Optimal Derivative Investment in Continuous Time

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Abstract

We determine the optimal investment in a risky asset, and in a set of derivatives written on this asset, when the underlying price process is a pure jump Lévy process. Both the infinite horizon problem with intermediate consumption and the finite horizon problem with no intermediate consumption are solved in closed form for investors with HARA utility when the statistical and risk-neutral price processes are in the variance gamma (VG) class of processes. We find that optimal derivative contracts are written on successive price relatives, paying a function of these relatives continuously through time. For statistical volatilities below risk-neutral ones, we show that the optimal derivative contract for intermediate levels of risk aversion is a collar that finances downside protection by selling off upside gain.
1 Introduction

Although standard valuation models treat derivatives as redundant securities, the large volume observed in derivatives markets suggests that derivatives offer advantages to many investors which cannot be replicated by dynamic trading in the underlying assets. This raises the question of the optimal position to be taken in derivatives when markets for the underlying are incomplete. Working in a single period model, Ross[23], Breeden and Litzenberger[3], Leland[15], Brennan and Solanki[4], Hakansson[12, 13], Carr and Madan[5], and others have solved for the optimal static positions taken in options when these instruments complete the market. Similarly, working in a continuous time model, Merton[19] has derived the optimal dynamic investment and consumption policy in an economy consisting of a riskless asset and several risky assets whose prices all follow geometric Brownian motion.

Since derivative price processes cannot be modelled as geometric Brownian motion, Merton’s results cannot be used without modification to describe the optimal policies in markets with derivatives. The purpose of this paper is thus to determine optimal dynamic investment and consumption policies in a continuous time economy with derivatives. Just as Merton was able to derive explicit optimal policies by assuming that investor’s utility functions display hyperbolic absolute risk aversion (HARA), we obtain explicit optimal policies for a HARA investor in an economy consisting of a riskless asset, a single risky asset, and European options written on the risky asset’s price. In Merton’s continuous time economy, the continuity of the price process precludes investor demand for options. Thus, in order to generate demand for options, we assume in contrast to Merton that the price of the underlying risky asset is not purely continuous over time.

In another classic paper, Merton[20] introduced the jump diffusion process as a vehicle for describing realistic asset price dynamics. As is well known, this process combines an infinite variation diffusion process with a finite variation Poisson process to describe prices exhibiting frequent small changes and infrequent large changes. Recently, Geman, Madan, and Yor[10] (GMY) proposed using pure jump Lévy processes\(^1\) as a tractable alternative to jump diffusions. By working with one class of processes rather than two, GMY were able to analytically characterize several pure jump processes

\(^1\)Lévy processes comprise the class of continuous time stochastic processes with homogeneous independent increments. The Lévy class includes the standard Brownian motion and standard Poisson process as important special cases.
which also qualitatively capture the behavior of stock prices. To capture the high frequency of price changes encountered in tick data, GM

Y study processes with an infinite arrival rate of jumps, so that the price process jumps infinitely often in any time interval. To capture the observation that most price changes are small, most of the jumps in the processes studied in GM

Y are arbitrarily small. However, in contrast to pure diffusions and jump diffusions, the variation of these processes may be kept finite, allowing derivatives contracts to be written on the variation of the process. The boundedness of the variation also permits an important decomposition of the pure jump Lévy process into price increases and decreases, allowing each process to be parametrized separately and allowing contracts to be written on either component. It also permits the continuous time variation to be chosen to match the observed variation, in the same manner that quadratic variation is often chosen to match observed volatility. These advantages are needless lost if one layers on an orthogonal diffusion component with its attendant infinite variation. In common with jump diffusions, the pure jump processes studied in GM

Y exhibit rare large moves. Since all price changes are due to jumps in these processes, it is easier to draw inferences about the frequency of large moves by extrapolating from the observed frequency of smaller moves.

