An introduction to Modern Portfolio Theory: Markowitz, CAP-M, APT and Black-Litterman

Dr. Graeme West
Financial Modelling Agency
www.finmod.co.za
graeme@finmod.co.za

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Contents

Some of these notes are based on parts of (Elton & Gruber 1995) and (Harrington 1987). However, the intention here has been to provide as much mathematical rigour as possible, which we believe these two texts avoid as much as possible.

Chapter 1

Markowitz portfolio theory

Definition 1 *The simple return on a financial instrument P. is* $\frac{P_t - P_{t-1}}{P_{t-1}}$.

This definition has a number of caveats:

- The time from $t-1$ to t is one business day. Thus it is the daily return. We could also be interested in monthly returns, annual returns, etc.
- P_t is a price. Sometimes a conversion needs to be made to the raw data in order to achieve this. For example, if we start with bond yields y_t, it doesn't make much sense to focus on the return as formulated above. Why?
- We need to worry about other income sources, such as dividends, coupons, etc.
- The continuous return is $\ln \frac{P_t}{P_{t-1}}$. This has better mathematical and modelling properties than the simple return above. For example, it is what occurs in all financial modelling and (hence) is what we used for calibrating a historical volatility calculator. However, it is bad for portfolios - see Tutorial. See (J.P.Morgan & Reuters December 18, 1996, TD4ePt2.pdf, §4.1) for additional information and clues.

1.1 Axioms of the theory

The Markowitz framework (Markowitz 1952) is often generically known as the mean-variance framework. The assumptions (axioms) of this model are

1. Investors base their decisions on expected return and risk, as measured by the mean and variance of the returns on various assets.
2. All investors have the same time horizon. In other words, they are concerned only with the utility of their terminal wealth, and not with the state of their portfolio beforehand, and this terminal time is the same for all investors.
3. All investors are in agreement as to the parameters necessary, and their values[1], in the investment decision

[1] This means that information is freely and simultaneously available to all market participants.

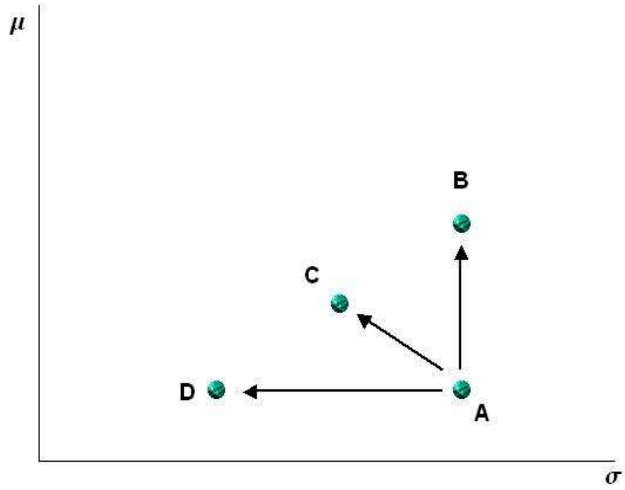


Figure 1.1: An individual will prefer B, C and D to A - but which one is most preferable?

making process, namely, the means, variances and correlations of returns on various investments. (We say the investors are homogeneous.)

4. Financial assets are arbitrarily fungible.

The basic tenant of the Markowitz theory is that knowing the mean and standard deviation of the returns on the portfolio is sufficient, and that our desire is to maximise the expected return and to minimise the standard deviation of the return. The standard deviation is the measure of riskiness of the portfolio.

One thing that is obvious (because individuals are utility maximisers), is that they will always switch from one investment to another which has the same expected return but less risk, or one which has the same risk but greater expected return, or one which has both greater expected return and less risk. See Figure 1.1.

1.2 The expected return and risk of a portfolio of assets

Suppose we have a portfolio with n assets, the i^{th} of which delivers a return $R_{t,i}$ at time t. This return has a mean $\mu_{t,i}$ and a variance $\sigma^2_{t,i}$. Suppose the proportion of the value of the portfolio that asset i makes up is w_i (so $\sum_{i=1}^n w_i = 1$).

What is the mean and standard deviation of the return R of the portfolio? All known values are assumed to be known at time t, and the t will be implicit in what follows. We can suppress the subscript t as long as we understand that all of the parameters are dynamic and we need to refresh the estimates on a daily basis.

$$\mu := \mathbb{E}\left[R\right] = \mathbb{E}\left[\sum_{i=1}^n w_i R_i\right] = \sum_{i=1}^n w_i \mathbb{E}\left[R_i\right] = \sum_{i=1}^n w_i \mu_i \tag{1.1}$$

and

$$\begin{aligned}
\sigma^2(R) &= \mathbb{E}\left[(R-\mu)^2\right] \\
&= \mathbb{E}\left[(\sum_{i=1}^{n} w_i(R_i-\mu_i))^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (R_i-\mu_i)(R_j-\mu_j)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[w_i w_j (R_i-\mu_i)(R_j-\mu_j)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{covar}(R_i, R_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} \\
&= w'\Sigma w
\end{aligned}$$

where $w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ and $\Sigma = [\sigma_{i,j}] = \begin{bmatrix} \sigma_{11} & \cdots & \cdots & \cdots & \sigma_{1n} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \sigma_{n1} & \cdots & \cdots & \cdots & \sigma_{nn} \end{bmatrix}$. This is called the covariance matrix.

So, the return on the portfolio has

$$\begin{aligned}
\mathbb{E}[R] &= w'\mu \\
\sigma(R) &= \sqrt{w'\Sigma w}.
\end{aligned}$$

Note that

- σ_{ij} is the covariance between R_i the return on asset i and R_j the return on asset j.
- $\sigma_i^2 = \sigma_{ii}$ is the variance of R_i.
- $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$ is the correlation of R_i and R_j.

We will denote

- the covariance matrix by Σ;
- the correlation matrix $[\rho_{ij}] = \begin{bmatrix} \rho_{11} & \cdots & \cdots & \cdots & \rho_{1n} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \rho_{n1} & \cdots & \cdots & \cdots & \rho_{nn} \end{bmatrix}$ by $\mathbb{P}$;

- the diagonal matrix of standard deviations $\begin{bmatrix} \sigma_1 & 0 & \cdots & \cdots & 0 \\ 0 & \sigma_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \sigma_n \end{bmatrix}$ by $\mathbb{D}$.

Then

$$\Sigma = \mathbb{DPD} \tag{1.2}$$

and so

$$\sigma(R) = \sqrt{w'\mathbb{DPD}w} \tag{1.3}$$

1.3 The benefits of diversification

Let us consider some special cases. Suppose the assets are all independent, in particular, they are uncorrelated, so $\rho_{ij} = \delta_{ij}$. (δ_{ij} is the indicator function.) Then $\sigma^2(R) = \sum_{i=1}^n w_i^2 \sigma_i^2$. Suppose further that the portfolio is equally weighted, so $w_i = \frac{1}{n}$ for every i. Then

$$\sigma^2(R) = \sum_{i=1}^n \frac{1}{n^2}\sigma_i^2 = \frac{1}{n}\sum_{i=1}^n \frac{\sigma_i^2}{n} \longrightarrow 0$$

as $n \longrightarrow \infty$. If we accept that variance is a measure of risk, then the risk goes to 0 as we obtain more and more assets.

Suppose now that the portfolio is equally weighted, but that the assets are not necessarily uncorrelated. Then

$$\begin{aligned} \sigma^2(R) &= \sum_{i=1}^n \sum_{j=1}^n \frac{1}{n^2}\sigma_{ij} \\ &= \frac{1}{n}\sum_{i=1}^n \frac{\sigma_i^2}{n} + \frac{n-1}{n}\sum_{i=1}^n \sum_{j=1, j\neq i}^n \frac{\sigma_{ij}}{n(n-1)} \\ &= \frac{1}{n}\overline{\sigma_i^2} + \frac{n-1}{n}\overline{\sigma_{ij,i\neq j}} \\ &\longrightarrow \overline{\sigma_{ij,i\neq j}} \text{ as } n \longrightarrow \infty \end{aligned}$$

The limit is the average covariance, which is a measure of the undiversifiable market risk.

