}\right) \,, \tag{30}$$

where $P^s$ is defined in expression (19).

## 4.4 Instability of value-at-risk components

At this point one may think that risk decomposition was successfully defined. However the risk components defined in equations (25) are usually not stable: in the historical simulation case they may vary significantly form one day to next; in Monte Carlo simulations may be completely different from one simulation run to the next.

It is easier to understand the nature of this instability noticing that each risk component, as defined in expressions (25), simply singles out the asset loss in one specific simulation (i.e. simulation $\bar{s}$). As such, from one day to next, or in a second Monte Carlo simulation, the specific values assumed by $u_i^{\bar{s}}$ might be very different from the previous one. Let's clarify even further with an example.

**Numerical example** Consider a portfolio of two assets in a historical simulation, with the following two scenarios $s_1$ and $s_2$:

$$u_1^{s_1} = 0.90\,, \qquad u_2^{s_1} = 0.81\,, \qquad u_1^{s_2} = 0.80\,, \qquad u_2^{s_2} = 0.90\,.$$

Suppose that, on the first day we have $W_1$=$W_2$=1000 \$, so that,

$$P^{s_1} \;=\; 1,000\,\$ * 0.90 + 1000\,\$ * 0.81 \;=\; 1,710\,\$\,, \qquad (31)$$

$$P^{s_2} \;=\; 1,000\,\$ * 0.80 + 1000\,\$ * 0.90 \;=\; 1,700\,\$\,. \qquad (32)$$

Assuming the value at risk to be be given by scenario $s_2=\bar{s}$, i.e. 300 \$, we have,

$$C_1 \;=\; 1,000\,\$ * (1 - 0.80) \;=\; 200\,\$\,, \qquad (33)$$

$$C_2 \;=\; 1,000\,\$ * (1 - 0.90) \;=\; 100\,\$\,. \qquad (34)$$

The loss at risk of 300 \$ is split 200 \$ to the first asset and 100 \$ to the second asset.

On a second day suppose we had a decrease of 10% in the value of the first asset, therefore $W_1$=900 \$ and an increase of the 10% in the value of the second asset so that $W_2$=1,100 \$, therefore,

$$P^{s_1} \;=\; 900\,\$ * 0.90 + 1100\,\$ * 0.81 = 1,701\,\$\,, \qquad (35)$$

$$P^{s_2} \;=\; 900\,\$ * 0.80 + 1100\,\$ * 0.90 = 1,700\,\$\,. \qquad (36)$$

Now let's assume that, in this case, $s_1=\bar{s}$ so that VaR is 299\$, hence,

$$C_1 \;=\; 900\,\$ * (1 - 0.90) \;=\; 90\,\$\,, \qquad (37)$$

$$C_2 \;=\; 1,100\,\$ * (1 - 0.81) \;=\; 209\,\$\,. \qquad (38)$$

Here the loss of 299 \$ was split to 90 \$ to the first asset and 209 \$ to the second one completely changing the portfolio risk components.

Even though this is an extreme example, as shown in section 8, similar events can happen in the real world and completely change the component VaR of historical simulation from one day to next.

# 5 Building a stable risk decomposition

In order to avoid the unstable behavior just described we can define the different risk components as the average of more than one scenarios. In the previous example we would have liked to split risk at about 150 \$ for each asset. This result can be accomplished rigorously introducing *average value at risk*, also denoted as average VaR or aVaR.

Given two percentiles we define average value at risk as the average loss for all simulations falling between the two percentiles. Formally, to define average VaR consider two percentile values, the lower percentile $\check{c}$, and the higher percentile $\hat{c}$, so that,

$$\check{c} \,<\, \hat{c}\,. \qquad (39)$$

For example one can choose $\check{c}$=98.5% and $\hat{c}$=99.5%. Then, see expression (20), find the two scenarios $\check{s}$ and $\hat{s}$ needed to compute value at risk with percentiles $\check{c}$ and $\hat{c}$. Now define the set $S$ of all simulations that have a portfolio value between $P^{\hat{s}}$ and $P^{\check{s}}$,

$$s \in S \qquad \text{if and only if} \qquad P^{\hat{s}} \,\le\, P^{s} \,\le\, P^{\check{s}}\,. \qquad (40)$$