Motivated by the advantages inherent in using the pure jump processes considered in GM

Y, this paper describes the problem of determining optimal investment and consumption policy when the price process of the risky asset is in this class. In order to generate explicit results, we focus on a particularly tractable subcase which is obtained from geometric brownian motion by randomizing the quadratic variation in the log of the price in a particular manner. More specifically, when this quadratic variation is given by a pure jump gamma process, then the resulting price process is a pure jump Lévy process termed the variance gamma (VG) process in [10],[17], [16], and [18].

For HARA investors in an economy with VG statistical and risk-neutral price processes, we provide explicit solutions for the optimal investment and consumption policy in both the infinite and finite horizon settings. In both settings, we show that the optimal derivative security is a contract written on the underlying asset’s instantaneous returns (as measured by log price relatives), rather than on its final price. The optimal contract pays a function of this return continuously through time. Furthermore, this function is usually kinked at the current stock price. Thus, in an economy in which the only derivatives available for trading are options written on prices, an implication of our analysis is that the greatest volume would be observed in short-term at-the-money options. If over-the-counter derivatives
are introduced, a further implication of our analysis is that such derivatives would be optimally written on future returns rather than prices. In fact, certain investment houses have recently started offering derivatives whose payoffs are tied to returns\(^2\) in response to investor demand. When statistical volatilities are below risk-neutral ones as is frequently the case, we show that the optimal derivative contract for intermediate levels of risk aversion is a collar that finances downside protection by selling off upside gain.

The outline of this paper is as follows. Section 2 presents the economic environment and the investor's consumption and investment problems for both an infinite and a finite horizon. The general solution of these problems is presented in section 3 while specific solutions for HARA utility and VG dynamics are given in section 4. Both solutions are discussed in section 5, while section 6 summarizes the paper and suggests extensions.

2 The Investor's Problem

In this section, we first describe the economy and then formulate the investor's problem in both the infinite and finite horizon settings.

2.1 The Economy

Consider an economy over the time interval \([0, \bar{T}]\), where \(\bar{T}\) may be infinite. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a stochastic basis given by \(F = \{\mathcal{F}_t \mid 0 \leq t \leq \bar{T}\}\), where \(\mathcal{F}_t\) is an increasing, right continuous, complete filtration of \(\sigma\)-fields of \(\mathcal{F}\), with \(\mathcal{F}_{\bar{T}} = \mathcal{F}\). All price processes are adapted to the stochastic basis \(F\). The assets traded in the economy at any time \(t \in [0, \bar{T}]\) are: i) a money market account \(B_t = e^{rt}\) representing the balance at \(t\) of an initial investment of a dollar growing at the constant rate \(r\); ii) a nondividend-paying stock with time \(t\) price, \(S_t\), and iii) European calls \(C_t(K, \bar{T})\) and puts \(P_t(K, \bar{T})\) of all strikes \(K > 0\) and all maturities \(T \in (t, \bar{T}]\).

The traditional literature on derivatives pricing and investment management generally supposes that investors can trade continuously and that the stock price process is a diffusion. To generate demand for derivatives, we instead consider a pure jump Lévy process. The stochastic component of these

\(^2\)For example, three recent innovations which satisfy these criteria are at-the-money forward-start options, passport options, and droptions (i.e. options on drops).
processes is characterized by a Lévy density\(^3\) which essentially replaces the scalar volatility of a diffusion process with a function describing the nonnegative arrival rate of jumps of all sizes. Figure 1 graphs a typical risk-neutral Lévy density for a log price process estimated from the prices of options on the S&P 500. The downward sloping volatility smirk observed when plotting implied volatilities against strikes is manifested in a thicker left tail, i.e. a higher arrival rate for downward jumps when compared to upward jumps of the same size. A deterministic positive drift can be added to this process to ensure that the expected return over any time interval is positive. The Lévy density shown integrates to infinity over any interval containing the origin. Thus, over any time interval, the resulting log price process will have an infinite number of jumps whose absolute size is below any specified level. Despite this infinite arrival rate, the Lévy density graphed describes a Lévy process with bounded variation. Unlike diffusion processes or jump diffusions, such processes can always be represented as the difference of two increasing processes, which can be interpreted as measuring the magnitude

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\(^3\)The Lévy density \(k(x)\) describing the arrival rate of jumps of size \(x\) in the log price is related to the homogeneous transition density \(p(x, t)\) by

\[
k(x) = \lim_{t \to 0} \frac{p(x, t)}{t}
\]
of positive and negative returns respectively.