1.4 Delineating efficient portfolios

Remember that in the theory we are dealing with now, the mean and standard deviation of the return on an asset or portfolio is all that is required for analysis. We plot competing portfolios with the standard deviation on the horizontal axis and expected return on the vertical axis (risk/return space).

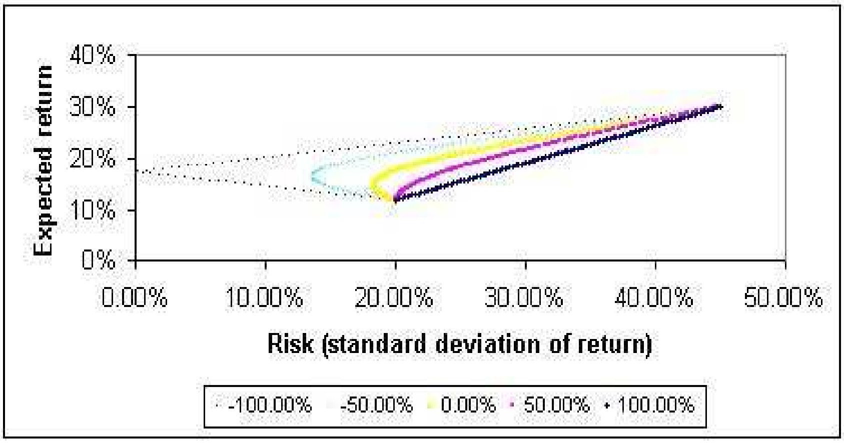


Figure 1.2: Two risky assets held long; differing correlations.

1.4.1 Only long positions allowed

Consider the case where we have a portfolio of two risky assets, both held long. Then

$$
\begin{aligned}
w_1,\ w_2 &\geq 0 \\
w_1 + w_2 &= 1 \\
\mathbb{E}[R] &= w_1\mu_1 + w_2\mu_2 \\
\sigma^2(R) &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&= w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2
\end{aligned}
$$

We can consider some special cases:

- If $\rho = 1$, then $\sigma^2(R) = (w_1\sigma_1 + w_2\sigma_2)^2$, so $\sigma(R) = w_1\sigma_1 + w_2\sigma_2$. Then we have a straight line in risk/return space.
- If $\rho = 0$, then $\sigma^2(R) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$. We have part of a hyperbola in risk/return space.
- If $\rho = -1$, then $\sigma^2(R) = (w_1\sigma_1 - w_2\sigma_2)^2$, so $\sigma(R) = |w_1\sigma_1 - w_2\sigma_2|$. Then we have a "hooked line" in risk/return space.

These possibilities are always paths connecting (σ_1, μ_1) to (σ_2, μ_2). The portfolio which has the least risk is called the minimum risk/variance portfolio. Since $w_2 = 1 - w_1$, $\sigma^2(R)$ is always an upwards facing quadratic in w_1, so the value of w_1 where this minimum occurs can be found by writing down the axis of symmetry of the parabola.[2] One gets

$$w_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{1.4}$$

[2]This might lead to a value of w_1 outside $[0, 1]$, in which case the minimum occurs at one of the endpoints.

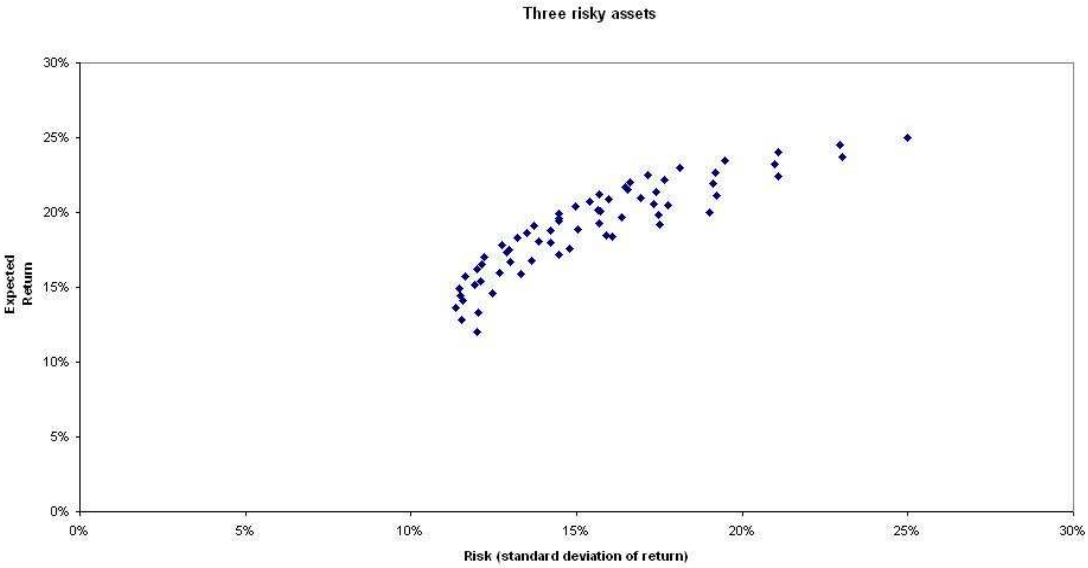


Figure 1.3: Three risky assets held long.

Since $\sigma^2(R)$ is greatest when $\rho = 1$ (all other factors fixed) the path always lies to the left of a straight line between two points on the path. This means that the curve is convex.

Now we suppose there are possibly more than two assets.

The set of attainable points in risk/return space is no longer a line segment or curve but a "cloud" of points. Given any two points in the set, a convex path within the set connects them.

The minimum risk portfolio exists, being the leftmost point on this set.

The maximum return portfolio exists, being the portfolio which consists only of the asset with the highest expected return.

Since any rational investor will

- Prefer a greater expected return to a smaller, risk being fixed,
- Prefer a smaller risk to a greater, expected return being fixed,

the set of points that we need consider (and the corresponding portfolios) form the so called efficient frontier.

Definition 2 *The efficient frontier is the set of portfolios on the upper left boundary of the attainable set, between the minimum variance portfolio and the maximum return portfolio.*

Which one of these portfolios will any investor choose? They will choose the one that maximises their utility. Thus, they will choose the portfolio marked in the diagram.

Everybody's utility curves are different (and unquantifiable, but that is another matter). Thus everybody will choose different portfolios.

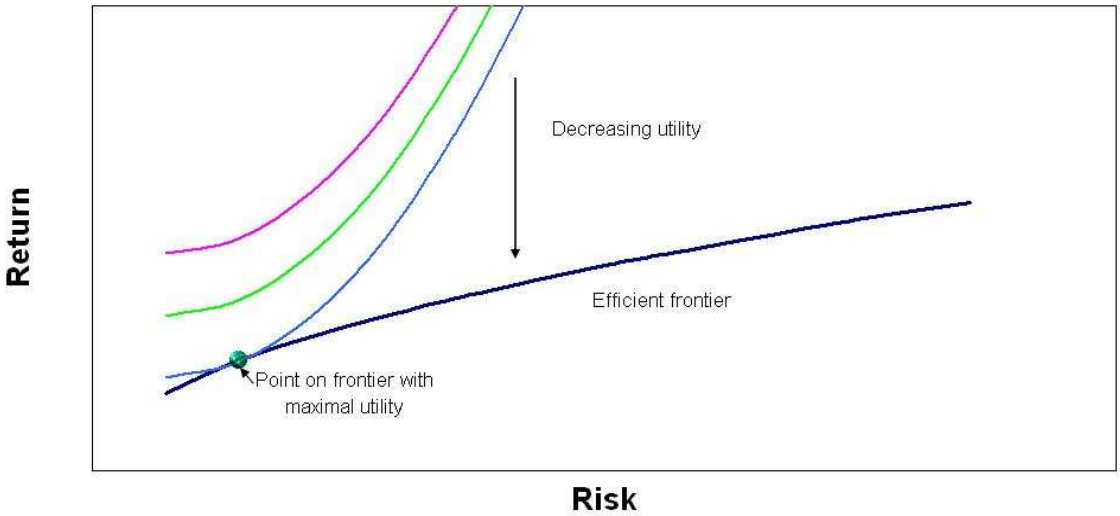


Figure 1.4: The portfolio chosen to maximise the utility of the individual investor

How to find the efficient frontier here is problematic. Optimisers are required. In the next section there will be no constraints on short selling, and the problem becomes one of pure linear algebra, and so can be solved in an understandable way.