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The average value at risk, denoted as $\mathcal{R}^{\mathrm{aVaR}}(P)$, or $\mathcal{R}^{\mathrm{aVaR}}_{\check{c}\hat{c}}(P)$ when explicitly dependent on $\check{c}$ and $\hat{c}$, is then defined as the average loss of all scenarios in set $S$, i.e.,

$$\mathcal{R}^{\mathrm{aVaR}}(P) = \max\left(W - \frac{1}{N^S}\sum_{s\in S} P^s, 0\right) , \tag{41}$$

where $N_s$ is the number of scenarios included in $S$.

Again we will assume the average VaR to be positive and explicitly write the loss term in equation (41) as,

$$W - \frac{1}{N^S}\sum_{s\in S} P^s = W_1 + \ldots + W_n - \frac{1}{N^S}\sum_{s\in S}\left(u_1^s W_1 + \ldots + u_n^s W_n\right) , \tag{42}$$

so that,

$$\mathcal{R}^{\mathrm{aVaR}}(P) = \sum_{i=1}^{n} C_i \, , \tag{43}$$

with,

$$C_i = \frac{1}{N^S}\sum_{s\in S}\left(1 - u_i^s\right) W_i \, , \qquad \text{for } i = 1, \ldots, n \, . \tag{44}$$

Comparing this expression with equation (25) we notice that the average VaR components are given by the average of the risk components of VaR for percentiles between $\check{c}$ and $\hat{c}$. Given a number of scenarios in $S$ sufficiently large, the averaging of VaR components just mentioned gives stability to the risk decomposition between one day and another, for historical simulations, and between one set of simulations and another for Monte Carlo simulations.

**Another technical note on fractional scenarios** We show here how to compute average VaR when either or both $N * (1 - \check{c})$ and $N * (1 - \hat{c})$ are not integers. For simplicity we will order scenarios so that,

$$P^s \geq P^{s+1} \, . \tag{45}$$

We then search for the unique four scenarios $s_1$, $s_2$, $s_3$, $s_4$, with $s_2 = s_1 + 1$, $s_4 = s_3 + 1$, and the corresponding percentiles $c_1$, $c_2$, $c_3$, and $c_4$, so that,

$$c_1 < \check{c} \leq c_2 \leq c_3 < \hat{c} \leq c_4 \, , \tag{46}$$

which implies,

$$P^{s_1} \geq P^{s_2} \geq P^{s_3} \geq P^{s_4} \, . \tag{47}$$

Noticing that,

$$\frac{c_2 - c_1}{\hat{c} - \check{c}}\left[\frac{c_2 - \check{c}}{c_2 - c_1} + s_3 - s_2 + \frac{\hat{c} - c_3}{c_4 - c_3}\right] = 1 \, , \tag{48}$$

we define the average value at risk to be the positive quantity,

$$\mathcal{R}^{\mathrm{aVaR}}(P) = W - \frac{c_2 - c_1}{\hat{c} - \check{c}}\left[\frac{c_2 - \check{c}}{c_2 - c_1} P^{s_1} + P^{s_2} + \ldots + P^{s_3} + \frac{\hat{c} - c_3}{c_4 - c_3} P^{s_4}\right] , \tag{49}$$

so that it can be shown the risk components to be given by,

$$C_i = \frac{c_2 - c_1}{\hat{c} - \check{c}}\left[\frac{c_2 - \check{c}}{c_2 - c_1}\left(1 - u_i^{s_1}\right) + \left(1 - u_i^{s_2}\right) + \ldots + \left(1 - u_i^{s_3}\right) + \frac{\hat{c} - c_3}{c_4 - c_3}\left(1 - u_i^{s_4}\right)\right] W_i \, . \tag{50}$$

| Risk function | $\check{c}$ | $\hat{c}$ |
|---|---|---|
| Value at Risk | 99% | 99% |
| Expected shortfall | 99% | 100% |
| Percentile-symmetric aVaR | 98.5% | 99.5% |
| Loss-symmetric aVaR | satisfying (53) | 99.5% |

Table 2: List of possible choices for $\check{c}$ and $\hat{c}$ for a given value-at-risk percentile $c$=99%.