In general, the jump processes we consider are right continuous with left limits. The price at time $t$ is $S_t$, and if there is a jump at time $t$, then the left limit is $S_{t-}$, and the magnitude of the jump is $\Delta S_t = S_t - S_{t-}$. We suppose that under the statistical measure $P$, the mean rate of return on the stock is a constant $\mu$, i.e. $ES_t = S_0e^{\mu t}$. As in the diffusion case, the drift $\alpha$ of the log of the stock price differs from $\mu$ by a term to be developed, which arises from Jensen’s inequality. The log of the stock price will also be a pure jump process, which may be written as:

$$\ln S_t = \ln S_0 + \alpha t + \sum_{0<s\leq t} \Delta X_s, \quad (1)$$

where $X_t$ is the process for the accumulated jumps in the log of the stock price, and:

$$\Delta X_t = \ln(S_t/S_{t-}).$$

We assume that $X$ is a Lévy process under the statistical measure $P$, i.e. that $X$ has independent and homogeneous increments with a Lévy density $k_P(x)$ describing the arrival rate of jumps of size $x$. It follows from the Lévy Khintchine theorem that the characteristic function of the random variable $X_t$ is$^4$:

$$\phi_{X_t}(u) \equiv E e^{iuX_t} = \exp \left[ -t \int_{-\infty}^{\infty} \left( 1 - e^{ixu} \right) k_P(x) dx \right]. \quad (2)$$

Exponentiating (1) and evaluating (2) at $u = 1/i$, we may write the price process as:

$$S_t = S_0 \exp(\mu t) \exp \left[ t \int_{-\infty}^{\infty} \left( 1 - e^{ix} \right) k_P(x) dx \right] \exp(X_t), \quad (3)$$

where we observe on comparing (1) and (3) that:

$$\alpha = \mu + \int_{-\infty}^{\infty} (1 - e^{ix})k_P(x) dx. \quad (4)$$

Thus, if $k_p$ is symmetric about zero, then $\alpha$ is below $\mu$ due to the convexity of the exponential and Jensen’s inequality.

$^4$To ensure that the integral is finite, we assume that the Lévy density integrates $|x|$. 

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This completes the specification of the money market account and the stock price process. We come now to the process for option prices. The arbitrage restrictions among options make it difficult to specify option price dynamics directly. Instead, we require that option prices be consistent with a martingale measure under which all discounted asset prices are martingales. Hence we suppose the existence of a probability measure \( Q \) equivalent to \( P \), such that under \( Q \), the processes \( S_t / B_t \), \( C_t(K, T) / B_t \), and \( P_t(T, K) / B_t \) are all martingales. This will be done next by specifying the dynamics of \( S_t \) under \( Q \) and then explicitly deriving option price dynamics as discounted expected payoffs under \( Q \).

Thus, we suppose that under \( Q \), the stock price process is also\(^5\) a Lévy process with Lévy density \( k_Q(x) \). Repeating the argument used to derive (3) implies that the stock price dynamics under \( Q \) are:

\[
S_t = S_0 \exp(rt) \exp \left[ t \int_{-\infty}^\infty (1 - e^x)k_Q(x) \, dx \right] \exp(X_t). \tag{5}
\]