1.4.2 Long and short positions allowed

If short sales are allowed, the condition that $w_i \geq 0$ disappears, while the condition that $\sum_{i=1}^{n} w_i = 1$ remains. Most parts of the above analysis remain or generalise easily.

There is now no upper bound to the efficient frontier: it does not end with the "maximum return" portfolio as defined above. This is because we can short sell the asset with low expected returns (or generally, some combination of assets with low expected returns) and use the funds to go long assets with higher returns. Since this can be done to an arbitrary level, the efficient frontier continues without bound.

Definition 3 *The efficient frontier with shorts allowed is the set of portfolios on the upper left boundary of the attainable set, from the minimum variance portfolio and increasing without bound.*

1.4.3 Risk free lending and borrowing allowed

This is simply another asset with $\sigma = 0$, and known return r say. Lending is long and borrowing is short.

Let w be the proportion of the portfolio invested in risky assets, with risk statistics μ_p, σ_p, and $1 - w$ in

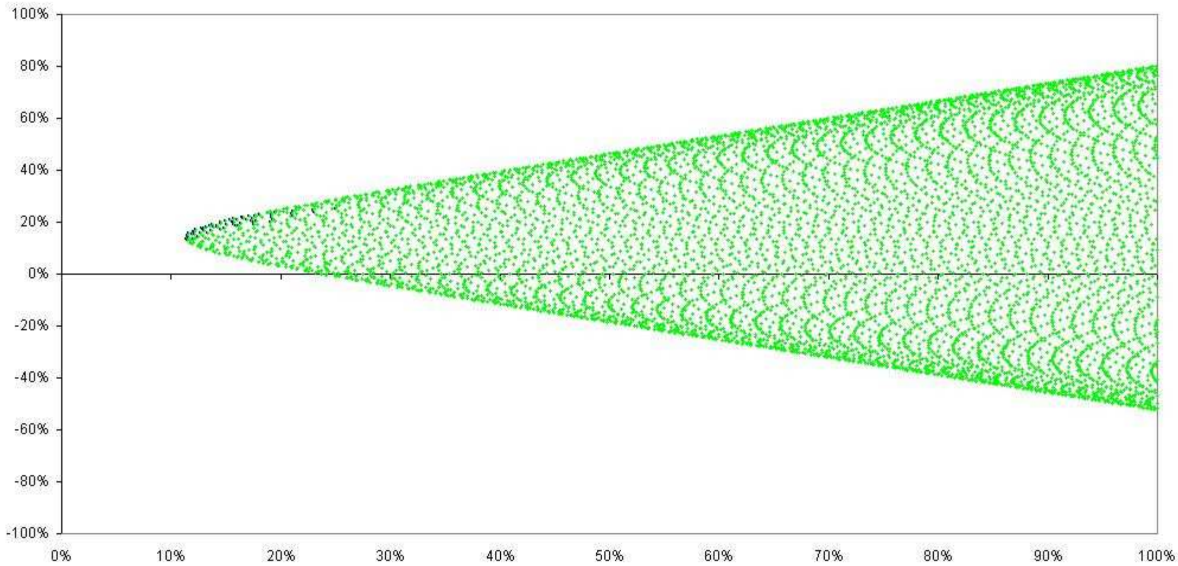


Figure 1.5: Three risky assets held long or short

the risk free account. Then

$$\begin{aligned}\mathbb{E}[R] &= (1-w)r + w\mu_p \\ &= r + w(\mu_p - r) \\ \sigma(R) &= w\sigma_p \\ \Rightarrow \mathbb{E}[R] &= r + \frac{\mu_p - r}{\sigma_p}\sigma(R)\end{aligned}$$

which is a straight line. When $w = 0$, we get the vertical intercept at $(0, r)$. The slope is $\frac{\mu_p - r}{\sigma_p}$. The rational investor, preferring higher returns to fewer returns at a fixed level of risk, will want this line to have as steep a gradient as possible.

Thus, we start with a very steep downward sloping line going through $(0, r)$ and rotate it anti-clockwise while we have an attainable portfolio. The 'last' portfolio is called the Optimum Portfolio of Risky Assets (OPRA). We will quantify it by observing that of all such lines going through the attainable region it is the one with maximal gradient. The investor can now place themselves anywhere on this line through an appropriate amount of lending or borrowing and using the remainder (which could be greater than one, if borrowing occurs - this is known as gearing) to buy the OPRA.

The efficient frontier is no longer the best we can do for portfolio selection, because we can always do better: we can place ourselves on the ray diagrammed, which is called the capital market line. Finding the capital market line becomes merely a function of knowing the risk free return and finding the OPRA.

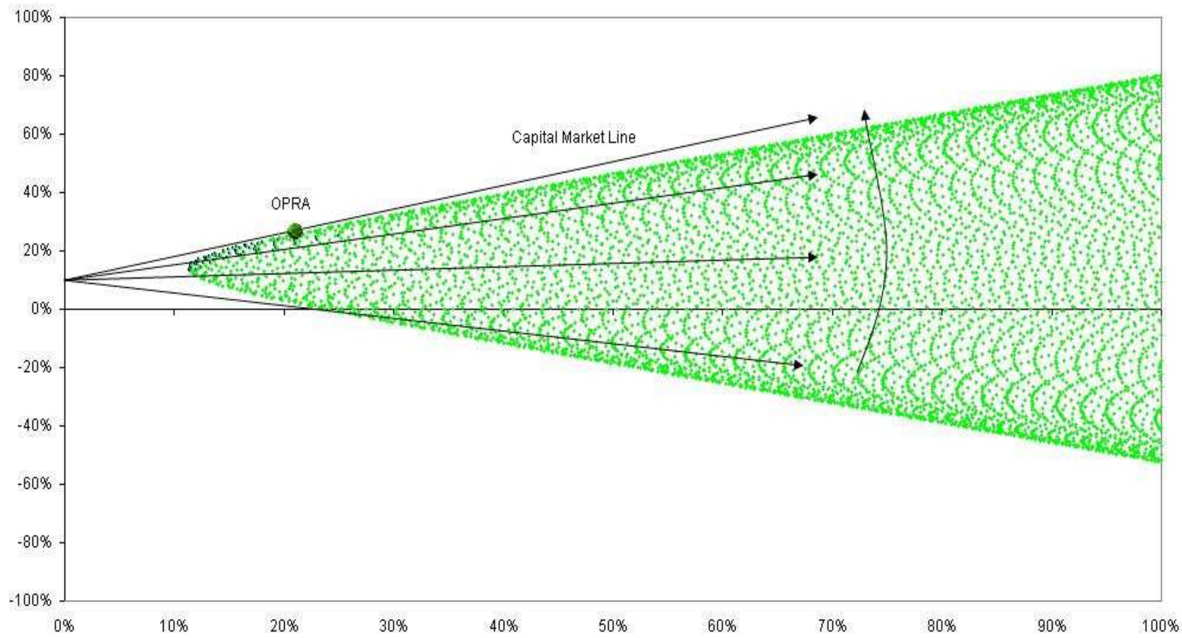


Figure 1.6: Finding the OPRA

1.5 Finding the Efficient frontier and in particular the OPRA

We only consider the case where short sales are allowed.

In §1.4 we saw that even with 3 assets it is not obvious computationally how to find the efficient frontier. In fact we can find the efficient frontier by manipulating the concept of OPRA.

What we do is hypothetically vary the risk free rate. For each risk free rate r we get an OPRA$_r$. All of these OPRA$_r$'s form the curved efficient frontier.