## 5.1 Choice of percentiles for average value at risk

We defined a risk decomposition for VaR that is unstable and a stable decomposition for average VaR. In this section we bring this two notions together to find a stable risk decomposition for VaR. The main disadvantage of average VaR is that two different percentiles are needed instead of the single one for VaR. Here given a choice of the percentile $c$, we define a number of possible choices for $\check{c}$ and $\hat{c}$ and examine the resulting risk measures.

The easiest choice for $\check{c}$ and $\hat{c}$ is to set $\check{c} = c$ and $\hat{c} = c$. With this choice average VaR matches exactly VaR. Clearly this choice was discussed for completeness, however, it should not be used in practice for the reasons already explained in subsection 4.2.

**Expected shortfall** Another choice is to set $\check{c} = c$ and $\hat{c}$=100% for any given $c$. It can be shown that with this choice the resulting risk measure, $\mathcal{R}^{\mathrm{aVaR}}_{\check{c},\hat{c}}(P)$, matches the expected shortfall of a portfolio. The expected shortfall, the average of all losses exceeding a given percentile, is a very useful risk measure and is widely used in many risk management applications. However, particularly when using historical simulations, there might be some foul scenarios in the far tails of the loss distribution, scenarios that would completely skew the risk decomposition results. For this reason, the use of expected shortfall decomposition is recommended only when we are absolutely certain that there are no artifacts in the tails of the simulated scenarios.

**Percentile-symmetric average VaR** Another choice, similar to expected shortfall, is to keep the difference between $\hat{c}$ and $\check{c}$ to be 1-$c$, centering the two percentiles around $c$ itself. This definition of average VaR has the advantage to avoid the inclusion in the averaging of scenarios in the far tails of the loss distribution. Specifically, choose the upper and lower percentiles so that[3],

$$\check{c} = c - \frac{1-c}{2}\,, \tag{51}$$

and,

$$\hat{c} = c + \frac{1-c}{2}\,. \tag{52}$$

For example if $c$=95 % we define $\hat{c}$=97.5 % and $\check{c}$=92.5 %. This definition results in a risk measure that is well suited to be decomposed and that does not have the shortcomings of expected shortfall. The main problem with this decomposition is that the sum of contributions is, in general, close to the portfolio Var but does not match it. Since the scenarios between $\check{c}$ and $c$ bring a different loss to the average than the scenarios between $c$ and $\hat{c}$, the average VaR defined in this way is slightly biased.

[3]This is the original definition of average VaR introduced by Dario Cintioli, in the year 2004, in the second version of the StatPro risk decomposition.

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### 5.2 Loss-symmetric average VaR

In order to remove the bias introduced by the percentile-symmetric average VaR, we can choose $\hat{c}$ as in equation (52) and choose $\check{c}$ so that it contains enough scenarios to bring the average VaR to match VaR. In other words write equation,

$$\mathcal{R}_c^{\text{VaR}}(P) = \mathcal{R}_{\check{c},\hat{c}}^{\text{aVaR}}(P)\,, \tag{53}$$

and solve this expression for $\check{c}$. In this way the losses between $\check{c}$ and $c$ will match those between $c$ and $\hat{c}$. In summary one first computes the standard value at risk, then sets $\hat{c}$ as in equation (52), solves for $\check{c}$ so that equation (53) is satisfied and finally computes risk contributions $C_i$ as in expressions (44). The end result is to obtain a risk decomposition of value at risk,

$$\mathcal{R}_c^{\text{VaR}}(P) = \sum_{i=1}^{n} C_i\,, \tag{54}$$

where the $C_i$'s are stable, do not contain potentially-harmful tail scenarios, and do not vary significantly from one set of simulations to another.