Call and put prices are given by \( C_t(K, T) = B_tE^Q \left[ \frac{(S_T - K)^+}{B_T} \mid \mathcal{F}_t \right] \) and \( P_t(K, T) = B_tE^Q \left[ \frac{(K - S_T)^+}{B_T} \mid \mathcal{F}_t \right] \) respectively. In fact, for any European contingent claim paying \( f(S_T) \) at time \( T \), our assumptions imply that its value process has the form \( V(t, S_t) \) with \( V(t, S_t) = B_tE^Q \left[ \frac{f(S_T)}{B_T} \mid \mathcal{F}_t \right] \). An application of Itô’s lemma for semi-martingales (Jacod and Shiryaev\(^{[14]}\), page 86) implies that in order for \( V(t, S_t) / B_t \) to be a martingale, \( V(t, S) \) must satisfy the integro-differential equation (i.d.e.):

\[
\frac{\partial}{\partial t} V(t, S) + \int_{-\infty}^\infty [V(t, Se^x) - V(t, S)] k_Q(x) \, dx + rS \frac{\partial}{\partial S} V(t, S) = rV(t, S). \tag{6}
\]

This fundamental i.d.e. is the pure jump counterpart to the classical Black Scholes partial differential equation. As in the classical case, the i.d.e. is a mathematical consequence of requiring that the risk-adjusted expected return of any claim be the riskless return. In our model, the ability of the investor to dynamically trade in a continuum of options enforces this no arbitrage condition. Explicit solutions for option values under the VG Lévy density can be found in Madan, Carr, and Chang\(^{[16]}\).

\(^5\)The existence of a general equilibrium economy in which the statistical and risk-neutral processes are both VG processes is documented in Madan and Milne\(^{[17]}\).
2.2 Investor Problem Specification

Having completed the description of the statistical and risk-neutral evolution of asset prices, we now turn to the formulation of the investor's consumption and investment problem. This problem consists of simultaneously choosing a consumption stream and a dynamic strategy in the money market account, the stock, and the doubly indexed continuum of options written on the stock. As our asset space is quite complex, we adapt a method introduced by Pliska[21] and Cox[7], where the focus is on the payoff rather than the position. In our continuous time setting, the investor's control will be the response of the log of his wealth to jumps in the log of underlying stock price. The dynamic trading strategy needed to induce this response will be taken up in the next subsection.

We define \( w(x,t) \) as the jump in the log of the investor's wealth at time \( t \) if the jump at \( t \) in the log of the stock price is \( x \). This wealth response function measures the percentage change in wealth resulting from an \( x \) percent change in the stock price at \( t \). The consumption and investment plan is specified by choosing the pair of processes \( [c(t), w(x,t)] \):

**Definition:** A consumption and investment plan \( [c(t), w(x,t)] \) is feasible if the resulting wealth process \( W(t), 0 \leq t \leq T \), defined by:

\[
W(t) = W_0 + \int_0^t rW(s)ds - \int_0^t c(s)ds
+ \int_0^t \int_{-\infty}^{\infty} W(s) \left[ e^{w(x,s)} - 1 \right] m(\omega;dx,ds) - kQ(x)dxds, \tag{7}
\]

is almost surely non-negative for \( t = T \). The measure \( m(\omega;dx,ds) \) in (7) is an integer valued random measure giving the number of jumps of size \( x \) occurring at time \( s \) over measurable subsets of \( \mathbb{R} \times \mathbb{R}_+ \) (see Jacod and Shiryaev[14] for further details).

Equation (7) defines the wealth process arising from the initial wealth \( W_0 \) and the consumption and investment plan \([c(\cdot), w(\cdot)]\). Changes in wealth arise from the difference between interest earned and the withdrawals made to finance consumption. Changes also arise from the difference between investment returns and the ongoing cost of acquiring them. The wealth transition equation (7) can be written in terms comparable to Merton's
original wealth transition equation⁶:

\[
W(t) = W_0 + \int_0^t rW(s)ds - \int_0^t c(s)ds \\
+ \int_0^t \int W(s)[e^{w(x,s)} - 1] [k_p(x)dx - k_Q(x)dx]ds \\
+ \int_0^t \int W(s)[e^{w(x,s)} - 1] [m(\omega; dx, ds) - k_P(x)dxds]. \tag{8}
\]

Thus, changes in wealth arise from earning interest, paying for consumption, earning a risk premium from risky investments, and facing the martingale component of the risky investments under the statistical measure \( P \). We now address the issue of identifying the dynamic trading strategy needed to sustain an arbitrary consumption and investment policy.