The problem thus reduces to finding the OPRA for any risk free rate r. We have

$$\frac{\partial}{\partial w_i} w'\Sigma w = 2\sum_{j=1}^{n} w_j \sigma_{ij} \tag{1.5}$$

Let $\theta = \frac{\mu_p - r}{\sigma_p}$. θ is known as the market price of risk of the portfolio; more on this later (in this course and

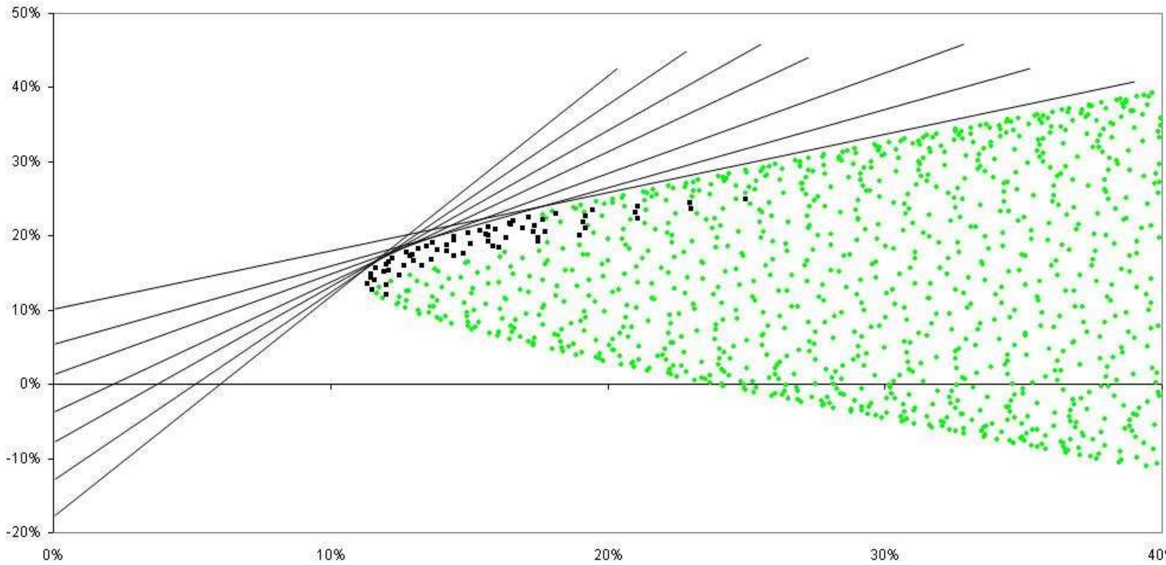


Figure 1.7: Finding the Efficient Frontier

elsewhere!). We need to maximise θ. The constraint is $\sum_{j=1}^n w_j = 1$.

$$\begin{aligned}
\theta &= \frac{\sum_{j=1}^n w_j(\mu_j - r)}{\sqrt{w'\Sigma w}} \\
\Rightarrow \frac{\partial\theta}{\partial w_i} &= \frac{\sqrt{w'\Sigma w}(\mu_i - r) - \sum_{j=1}^n w_j(\mu_j - r)\frac{2\sum_{j=1}^n w_j\sigma_{ij}}{2\sqrt{w'\Sigma w}}}{w'\Sigma w} \\
\Rightarrow 0 &= \mu_i - r - \frac{\sum_{j=1}^n w_j(\mu_j - r)}{w'\Sigma w}\sum_{j=1}^n w_j\sigma_{ij} \\
\Rightarrow \underline{0} &= \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} - r\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \lambda\Sigma\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}
\end{aligned}$$

where it happens that

$$\lambda = \frac{\sum_{j=1}^n w_j(\mu_j - r)}{w'\Sigma w}. \tag{1.6}$$

λ is known as the Merton proportion. Thus

$$\lambda\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \Sigma^{-1}\begin{bmatrix} \mu_1 - r \\ \mu_2 - r \\ \vdots \\ \mu_n - r \end{bmatrix} \tag{1.7}$$

and we can solve for $w_1,\ w_2,\ \ldots,\ w_n$, using the fact that $\sum_{j=1}^n w_j = 1$. Having done so, the point (σ, μ) calculated in the usual way with these weights is on the efficient frontier.

To summarise

- By varying r, we can find the efficient frontier,
- By fixing r (to be what it really is) we can find the OPRA,
- The OPRA and r together give us the capital market line.

1.6 Calibrating the model

Nobody uses the Markowitz model anymore. However, if we were to use a toy model with real data, the parameters that are required might be found using the EWMA method. Likewise for CAP-M, which features in the next chapter. For convenience, these methods are repeated below:

The data available is $x_0,\ x_1,\ \ldots,\ x_t$:

$$\begin{aligned} p_i &= \ln \frac{x_i}{x_{i-1}} \quad (1 \le i \le t) && (1.8) \\ \sigma(0) &= \sqrt{10 \sum_{i=1}^{25} p_i^2} && (1.9) \\ \sigma(i) &= \sqrt{\lambda \sigma^2(i-1) + (1-\lambda) p_i^2 250} \quad (1 \le i \le t) && (1.10) \end{aligned}$$

The rolling calculator for covolatility (i.e. annualised covariance) - see (Hull 2002, §17.7) - is

$$\begin{aligned} \mathrm{covol}_0(x,y) &= \left(\sum_{i=1}^{25} p_i(x) p_i(y) \right) 10 && (1.11) \\ \mathrm{covol}_i(x,y) &= \lambda \mathrm{covol}_{i-1}(x,y) + (1-\lambda) p_i(x) p_i(y) 250 \quad (1 \le i \le t) && (1.12) \end{aligned}$$

Following on from this, the derived calculators are

$$\begin{aligned} \rho_i(x,y) &= \frac{\mathrm{covol}_i(x,y)}{\sigma_i(x)\sigma_i(y)} && (1.13) \\ \beta_i(x,y) &= \frac{\mathrm{covol}_i(x,y)}{\sigma_i(x)^2} && (1.14) \end{aligned}$$

the latter since the CAP-M β is the linear coefficient in the regression equation in which y is the dependent variable and x is the independent variable. The CAP-M intercept coefficient α has to be found via rolling calculators. Thus

$$\begin{aligned} \overline{p_1(x)} &= 10 \sum_{i=1}^{25} p_i(x) && (1.15) \\ \overline{p_i(x)} &= \lambda \overline{p_{i-1}(x)} + (1-\lambda) p_i(x) 250 \quad (1 \le i \le t) && (1.16) \end{aligned}$$

and likewise for $p(y)$. Then

$$\alpha_i(x,y) = \overline{p_i(y)} - \beta_i(x,y) \overline{p_i(x)} \tag{1.17}$$

However, the historical approach would not be used for finding the expected return: the above measure is historical, and while one can claim that history will provide a good estimate for the other parameters, it is unlikely to provide a good measure for expected returns. Here, we might be more reliant on subjective or econometric criteria. No information in this regard is provided by (Elton & Gruber 1995) or (Harrington 1987). All that is provided in (Elton & Gruber 1995) are some toy examples with annual (historical) returns being used as predictors of drift. This is indeed a serious shortcoming of the model.

We will see a partial answer to this question in Chapter 4.

Chapter 2

Capital Asset Pricing Model

2.1 The single index model

The fundamental objection to the Markowitz theory is the need for $2n + \binom{n}{2}$ parameters. While typically in this course we use values derived from historical data analysis, for an institution to add value they will need to forecast parameters, which is impractical if the number of required parameters is large.

The model of Sharpe (Sharpe 1964) is the first simplified model - the simplification is in the data requirements - and led to Markowitz and Sharpe winning the Nobel Prize in 1990. All that is required is parameter estimation of how the security will behave relative to the market. Estimation of pairwise behaviour is not required.

The model starts with a regression equation

$$R_i(t) = \alpha_i + \beta_i R(t) + e_i(t) \tag{2.1}$$

where

$R(t)$	the return at time t for the market
i	the index for a single security
$R_i(t)$	the return on the single security i at time t
α_i	the α-parameter of security i
β_i	the β-parameter of security i
$e_i(t)$	a random variable, with expectation 0, and independent from $R(t)$.