**Existence and uniqueness of a solution** When equation (53) has more than one solutions for the percentile $\check{c}$, we should choose the solution that yields the smallest percentile.

In some rare cases with a very flat loss distribution, equation (53) does not have any solution, this is because $\hat{c}$, chosen with definition (52), is too high. In these cases we should try with a smaller $\hat{c}$, for example given by,

$$\hat{c} = c + \frac{1-c}{k}\,, \tag{55}$$

with $k = 3$, 4, 5, and so on[4]. We then choose the percentile $\hat{c}$ obtained from the smallest $k$ for which equation (53) has a solution.

In table 2 we list the possible suggestions for the upper and lower percentiles of average VaR in the case of $c$=99%.

## 6 Risk decomposition for a generic risk function

So far we defined portfolio risk in a number of different ways: as portfolio volatility, as value at risk, as expected shortfall, and so on. What all these definitions of risk have in common? They are all risk measures. In this section we consider the risk decomposition of a generic risk measure $\mathcal{R}$. It turns out that if the risk measure has the property of positive homogeneity we can provide a risk decomposition for it.

In order to compute the additive components for a generic risk function, we need to determine the increase, or decrease, of risk obtained by adding a small amount of an asset: *marginal risk*.

### 6.1 Marginal Risk

Intuitively the marginal risk of a portfolio with respect to an asset, or one of its segments, is the increment in risk that we obtain by buying a small amount of that asset. According to reference [4], marginal risk is defined to be the change in portfolio risk resulting from taking an additional dollar of exposure in a given segment.

[4]Notice that equation (52) is a special case of equation (55) with $k = 2$.

Consider portfolio risk written explicitly as a function of its asset outstanding values: $\mathcal{R}(W_1, \ldots, W_n)$. Given an asset identified by the index $i$, let us increase the exposure in this asset by an additional amount $\varepsilon$. The new portfolio risk is then $\mathcal{R}(W_1, \ldots, W_i + \varepsilon, \ldots, W_n)$. When the risk function $\mathcal{R}$ is smooth in its argument $W_i$ using the Taylor rule we have,

$$\mathcal{R}(W_1, \ldots, W_i + \varepsilon, \ldots, W_n) = \mathcal{R}(W_1, \ldots, W_i, \ldots, W_n) + \frac{\partial \mathcal{R}}{\partial W_i}\varepsilon + \ldots \,, \tag{56}$$

so that, neglecting contributions coming from higher order derivatives, the portfolio risk increases by an amount

$$\frac{\partial \mathcal{R}}{\partial W_i}\varepsilon \,. \tag{57}$$

Marginal risk is defined by this risk increase normalized by the increment size $\varepsilon$ in the limit of vanishingly small $\varepsilon$. In other words the marginal risk $M_i$ of a portfolio with respect an asset $i$, is the partial derivative of the portfolio risk with respect to the asset value, i.e.,

$$M_i = \frac{\partial \mathcal{R}}{\partial W_i} \,. \tag{58}$$

**Examples of marginal-risk computations** Marginal risk in the covariance framework was already computed in equation (14),

$$M_i = \frac{\partial \mathcal{R}^{\text{cov}}}{\partial W_i} = \sum_{j=1}^{n} \frac{\Gamma_{ij}}{\sigma} W_j \,. \tag{59}$$

The computation of risk in the generic simulation framework yields marginal risk for VaR[5] to be computed directly from equation (23),

$$M_i = \frac{\partial \mathcal{R}^{\text{VaR}}}{\partial W_i} = 1 - \bar{u}_i \,, \tag{60}$$

and that for average value at risk from equations (43) and (44),

$$M_i = \frac{\partial \mathcal{R}^{\text{aVaR}}}{\partial W_i} = \frac{1}{N^S} \sum_{s \in S} (1 - u_i^s) \,. \tag{61}$$

Comparing these three different expressions for marginal risk with the corresponding expressions for component risk we notice that in all three cases we have,

$$C_i = M_i \, W_i \,. \tag{62}$$

As we will show next, it turns out that this expression for component risk is quite general and can be derived from Euler's theorem on homogeneous functions.