2.3 Replicating Payoffs

The ability to dynamically trade in the money market account, the stock, and in all European options allows an investor to generate any desired consumption stream and wealth response function consistent with initial wealth and which depends on the observed state variables. Since the consumption is locally deterministic, any desired consumption stream can be achieved by the appropriate dynamic investment in the money market account. In contrast, since wealth responds in a non-linear and non-stationary fashion to jumps in the stock price, dynamic investment in stock and options is also required in order to span any given wealth response function. In this section, we identify two approaches to synthesizing desired wealth responses which differ from each other in important ways. The first method involves investing in only instruments with infinitesimally short maturities, while the second more realistic method invests in only longer dated instruments. Since the first method is less computationally intensive than the second, it can be regarded as a convenient analytic approximation when rolling over short maturity options.

⁶This equation is:

\[
W(t) = W_0 + \int_0^t rW(s)ds - \int_0^t c(s)ds + \int_0^t a(\mu - r)W(s)ds + \int_0^t a\sigma W(s)dB(s), \text{ for } B
\]
a standard Brownian motion, and where \( a \) is the fraction of wealth invested in the risky asset with mean return \( \mu \) and volatility \( \sigma \).
In our first method, the vehicles for investment are unit bonds, the stock, and the entire spectrum of out-of-the-money options, where the latter have an infinitesimally small maturity. Just as rollover strategies in short-term bonds are modeled by a fictitious money market account, we require rollover strategies in these short-term out-of-the-money options to sustain our consumption and investment policy. Let \( N_t^b, N_t^s, \) and \( N_t^o(K) \) denote the number of bonds, stocks, and out-of-the-money options held by the investor at time \( t \). Thus, letting \( S_t \) denote the pre-jump stock price at \( t \), \( N_t^o(K) \) gives the number of puts held at all strikes \( K < S_t \) and the number of calls held at all strikes \( K > S_t \). Let \( f(S) = W(t)e^{w_x(\ln(S)/S_t, t)} \) denote the post-jump wealth when regarded as a function of the post-jump stock price \( S \). For a continuously rebalanced position to be delivering the required exposure, we must have:

\[
f(S) = N_t^b + N_t^s S + \int_0^{S_t} N_t^o(K)(K - S)^+ dK + \int_{S_t}^{\infty} N_t^o(K)(S - K)^+ dK.
\]

Carr and Madan[5] prove that for any twice differentiable function \( \phi(S) \):

\[
\phi(S) = [\phi(S_t) - \phi'(S_t) S_t + \phi''(S_t) S_t^2] + \int_0^{S_t} \phi''(K)(K - S)^+ dK + \int_{S_t}^{\infty} \phi''(K)(S - K)^+ dK.
\]

Since \( f(S) = W(t)e^{w_x(\ln(S)/S_t, t)} \), the positions \( N_t^b, N_t^s, \) and \( N_t^o(K) \) are given by:

\[
N_t^b = f(S_t) - f'(S_t) S_t = W(t)e^{w_x(0, t)}[1 - w_x(0, t)]
\]

\[
N_t^s = f'(S_t) = W(t)e^{w_x(0, t)} \frac{w_x(0, t)}{S_t}
\]

\[
N_t^o(K) = f''(K) = W(t)e^{w_x(\ln(K)/S_t, t)}[w_x(\ln(K)/S_t, t) - 1] \frac{1}{S_t^2}.
\]