This is all purely regression analysis: $R_i(t)$ (dependent variable) is regressed in t against $R(t)$ (independent variable), with linear term β_i and constant coefficient α_i. In sample, regression analysis ensures that the $e_i(t)$ have sample mean 0 and that the $R(t)$ are uncorrelated to $e_i(t)$.

Our regression analysis will be performed using Exponential Weighted Moving Averages.

The fundamental assumption of a single index model is that the e_i are independent in i - in other words, e_i and e_j are independent for $i \neq j$. Hence equities move together systematically only because of market movement. There is nothing in the regression analysis that ensures that this will be true in sample. It doesn't even make sense practically. After all, resource stocks might move together in some statistically significant sense, finance stocks likewise, etc. Thus, the major generalisations of the CAP-M model were multifactor models. There is even a concept of multi-linear regression in the EWMA scheme, which can be used here for calibration. We won't consider these multifactor models in this course. See (Elton & Gruber 1995, Chapter 8), for example.

Note that

$$\mathbb{E}\left[R_i(t)\right] = \alpha_i + \beta_i \mathbb{E}\left[R(t)\right]$$

and so

$$\mu_i = \alpha_i + \beta_i \mu \tag{2.2}$$

It follows that

$$\mathbb{E}\left[(R_i - \overline{R_i})^2\right] = \mathbb{E}\left[(\beta_i(R - \overline{R}) + e_i)^2\right] = \beta_i^2 \sigma^2(R) + \sigma^2(e_i)$$

and so

$$\sigma_i^2 = \beta_i^2 \sigma^2(R) + \sigma^2(e_i). \tag{2.3}$$

Similarly we have

$$\mathbb{E}\left[(R_i - \overline{R_i})(R_j - \overline{R_j})\right] = \mathbb{E}\left[(\beta_i(R - \overline{R}) + e_i)(\beta_j(R - \overline{R}) + e_j)\right] = \beta_i \beta_j \sigma^2(R)$$

and so

$$\sigma_{ij} = \beta_i \beta_j \sigma^2(R). \tag{2.4}$$

Hence, the covariance matrix of n of these assets is given by

$$\begin{aligned}
\Sigma &= \begin{bmatrix} \beta_1^2\sigma^2(R) + \sigma^2(e_1) & \beta_1\beta_2\sigma^2(R) & \cdots & \beta_1\beta_n\sigma^2(R) \\ \beta_2\beta_1\sigma^2(R) & \beta_2^2\sigma^2(R) + \sigma^2(e_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ \beta_n\beta_1\sigma^2(R) & \dots & \dots & \beta_n^2\sigma^2(R) + \sigma^2(e_n) \end{bmatrix} \\
&= \underline{\beta}\,\underline{\beta}'\sigma^2(R) + \begin{bmatrix} \sigma^2(e_1) & 0 & \cdots & \cdots & 0 \\ 0 & \sigma^2(e_2) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \sigma^2(e_n) \end{bmatrix} \\
&:= \underline{\beta}\,\underline{\beta}'\sigma^2(R) + \Sigma_e
\end{aligned} \tag{2.5}$$

We are now in a situation to consider portfolios. Suppose we have a portfolio with weights $w_1,\ w_2,\ \ldots,\ w_n$, and model parameters $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$ and $\beta_1,\ \beta_2,\ \ldots,\ \beta_n$. Let

$$\alpha_P = \sum_{i=1}^n w_i \alpha_i \tag{2.6}$$

$$\beta_P = \sum_{i=1}^n w_i \beta_i \tag{2.7}$$

This definition is motivated by the fact that

$$\mathbb{E}\left[R_P(t)\right] = \alpha_P + \beta_P \mathbb{E}\left[R(t)\right]. \tag{2.8}$$

As a crucial special case of this we can suppose P is the market portfolio M ie. the portfolio that makes up the market M that is being used in this analysis. Then $\alpha_P = 0$ and $\beta_P = 1$. Thus an equity with $\beta > 1$ is to be thought of as being more risky than the market and one with $\beta < 1$ is to be thought of as being less risky than the market.

The efficient frontier can now be found using the same techniques as that for the Markowitz model. However the number of parameters now required for estimation is $3n+2$: the α's, β's, $\sigma(e)$'s, the volatility of the return on the market and the expected return on the market.

2.2 CAP-M and diversification

For any portfolio P

$$\begin{aligned}
\sigma^2(P) &= w'\Sigma w \\
&= w'(\underline{\beta}\,\underline{\beta}'\sigma^2(R) + \Sigma_e)w \\
&= \left(\sum_{i=1}^{n} w_i\beta_i\right)^2 \sigma^2(R) + \sum_{i=1}^{n} w_i^2\sigma^2(e_i)
\end{aligned}$$

Now let $w_i = \frac{1}{n}$. Then

$$\sigma^2(P) = \overline{\beta}^2\sigma^2(R) + \frac{1}{n}\overline{\sigma^2(e.)}$$

and so

$$\sigma(P) \longrightarrow \overline{\beta}\,\sigma(R).$$

This reaffirms that β_i is a measure of the contribution of the i^{th} security to the risk of the portfolio. $\beta_i\,\sigma(R)$ is called the market, or undiversifiable, risk of security i. $\sigma(e_i)$ is called the non-market risk, unsystematic risk, unique risk or residual risk of equity i. This risk is diversifiable.

2.3 The Sharpe-Lintner-Mossin CAP-M

The CAP-M is what is known as an equilibrium model. The market participants as a whole act to put the market into equilibrium.

A number of additional simplifying assumptions (over and above those of Markowitz) are made in the CAP-M which are thought to be not too far removed from reality, yet are useful in order to simplify (or even make possible) the derivation of the model. Of course, a set of such assumptions is necessary in any economic model. In this model, they are:

1. Short sales are allowed.

2. There is a risk free rate for lending and borrowing money. The rate is the same for lending and borrowing, and investors have any amount of credit.

3. There are no transaction costs in the buying and selling of capital assets.

4. Similarly, there are no income or capital gains taxes.
5. The market consists of all assets. (No assets are exclusively private property.)

2.4 The intuition of CAPM

By homogeneity, everybody has the same r and the same OPRA. This is then the market.

Now accept that β is the appropriate measure of risk of a security. This is intuitive because all investors are holding some amount of the market, and the non-systematic risk has been diversified away. By classic no-arbitrage considerations, all securities lie on a straight line when plotted in $\beta - \mu$ space.[1] Two such portfolios are the riskless asset alone and the market alone. Thus $(0, r)$ and $(1, \mu_M)$ are points on this line in $\beta - \mu$ space, so the line has equation

$$\mu_i - r = \beta_i(\mu_M - r) \tag{2.9}$$

which is known as the equation of the security market line, and 'is' CAP-M.

2.5 A formal proof of CAPM

Recall from §1.5 that when finding the optimal portfolio,

$$\lambda\Sigma w = \underline{\mu} - r\mathbf{1} \tag{2.10}$$

and so

$$\lambda w'\Sigma w = w'(\underline{\mu} - r\mathbf{1}). \tag{2.11}$$

Since we have homogeneity, all investors select the same optimum portfolio. Thus in equilibrium this portfolio is the market i.e. all securities are represented and their weights are the weights they have in the market. Thus, $w'\Sigma w = \sigma^2(M)$ and so

$$\lambda\sigma^2(M) = \mu_M - r \tag{2.12}$$

and we have once again solved for λ. But also,

$$\begin{aligned}
\sum_{j=1}^{n} w_j \sigma_{ij} &= \sum_{j=1}^{n} w_j \mathbb{E}\left[(R_i - \mu_i)(R_j - \mu_j)\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{n} w_j (R_i - \mu_i)(R_j - \mu_j)\right] \\
&= \mathbb{E}\left[(R_i - \mu_i)(R_M - \mu_M)\right] \\
&= \sigma_{iM}.
\end{aligned}$$

We now substitute into the i^{th} line of (2.10):

$$\frac{\mu_M - r}{\sigma^2(M)}\,\sigma_{iM} = \mu_i - r$$

[1] When plotting these, if we have a security say above the line, we buy it and go short a security on or below the line. We can arrange to have a portfolio with no risk (zero β), no investment required, and positive return.

or

$$\mu_i = r + \frac{\mu_M - r}{\sigma^2(M)}\sigma_{iM} = r + \beta_i(\mu_M - r) \tag{2.13}$$

which is the same as (2.9).