## 6.2 Homogeneous risk functions and risk decomposition

We define a generic risk measure $\mathcal{R}$ as a function from the portfolio domain to the cash domain. We assume to be able to compute risk for any possible portfolio. Among the many properties that a

[5] Value at risk as defined in the present paper presents a finite number of discontinuities for the first derivatives. The results presented here, however, are still valid when computing the derivatives looking at the VaR function as a distribution.

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risk measure needs to satisfy in order to be considered sound (see reference [9] for more details), the interesting property in this context is positive homogeneity. A risk measures is said to be *positively homogeneous* if, for each positive number $x$, we have,

$$\mathcal{R}\,(x\,W_1,\ldots,x\,W_n) = x\,\mathcal{R}\,(W_1,\ldots,W_n)\ . \tag{63}$$

For example when $x$ is ten, the above definition implies that a portfolio that holds each asset in a quantity that is ten times bigger than our portfolio, is exactly ten times riskier. Examples of risk functions $\mathcal{R}$ that satisfy positive homogeneity are: value at risk, tracking error, average VaR, and expected shortfall.

Euler's theorem states that a positive homogeneous function, in the domain where it is positive, satisfies Eurler's equation:

$$\mathcal{R} = W_1\frac{\partial\mathcal{R}}{\partial W_1} + \ldots + W_n\frac{\partial\mathcal{R}}{\partial W_n}\ . \tag{64}$$

Substituting in this equation definition (58) we obtain,

$$\mathcal{R} = W_1 M_1 + \ldots + W_n M_n\ . \tag{65}$$

Hence, it is always possible to decompose any positively homogeneous risk measure as,

$$\mathcal{R} = C_1 + \ldots + C_n\ , \tag{66}$$

with the components $C_i$'s given by

$$C_i = W_i M_i \qquad\qquad \text{for } i = 1,\ \ldots\ n\ . \tag{67}$$

The different decomposition of previous sections were just particular cases of this general statement.

# 7 Risk decomposition by segments

The method described so far in this section, ultimately leading to equation (54), has enable us to decompose portfolio risk as the sum of contributions coming from the portfolio assets. We can now group the terms of equation (54) together according to the segment they belong to, so that we can write,

$$\mathcal{R}^{\mathrm{VaR}}(P) = C_a + C_b + \ldots\ \ , \tag{68}$$

where for each sector, e.g. sector $a$, we have,

$$C_a = \sum_{i\in a} C_i\ . \tag{69}$$

Equation (16) provides the risk decomposition of a portfolio by sector.

## 7.1 Top-down risk decomposition

We have shown how to compute the additive risk components of each asset and segment for a generic risk measure. Here we show how these results can be applied to the computation of marginal and component risk at segment level without computing them at asset level.

As already shown previously the terms in equation (66) can be grouped by sector to give

$$\mathcal{R} = C_a + C_b + \ldots\ , \tag{70}$$

| $i$ | **Code** | **Description** | $W_i$ |
|---|---|---|---|
| 1 | US5949181045 | Microsoft Corp | 100,000 $ |
| 2 | US594918AC82 | MSFT 4.200 01-Jun-19 | 100,000 $ |
| 3 | DJZ0 | DJ IND AVG DEC0 | 100,000 $ |

Table 3: The investment universe composed of a stock, $i$=1, a corporate bond, $i$=2, and (the equity leg of) an index futures, $i$=3.

where for each sector, e.g. sector $a$, we have,

$$C_a = \sum_{i \in a} W_i M_i \, . \tag{71}$$

Inspired by expression (67), for the generic sector $a$ for which $W_a$ is not null, we can define the marginal risk $M_a$ to satisfy,

$$C_a = W_a M_a \, , \tag{72}$$

where $W_a$ is the exposure of sector $a$ as defined by expression (4). More explicitly, we have,

$$M_a = \frac{C_a}{W_a} = \sum_{i \in a} \frac{W_i}{W_a} M_i \, . \tag{73}$$

Since $W_i/W_a$ is the relative weight of asset $i$ in segment $a$, equation (73) states that the marginal risk of a segment is given by the sum of marginal risk of each asset in that segment multiplied by its relative weight.