In our second method, we restrict attention to bonds and options maturing at some fixed time \( T > t \). Since (9) shows how to span a given payoff function \( \phi \) with static positions in bonds, stocks, and options, we need only determine the \( \phi \) which delivers the appropriate wealth response. From (3), the time \( T \) stock price is given by:

\[
S_T = S_t e^{\alpha(T - t) + Y_{t, T}},
\]
where \( Y_{t,T} \equiv X_T - X_t \) accumulates the jumps between \( t \) and \( T \) in the log of the price. Let \( V_t^\phi(S_t) \) denote the value at \( t \) of the payoff \( \phi \) when considered as a function of the stock price:

\[
V_t^\phi(S_t) = e^{-r(T-t)} \int_{-\infty}^{\infty} \phi(S_t e^{\alpha(T-t)+y}) q(y) dy,
\]

where \( q(y) \) is the risk-neutral probability density function of \( Y \). After the jump of size \( x \) at \( t \), the stock price is \( S_t e^x \) and hence the value of the payoff \( \phi \) will be:

\[
V_t^\phi(S_t e^x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \phi(S_t e^{x+\alpha(T-t)+y}) q(y) dy.
\]

To sustain a given wealth response function \( w(x, t) \), we require that \( \phi \) be chosen so that the post jump value \( V_t^\phi(S_t e^x) \) matches the post jump wealth \( \tilde{W}_t(x) \equiv W(t) e^{w(x)} \), i.e.:

\[
\tilde{W}_t(x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \phi(S_t e^{x+\alpha(T-t)+y}) q(y) dy.
\]

Let \( u = x + y \) be a change of variable in the integral and let \( g(u) = e^{-r(T-t)} \phi(S_t e^{\alpha(T-t)+u}) \) be the discounted payoff as a function of \( u \). Then:

\[
\tilde{W}_t(x) = \int_{-\infty}^{\infty} g(u) q(u - x) du.
\]

To solve this integral equation for \( g \), take Fourier transforms\(^7\) of both sides:

\[
\mathcal{F}_w(\omega) = \mathcal{F}_g(\omega) \mathcal{F}_q(-\omega).
\]

Solving for \( \mathcal{F}_g(\omega) \) and inverting yields \( g \) and hence \( \phi \). We note that the terminal payoff \( \phi \) will change as \( t \) evolves, so dynamic trading in options is required. Furthermore, as \( t \) passes \( T \), rolling over into some later maturity will be required. Since this subsection has demonstrated that any wealth response function can be created by dynamic trading and rolling over either long or short dated options, we next address which function will be chosen.

\(^7\)Unfortunately, the functions \( \tilde{W}_t(x), g(u) \) or \( q(u) \) may not be \( L^2 \). In this case, one can dampen the function by multiplying it by a suitable exponential.
2.4 Infinite Horizon Consumption and Investment Problem

The infinite horizon problem is appropriate for many institutional investors (eg. mutual funds), which have no fixed liquidation time. It is also appropriate for individual investors acting on behalf of their beneficiaries. Following Merton[19], we suppose that an investor in the economy of section 2.1 has a preference ordering over potential consumption streams \( c = \{c(t), 0 < t < T\} \) given by the expected utility of the consumption stream. Let the instantaneous flow rate of utility at time \( t \) be \( u[c(t)] \) for some concave utility function \( u[\cdot] \). The discounted utility of the consumption stream \( c \) over the infinite horizon is:

\[
U = \int_{0}^{\infty} e^{-\beta t} u[c(t)] dt,
\]

where \( \beta \) is the pure rate of time preference for the investor.

The investor's infinite horizon problem may now be formalized for the stock price driven by a Lévy process as:

**Program A**

\[
\max_{[\tilde{e}(\cdot), \\tilde{w}(\cdot)]} U = \mathbb{E}^{P} \left\{ \int_{0}^{\infty} e^{-\beta s} u[c(s)] ds \right\}
\]

subject to:

\[
W(t) = W_0 + \int_{0}^{t} rW(s) ds - \int_{0}^{t} c(s) ds
\]

\[
+ \int_{0}^{t} \int_{-\infty}^{\infty} W(s-) \left[ e^{w(x,s)} - 1 \right] [m(\omega; dx, ds) - k_Q(x) dx ds],
\]

and \( W(\infty) \geq 0 \) almost surely.