2.6 CAPM and prices

Here we manipulate the Capital Asset Pricing Model into an equivalent equation of prices, rather than returns. Suppose we are at time t, and the horizon is at $t+1$. Let us denote prices by P. Thus $R = \frac{P_{t+1}-P_t}{P_t}$.

It follows from this that $\sigma_{iM} = \frac{\mathrm{covar}(P_{t+1,i},P_{t+1,M})}{P_{t,i}P_{t,M}}$ and $\sigma_M = \frac{\sigma(P_{t+1,M})}{P_{t,M}}$. Substituting into (2.13) we get

$$\frac{\overline{P_{t+1,i}} - P_{t,i}}{P_{t,i}} = r + \left(\frac{\overline{P_{t+1,M}} - P_{t,M}}{P_{t,M}} - r\right) \frac{\mathrm{covar}(P_{t+1,i}, P_{t+1,M})}{\sigma^2(P_{t+1,M})} \frac{P_{t,M}}{P_{t,i}}.$$

Multiplying through by $P_{t,i}$, we have

$$\overline{P_{t+1,i}} - P_i(t) = P_i(t)r + \left(\overline{P_{t+1,M}} - P_{t,M} - rP_{t,M}\right) \frac{\mathrm{covar}(P_{t+1,i}, P_{t+1,M})}{\sigma^2(P_{t+1,M})}.$$

Rearranging, we get

$$\overline{P_{t+1,i}} - (\overline{P_{t+1,M}} - P_{t,M} - rP_{t,M}) \frac{\mathrm{covar}(P_{t+1,i}, P_{t+1,M})}{\sigma^2(P_{t+1,M})} = P_{t,i} + P_{t,i}r,$$

or

$$P_{t,i} = \frac{1}{1+r}\left[\overline{P_{t+1,i}} - (\overline{P_{t+1,M}} - (1+r)P_{t,M}) \frac{\mathrm{covar}(P_{t+1,i}, P_{t+1,M})}{\sigma^2(P_{t+1,M})}\right] \tag{2.14}$$

which is the price equilibrium equation.

Such equations are common in Mathematics of Finance. $\frac{1}{1+r}$ represents the risk-free discounting function, $\overline{P_{t+1,i}}$ is the expected value of the asset, and the remaining negative term indicates a compensating factor for the investor's willingness to take on risk. The term in square brackets is known as the certainty equivalent.

$$\frac{\overline{P_{t+1,M}} - (1+r)P_{t,M}}{\sigma(P_{t+1,M})} \tag{2.15}$$

is the market price of risk, while

$$\frac{\mathrm{covar}(P_{t+1,i}, P_{t+1,M})}{\sigma(P_{t+1,M})} \tag{2.16}$$

is the price of risk associated with the individual security.

Chapter 3

Arbitrage Pricing Theory

The first generalisations of CAP-M were multi-index models. Later Steven Ross (Ross 1976) developed a different approach to explain the pricing of assets: pricing can be influenced by any 'abstract' factors.

3.1 The Theory

The axioms of the Arbitrage Pricing Theory are

1. Investors seek return tempered by risk; they are risk-averse and seek to maximise their terminal wealth.
2. There is a risk free rate for lending and borrowing money.
3. There are no market frictions.
4. Investors agree on the number and identity of the factors that are systematically important in pricing assets.
5. There are no riskless arbitrage pricing opportunities.

The model starts with a linear equation

$$R_i(t) = a_i + \sum_{j=1}^{J} b_{ij} I_j(t) + e_i(t) \tag{3.1}$$

where

$I_j(t)$	the value at time t of index j, $J \leq n-2$.
i	the index for a single security
$R_i(t)$	the return on the single security i at time t
a_i	the a-parameter of security i
b_{ij}	the sensitivity of the return on security i to the level of index j
$e_i(t)$	a random variable, with expectation 0, and ....

the e_i are assumed independent of each other, and independent of I_j.

This appears to be a multi-factor version of the CAP-M model where the returns are sensitive to the levels of indices, rather than to the returns of the single index (the market). However, we now take a quite different turn, because typical APT indices include (Roll & Ross Fall 1983):

- unanticipated changes in inflation
- unanticipated changed in industrial production
- unanticipated changes in risk premia, as measured in corporate bond spreads
- unanticipated changes in the slope and level of the term structure of interest rates

The APT theory involves a derivation of an equilibrium model, via an assumption of homogeneous expectations.

Suppose we have a portfolio P with weights $w_1,\ w_2,\ \ldots,\ w_n$. Using the model, we have

$$\begin{aligned} R_P &= \sum_{i=1}^{n} w_i R_i \\ &= \sum_{i=1}^{n} w_i \left(a_i + \sum_{j=1}^{J} b_{ij} I_j + e_i \right) \\ &= \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} \sum_{j=1}^{J} w_i b_{ij} I_j + \sum_{i=1}^{n} w_i e_i \end{aligned}$$

As a consequence, we define/have

$$\begin{aligned} a_P &= \sum_{i=1}^{n} w_i a_i \\ b_{Pj} &= \sum_{i=1}^{n} w_i b_{ij} \\ e_P &= \sum_{i=1}^{n} w_i e_i \\ R_P &= a_P + \sum_{j=1}^{J} b_{Pj} I_j + e_P \\ \overline{R_P} &= a_P + \sum_{j=1}^{J} b_{Pj} \overline{I_j}. \end{aligned}$$

Since e_i can be diversified away in the portfolio, the investor must not expect reward for e_i. Thus the investor is only concerned about the risk parameters b_{Pj} and the values of $\overline{I_j}$. The b_{Pj} are the appropriate measure of risk for a portfolio, and if two portfolio's have the same b's then they have the same risk.

Consider $b_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$, $b_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix}$, $\ldots$, $b_J = \begin{bmatrix} b_{1J} \\ b_{2J} \\ \vdots \\ b_{nJ} \end{bmatrix}$. This forms a set of J vectors in $\mathbb{R}^n$. Append $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$ to this set, and call it S. Let $\mathbb{E}[R] = \begin{bmatrix} \overline{R_1} \\ \overline{R_2} \\ \vdots \\ \overline{R_n} \end{bmatrix} \in \mathbb{R}^n$ be the vector of expected returns of the n securities in the market.

Suppose $w \in S^\perp$, and consider the portfolio P induced by $(w_1,\ w_2,\ \ldots,\ w_n)$. Since $w \perp \mathbf{1}$, we have $\sum_{i=1}^n w_i = 0$, so there is no cost to entering into P. Since $w \perp b_j$, we have $\sum_{i=1}^n w_i b_{ij} = 0$ for $1 \leq j \leq J$, so P bears no index risk. Also, by some diversification arguments, we can assume the residual risk is 0. Thus, P has no expected return, so $\sum_{i=1}^n w_i \overline{R_i} = 0$, and $w \perp \mathbb{E}[R]$.