Marginal risk was defined to be the extra risk taken for a small increment in a certain asset normalized by the increment. Such definition holds also for marginal risk at sector level since, because of equation (73), we have,

$$M_a = \sum_{i \in a} \frac{W_i}{W_a} \frac{\partial \mathcal{R}}{\partial W_i} = \lim_{\alpha \to 0} \frac{1}{\alpha} \left[ \mathcal{R} \left( W + \alpha \sum_{i \in a} \frac{W_i}{W_a} \right) - \mathcal{R}(W) \right] \, . \tag{74}$$

In other words for $\alpha$=1, the marginal risk of a sector is given by the difference between the risk of a portfolio with an extra cash in that segment and the original portfolio risk. Notice that in the modified portfolio the extra cash is split up in the segment with the quantities $W_i$'s modified so that,

$$W_i \rightarrow W_i + \frac{W_i}{W_a} \, . \tag{75}$$

if $i \in a$, and unmodified otherwise.

In the top-down approach to risk decomposition we can directly compute $M_a$ and $C_a$ without computing risk component and marginal risk at asset level. Proceed in the following way: for any given sector $a$ with $W_a$ not null, first compute sector-level marginal risk $M_a$, as given by equation (74), then use definition (72) to compute sector-level component risk $C_a$.

## 8 Numerical example

In this section we consider a simple numerical example useful to obtain a deeper understanding of the results obtained in this paper. Even thought the portfolio analyzed is very simple, the risk computation is performed using real-world data and the results obtained represent a sample of every-day risk management.

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| Risk Measure | Stock | Bond | Futures |
|---|---|---|---|
| Value at risk | 7,834 $ | 2,484 $ | 5,821 $ |
| Expected shortfall | 8,897 $ | 2,771 $ | 7,113 $ |
| Perc.-symmetric aVaR | 7,509 $ | 2,525 $ | 6,195 $ |

Table 4: Risk computed with 500 historical simulations for each asset in the universe when held by itself, i.e. not in the portfolio. The value at risk and expected shortfall were computed with a percentile $c$ equal 99%. The average value at risk (aVar) was computed using the percentile-symmetric definition of subsection 5.1 with $\check{c}$=98.5% and $\hat{c} = 99.5\%$

**Simulation details** We consider an investment universe composed of three assets (hence $n$=3). The first asset is a stock, the second one is an investment-grade corporate bond, and the last one is the equity leg of an index futures (referred from here on as a futures). Table 3 summarizes the universe composition together with the outstanding values of each asset in the portfolio.

To perform the risk analysis on the given portfolio we used the historical-simulation method described in reference [7], the credit risk of the corporate bond was accounted for by using the method of reference [1]. The simulations were performed using about two years of history, corresponding to exactly $N$=500 simulations.

**Risk analysis** In table 4 we show a summary of the risk results obtained for each single asset when held outside the portfolio. The computation of risk was performed with a confidence level of $c$=99%. At single-asset level we notice a moderate risk for the bond, a bigger risk for the index futures, and an even bigger risk for the stock: these results are expected. Comparing the different risk measures, while expected shortfall always gives bigger risk than both VaR and average VaR, average VaR is bigger than VaR for the equity and the futures but is smaller for the bond.

We performed the computation of portfolio risk using the method described in sections 4 and 5. Again as expected, we found that the diversification effect resulted in a total portfolio risk smaller then the sum of the single-asset risks. The different measures of portfolio risk are listed in the left-most column of table 6. In table 5 we list the values obtained for the simulated scenarios $u_i^s$ for the top eight portfolio biggest losses, together with the corresponding scenario percentile. Notice how in the eight worse scenarios the stock and the index are always at a loss, i.e. $u_{1,3}^s < 1$, while the bond is loosing money in some scenarios ($u_2^s < 1$) and making money in other scenarios ($u_2^s > 1$). Intuitively, these gains in the bond price corresponding to stock market losses are an example of the natural dynamics existing between equity and fixed-income markets.