While the infinite horizon problem is appropriate for many institutional investors who need to service a payout stream over an infinite horizon, many investors do not require intermediate consumption and do have a rough idea of their horizon. For such investors, a finite horizon formulation with no intermediate consumption is more appropriate and so we turn to the formulation of this problem.
2.5 The Finite Horizon Investment Problem

In the finite horizon problem, the investor’s objective function is amended to:

\[ U = E^P \{ u[W(T)] \}, \tag{15} \]

for a horizon of \( T \) and a concave utility function \( u[\cdot] \) defined over terminal wealth. The wealth transition equation is identical to (7) except that the investor does not have to choose an optimal consumption stream \( c(\cdot) \). Thus, the investor’s problem may now be formalized for the stock price driven by a Lévy process as:

**Program B**

\[
\max_{[w(\cdot)]} U = E^P \{ u[W(T)] \}
\]

subject to:

\[
W(t) = W_0 + \int_0^t rW(s-) ds + \int_0^\infty \int_{-\infty}^0 W(s_-) \left[ e^{W(x,s)} - 1 \right] [m(\omega; dx, ds) - kQ(x)dxds],
\]

and \( W(T) \geq 0 \) almost surely.

As the horizon lengthens, we anticipate that investors will behave in a more risk-tolerant manner, since there is more time to recover from adverse outcomes.

Both investment programs are examples of continuous time Markov control problems that may be solved using the methods in Rishel[22]. The first order conditions allow the optimal solution to be expressed in terms of the infinitesimal generator of the underlying Markov process. We take up the general form of this solution in the next section.

3 General Solutions of the Consumption/Investment Problem

We first present the procedure for the solution of the infinite horizon problem, and then consider the procedure for the finite horizon problem.
3.1 General Solution of the Infinite Horizon Problem

In the infinite horizon problem, the time homogeneity of the process under both measures $P$ and $Q$ implies that the optimal policy $w$ can depend only on $x$ and is independent of $t$. Thus, the solution of program A with initial wealth $W(0) = W$ results in an optimized expected utility which can be denoted by $J(W)$. Let the infinitesimal generator of the Markov wealth process under the measure $P$ and controls $[c(\cdot), w(\cdot)]$ be denoted by $A^{c,w}$. Given our assumption that the process is one dimensional Markov, this generator is defined by (see Garroni and Menaldi[9], page 50, or Gihman and Skorohod[11], page 291):

$$A^{c,w}[\varphi](W) = \left\{ rW - c - \int_{-\infty}^{\infty} W \left[ e^{w(x)} - 1 \right] k_Q(x) dx \right\} \varphi_W$$

$$+ \int_{-\infty}^{\infty} \left[ \varphi(We^{w(x)}) - \varphi(W) \right] k_P(x) dx. \quad (16)$$

It is shown in Rishel[22], equations 8.20 and 8.21, that under the optimal controls $[c^*(\cdot), w^*(\cdot)]$:

$$A^{c^*,w^*}[J] - \beta J + u[c^*(\cdot)] = 0, \quad (17)$$

and the optimal controls are given by:

$$c^*, w^* = \arg \max_{c,w} \left\{ A^{c,w}[J] - \beta J + u[c] \right\}. \quad (18)$$

Substitution of the generator (16) into (18) defines the instantaneous optimization problem determining the optimal controls $c^*, w^*$ as:

$$c^*, w^* = \arg \max_{c,w} \left\{ \left\{ rW - c - \int_{-\infty}^{\infty} W \left[ e^{w(x)} - 1 \right] k_Q(x) dx \right\} J_W + \right.$$  

$$\int_{-\infty}^{\infty} \left[ J(We^{w(x)}) - J(W) \right] k_P(x) dx - \beta J + u[c] \left\}. \quad (19)$$