Thus, $S^\perp \subset \mathbb{E}[R]^\perp$, and so $\mathbb{E}[R] \in S^{\perp\perp} = \operatorname{span} S$ and so

$$\mathbb{E}[R] = \lambda \mathbf{1} + \sum_{j=1}^J \lambda_j b_j.$$

for some $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_J \in \mathbb{R}$. Thus

$$\overline{R_i} = \lambda + \sum_{j=1}^J \lambda_j b_{ij} \tag{3.2}$$

for $1 \leq i \leq n$ and some $\lambda, \lambda_1,\ \lambda_2,\ \ldots,\ \lambda_J \in \mathbb{R}$. Note that this holds for every i, with λ and $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_J$ the same each time. It remains to find out what they are. Note that for any portfolio P,

$$\overline{R_P} = \lambda + \sum_{j=1}^J \lambda_j b_{Pj}.$$

Considering a riskless portfolio, with $b_{Pj} = 0$ for each j, we have $\lambda = r$. Hence

$$\overline{R_P} = r + \sum_{j=1}^J \lambda_j b_{Pj}.$$

Choosing a portfolio with $b_{Pj} = 1$ once only, and $b_{Pj} = 0$ otherwise - this is the portfolio exposed only to index j, so we must think of it as being a position in index j itself - we have

$$\overline{R_{I_j}} = r + \lambda_j$$

and so $\lambda_j = \overline{R_{I_j}} - r$. Thus

$$\overline{R_i} = r + \sum_{j=1}^J (\overline{R_{I_j}} - r) b_{ij}. \tag{3.3}$$

3.2 Consistency of APT and CAPM

CAP-M says that

$$\overline{R_{I_j}} = r + \beta_{I_j}(\overline{R_M} - r).$$

Hence

$$\overline{R_i} = r + \sum_{j=1}^{J} (\overline{R_{I_j}} - r) b_{ij} = r + \sum_{j=1}^{J} \beta_{I_j} (\overline{R_M} - r) b_{ij} := r + \beta_i (\overline{R_M} - r)$$

where

$$\beta_i = \sum_{j=1}^{J} \beta_{I_j} b_{ij}$$

Chapter 4

The Black-Litterman model

This model was introduced in (Black & Litterman 1992).

4.1 Another look at Markowitz and CAPM

Let us start by revisiting expected returns of each stock under the Markowitz model and under CAP-M.

Under the Markowitz model and in equilibrium, the ratio

$$\frac{\mu_i}{\frac{\partial}{\partial w_i}\sqrt{w'\Sigma w}} := \frac{\mu_i}{\Delta_i} \tag{4.1}$$

is a constant. See (Litterman 2003, Chapter 2). More precisely, it should be a constant, for if

$$\frac{\mu_1}{\Delta_1} < \frac{\mu_2}{\Delta_2}$$

then by decreasing the holding of asset 1 and increasing the holding of asset 2, we can obtain an increase in the expected return of the portfolio without an increase in risk. To be precise, sell ϵ worth of asset 1 and buy $\epsilon\frac{\Delta_1}{\Delta_2}$ worth of asset 2.

The change in expected cash return is $-\epsilon\mu_1 + \epsilon\frac{\Delta_1}{\Delta_2}\mu_2 > 0$. However, the change in risk is given (using Taylor series) by $-\epsilon\Delta_1 + \epsilon\frac{\Delta_1}{\Delta_2}\Delta_2 = 0$ to first order. Thus, we have an improved position, with the same risk but greater return.

Now,

$$\begin{aligned}\frac{\partial}{\partial w_i}\sqrt{w'\Sigma w} &= \frac{1}{2\sqrt{w'\Sigma w}}2\sum_{j=1}^{n} w_j\sigma_{ij} \\ &= \frac{1}{\sqrt{w'\Sigma w}}\sum_{j=1}^{n} w_j\sigma_{ij}\end{aligned}$$

and so

$$\underline{\mu} = K\Sigma\underline{w} \tag{4.2}$$

for some constant K, sometimes called the risk aversion coefficient. Here as before $\underline{w}$ is the vector of market capitalisation weights, and Σ will be the (historical) covariance matrix.

This approach actually enables us to reduce the estimation of expected returns to the choice of risk-aversion coefficient. Having chosen our own personal risk aversion coefficient, we can now derive the portfolio we must hold via (1.7).

4.2 Expected returns under Black-Litterman

The Black-Litterman model essentially allows us to have some views on some of the stocks in the portfolio. Thus, the investor is not asked to specify a vector of expected excess returns, one for each asset. Rather, the investor focuses on one or more views.

If, for example, we have one view, namely that one of the stocks is going to perform better than the return found in the previous section, the portfolio with views will be broadly similar to the market portfolio, but will be overweight in that share. This overweightedness is known as a tilt. It will in fact also have tilts towards shares that are strongly correlated to it. Moreover, the degree of tilt will be a function of the confidence that we have in that view.

Let μ be the equilibrium return vector found above. The vector of expected returns including views is given by

$$\underline{\mu}_v = \left[(\tau\Sigma)^{-1} + P'D^{-1}P\right]^{-1} \left[(\tau\Sigma)^{-1}\underline{\mu} + P'D^{-1}V\right] \tag{4.3}$$

where

- $\underline{\mu}_v$ is the new (posterior) expected return vector,
- τ is a scalar,
- Σ is the covariance matrix of returns,
- P is a $k \times n$ matrix identifying the stocks in the k different views,
- D is a $k \times k$ diagonal matrix of error terms in expressed views representing the level of confidence in each view (diagonal since these error terms are residuals, and so assumed independent of each other),
- V is the $k \times 1$ view vector.

We will focus on three different types of views (Idzorek 2002):

1. A will have an absolute return of 10% (Confidence of View = 50%);
2. A will outperform B by 3% (Confidence of View = 65%);
3. A market weighted portfolio of A and B will outperform a market weighted portfolio of C, D and E by 1.5% (Confidence of View = 30%).

and we can comment on these views as follows:

1. If A has an equilibrium return of less than 10%, the BL portfolio will tilt towards A, if more than 10%, the BL portfolio will tilt away;

2. If A has an equilibrium return of less than 3% more than B, the BL portfolio will tilt towards A and away from B, if more than 3% more than B, the BL portfolio will tilt away from A and towards B;

3. As above, but we compare the weighted equilibrium return of A and B with the weighted equilibrium return of C, D and E.

Now, in the above example, $V = \begin{bmatrix} 0.10 \\ 0.03 \\ 0.15 \end{bmatrix}$, $D = \begin{bmatrix} 0.50^{-1}c & 0 & 0 \\ 0 & 0.65^{-1}c & 0 \\ 0 & 0 & 0.30^{-1}c \end{bmatrix}$ where c is a calibration factor to be discussed shortly, and

$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ \frac{w_A}{w_A+w_B} & \frac{w_B}{w_A+w_B} & -\frac{w_C}{w_C+w_D+w_E} & -\frac{w_D}{w_C+w_D+w_E} & -\frac{w_E}{w_C+w_D+w_E} & \cdots \end{bmatrix}$ where the $w.$ are market capitalisation weights.

The scalar τ is more or less inversely proportional to the relative weight given to the implied equilibrium returns.

The only issues outstanding are the parameter τ and the calibration factor c. These problems are in fact related, and the literature is not clear about resolving this problem. According to (He & Litterman 1999), (Idzorek 2002) we have

$$c = \tfrac{1}{2}\,\mathbf{1}_k' P \Sigma P' \mathbf{1}_k \tag{4.4}$$

$$\tau = \frac{\text{trace}(D)}{2kc} \tag{4.5}$$

Having determined the expected returns, we choose the portfolio with weights given by

$$w = K\Sigma^{-1}(\underline{\mu}_v - r\mathbf{1}_n) \tag{4.6}$$

where again K is the risk aversion parameter (and again K can be eliminated if desired).

If all views are relative views then only the stocks which are part of those views will have their weights affected. If there are absolute views then all the stocks will have change in returns and weights, since each individual return is linked to the other returns via the covariance matrix of returns.

As usual as soon as there are constraints on short selling the model is much more difficult to use. One possibility is to use the vector of returns with views in a Markowitz model which has those short selling constraints included, as in §1.4.1.