**Risk decomposition** More interesting are perhaps the results for the additive risk components obtained using the methods described earlier in this paper. Intuitively, while we expect the stock and the index futures to give a big positive contribution to risk, the bond should introduce enough diversification to provide a negative risk component.

We provide in table 6 the results for the risk components computed for four different risk measures. Notice that the bond component of risk is positive for VaR and negative for all the others risk measure. This happens because the bond component of value at risk was computed using only scenario 496 (see table 5). A slight variation of market conditions could bring this components to either scenario 495 or scenario 497, in both case changing the sign of the component to negative. Expected shortfall and average VaR do not suffer from this shortcoming since an average is performed on more than one scenario. The expected shortfall by definition gives higher values, in absolute terms, for the risk components. While quantitatively both flavors of average VaR give similar results, the unbiased average VaR components have the advantage to exactly sum up to value at risk.

| $c$ | $s$ | $u_1^s$ | $u_2^s$ | $u_3^s$ |
|---|---|---|---|---|
| 99.8% | 500 | 0.9128 | 1.0145 | 0.9298 |
| 99.6% | 499 | 0.9217 | 0.9996 | 0.9418 |
| 99.4% | 498 | 0.8829 | 0.9929 | 0.9877 |
| 99.2% | 497 | 0.9204 | 1.0255 | 0.9235 |
| 99.0% | 496 | 0.9326 | 0.9920 | 0.9485 |
| 98.8% | 495 | 0.9402 | 1.0162 | 0.9210 |
| 98.6% | 494 | 0.9691 | 0.9892 | 0.9284 |
| 98.4% | 493 | 0.9384 | 1.0011 | 0.9485 |

Table 5: Numerical simulation results for the stock, the corporate bond, and the futures. The scenarios are ordered so that equations (46) and (47) are satisfied.

| **Risk Measure** | **Stock** | **Bond** | **Futures** | **Portfolio** |
|---|---|---|---|---|
| VaR | 6,744 $ | 803 $ | 5,150 $ | 12,697 $ |
| Expected shortfall | 8,595 $ | -488 $ | 5,376 $ | 13,484 $ |
| Perc.-symmetric aVaR | 7,080 $ | -269 $ | 5,764 $ | 12,575 $ |
| Average VaR unbiased | 7,162 $ | -283 $ | 5,819 $ | 12,697 $ |

Table 6: Portfolio risk and risk components computed using the different methods described in section 5. In the computation of loss-symmetric average VaR the lower percentile $\check{c}$, satisfying equation (53), had a numerical value of 98.5984%.

# 9 Conclusions

In this paper we have shown how to compute the additive components of risk in portfolio simulations for a number of risk measures.

All the results obtained in this paper for the loss-side of the portfolio distribution can be applied, with few modifications, to the gain side of the distribution. Hence, we can define a decomposition for the potential upside, average upside, and so on.

In reference [5] we show an extension of this method to the computation of risk components and marginal risk of a portfolio relative to a benchmark.

We have seen how it is possible to estimate how much portfolio risk can be attributed to a sector, a sub-sector, or even to a single asset. However consider, for example, a portfolio of convertible bonds for which we want to go even further and compute how much risk comes from interest rates, equities, and credit spread (i.e. the risk factors). In this case we need a tool to analyze risk components at risk-factor level: the so called risk-factor decomposition.

When the dependence of the asset price can be linearized on its risk factors, it is possible to apply the results of section 6 to obtain such a decomposition. Indeed a similar linearized solution was found, for example, in reference [8].

It turns out that the approach introduced in this paper can do much better. As shown by reference [6], it is possible to compute a non-linear risk-factor decomposition. Such method, for example, can be combined with the simulation method described in reference [1] to compute the component of portfolio risk coming from credit risk.

Finally, the method presented here should be useful to practitioners because is general enough, so that it can be used on a wide range of different portfolios, and simple enough to be understood by all the people involved in the risk management process.

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