In the following example we find our optimal portfolio with the only extraneous input being the risk aversion factor K.

w	σ	Correl	AGL	AMS	BIL	IMP	RCH	SAP	SOL
36.66%	31.24%	AGL	100%	56%	78%	44%	49%	55%	42%
4.30%	37.80%	AMS	56%	100%	49%	61%	38%	46%	35%
23.37%	32.08%	BIL	78%	49%	100%	39%	47%	50%	43%
6.55%	38.25%	IMP	44%	61%	39%	100%	38%	37%	32%
15.05%	34.96%	RCH	49%	38%	47%	38%	100%	41%	37%
3.54%	30.32%	SAP	55%	46%	50%	37%	41%	100%	50%
10.53%	30.78%	SOL	42%	35%	43%	32%	37%	50%	100%

Σ	AGL	AMS	BIL	IMP	RCH	SAP	SOL
AGL	0.098	0.066	0.078	0.053	0.053	0.052	0.041
AMS	0.066	0.143	0.059	0.088	0.050	0.052	0.041
BIL	0.078	0.059	0.103	0.048	0.053	0.048	0.042
IMP	0.053	0.088	0.048	0.146	0.050	0.043	0.037
RCH	0.053	0.050	0.053	0.050	0.122	0.043	0.039
SAP	0.052	0.052	0.048	0.043	0.043	0.092	0.046
SOL	0.041	0.041	0.042	0.037	0.039	0.046	0.095

Σ^{-1}	AGL	AMS	BIL	IMP	RCH	SAP	SOL
AGL	31.7	-4.4	-17.9	-0.8	-2.3	-4.2	-0.5
AMS	-4.4	13.7	-0.6	-5.6	-0.3	-2.0	-0.4
BIL	-17.9	-0.6	26.3	-0.2	-2.1	-1.2	-2.2
IMP	-0.8	-5.6	-0.2	11.4	-1.5	-0.6	-0.7
RCH	-2.3	-0.3	-2.1	-1.5	11.9	-1.5	-1.5
SAP	-4.2	-2.0	-1.2	-0.6	-1.5	18.5	-5.0
SOL	-0.5	-0.4	-2.2	-0.7	-1.5	-5.0	15.3

$\mu = K\Sigma w$	$\mu - r$	$\Sigma^{-1}(\mu - r)$	w
10.44%	2.9%	0.39119772	498%
8.92%	1.4%	0.03180503	41%
10.16%	2.7%	0.1696619	216%
8.00%	0.5%	-0.0580698	-74%
8.60%	1.1%	0.01717531	22%
6.99%	-0.5%	-0.2498161	-318%
6.51%	-1.0%	-0.2234732	-285%
	λ	0.07848087	

rfr	7.50%
K	1.4

Now we go on to the situation where in addition we have three views:

1. SOL will have a return of 12% (confidence 60%).
2. AGL will outperfrom BIL by 2% (confidence 50%).
3. AGL, AMS, BIL, IMP will outperform SAP, SOL by 5% (confidence 70%).

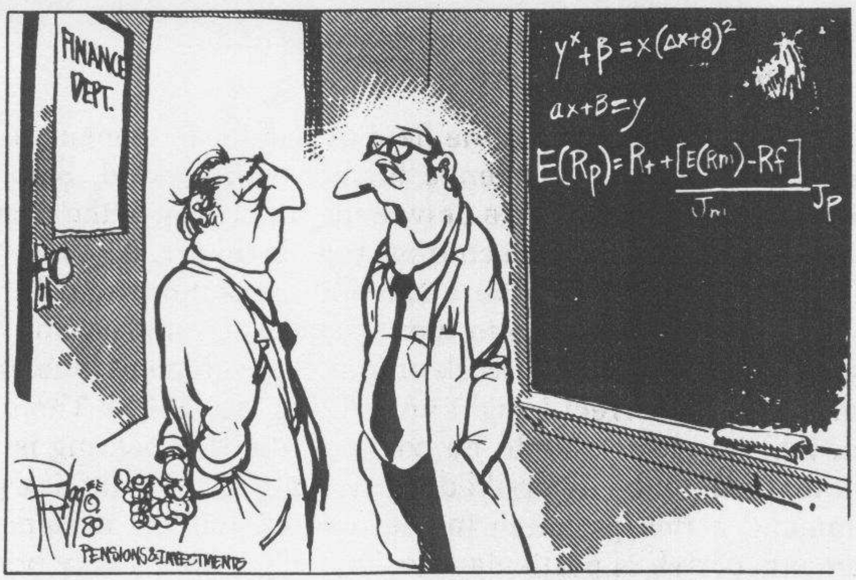
V	Conf		AGL	AGL	AGL	AGL	AGL	AGL	AGL
12%	60%		0	0	0	0	0	0	1
2%	50%		1	0	-1	0	0	0	0
5%	70%		0.52	0.06	0.33	0.09	0.00	-0.25	-0.75

c	6.49%	D	0.11	0	0
τ	85%		0	0.13	0
			0	0	0.09

$(\tau\Sigma)^{-1}$	AGL	AGL	AGL	AGL	AGL	AGL	AGL
AGL	37.4	-5.2	-21.1	-0.9	-2.7	-5.0	-0.6
AMS	-5.2	16.1	-0.7	-6.6	-0.4	-2.3	-0.5
BIL	-21.1	-0.7	31.0	-0.3	-2.5	-1.5	-2.5
IMP	-0.9	-6.6	-0.3	13.5	-1.8	-0.7	-0.9
RCH	-2.7	-0.4	-2.5	-1.8	14.0	-1.8	-1.8
SAP	-5.0	-2.3	-1.5	-0.7	-1.8	21.8	-5.9
SOL	-0.6	-0.5	-2.5	-0.9	-1.8	-5.9	18.0

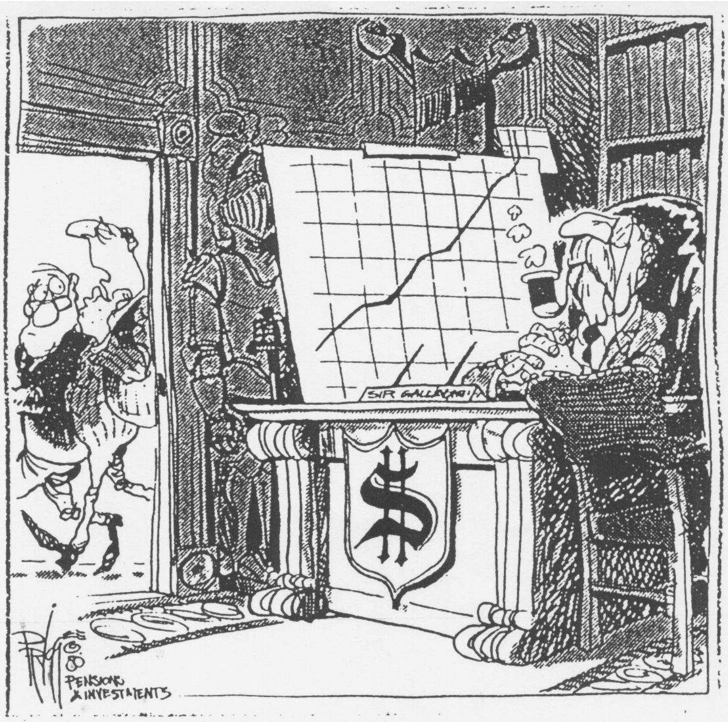
$P'D^{-1}P$	AGL	AMS	BIL	IMP	RCH	SAP	SOL
AGL	10.596	0.339	-5.869	0.516	0	-1.404	-4.178
AMS	0.339	0.040	0.216	0.061	0	-0.165	-0.490
BIL	-5.869	0.216	8.883	0.329	0	-0.895	-2.664
IMP	0.516	0.061	0.329	0.092	0	-0.251	-0.747
RCH	0	0	0	0	0	0	0
SAP	-1.404	-0.165	-0.895	-0.251	0	0.683	2.032
SOL	-4.178	-0.490	-2.664	-0.747	0	2.032	15.299

	$(\tau\Sigma)^{-1}\mu$	$P'D^{-1}V$	μ_ν	$\mu_\nu - r$	w	w
AGL	60%	43%	12.36%	4.9%	78%	157%
AMS	7%	3%	10.51%	3.0%	6%	12%
BIL	39%	2%	11.70%	4.2%	19%	37%
IMP	11%	5%	9.42%	1.9%	-6%	-12%
RCH	25%	0%	9.88%	2.4%	2%	5%
SAP	6%	-14%	8.18%	0.7%	-40%	-81%
SOL	17%	71%	8.49%	1.0%	-9%	-17%
					50%	



"Does it bother you at all that when you say MPT quickly it comes out 'empty'?"

Source: ***Pensions and Investments,*** **September 15, 1980.**



"He uses MPT—Medieval Portfolio Theory."

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