Differentiating with respect to $c$ and $w$ yields the first order conditions:

$$J_W = u'[c^*], \quad (20)$$

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and
\[ J_W(W e^{w^*(x)}) \frac{k_p(x)}{k_Q(x)} = J_W(W), \]  
(21)
respectively. Solving for the optimal consumption and the optimal wealth response yields:
\[ c^* = (u')^{-1}[J_W] \]  
(22)
and:
\[ w^*(x) = \ln \left[ (J_W)^{-1} \left( J_W(W) \frac{k_Q(x)}{k_P(x)} \right) \right] - \ln(W). \]  
(23)
If (22) and (23) are substituted in (17), then a nonlinear integro-differential equation (i.d.e.) arises for the indirect utility function \( J \). The problem taken up in Section 4 is to restrict preferences, beliefs, and price processes in such a way that this i.d.e. can be solved in closed form.

Equation (21) defining the optimal wealth response function has a useful economic interpretation. The random marginal utility per dollar when the jump size is \( x \) and the wealth response function is \( w^*(x) \) is \( J_W(W e^{w^*(x)}) \). The marginal utility expected in this state is \( J_W(W e^{w(x)})k_p(x) \), since \( k_p(x) \) provides the arrival rate of the state. The ex-ante cost of obtaining ex-post payoffs in state \( x \) is \( k_Q(x) \). Thus the ratio \( J_W(W e^{w(x)})k_p(x)/k_Q(x) \) is the expected marginal utility per ex-ante dollar invested in state \( x \). Hence, (21) expresses the classical Marshallian principle that the optimal policy is determined so that the expected marginal utility earned per ex-ante dollar spent is equal across all states.

### 3.2 General Solution of Finite Horizon Problem

The solution of program B with initial wealth \( W(0) = W \) and \( \Upsilon = T \) results in an optimized expected utility denoted by \( H(W,T) \). For convenience, we specify this indirect utility function as a function of calendar time rather than maturity:
\[ J(t,W) \equiv H(W,T-t). \]

Let the infinitessimal generator of the Markov wealth process under the measure \( P \) and control \( w(\cdot,\cdot) \) be once again denoted by \( A^w \). The generator now applies to functions that depend on both wealth and time and is given by:
\[ A^w[\varphi](t,W) \equiv \varphi_t + \left\{ \frac{rW - \int_{-\infty}^{\infty} W e^{w(x,t)} - 1}{k_Q(x)} \right\} \varphi_W \]
\[ + \int_{-\infty}^{\infty} \left[ \varphi(t, W e^{w(x,t)}) - \varphi(t, W) \right] k_P(x) dx. \] (24)

In contrast to the infinite horizon problem, \( A^w \) does not depend on the consumption path, but does depend on \( t \) as does the wealth response function, \( W(\cdot, \cdot) \). By Rishel\[22\], under the optimal control \( w^*(x,t) \), we must have that:

\[ A^w[J] = 0, \] (25)

and the optimal control is given by:

\[ w^* = \arg \max_{\{A^w[J]\}}. \] (26)

In addition we must have that at the horizon date, the \( J \) function coincides with the terminal utility function:

\[ J(W, T) = u(W). \] (27)

Substitution of the generator (24) into (26) defines the instantaneous optimization problem determining the optimal control \( w^* \) as:

\[ w^* = \arg \max_{\{A^w[J]\}} \left\{ J_t + \left\{ rW - \int_{-\infty}^{\infty} W \left[ e^{w(x,t)} - 1 \right] k_Q(x) dx \right\} J_W + \int_{-\infty}^{\infty} \left[ J(t, W e^{w(x,t)}) - J(t, W) \right] k_P(x) dx \right\} \] (28)

Differentiating with respect to \( w \) yields the first order condition:

\[ J_W (t, W e^{w^*(x,t)}) \frac{k_P(x)}{k_Q(x)} = J_W (t, W). \] (29)

Even though this condition is formally equivalent to that for the infinite horizon problem, we note that \( J \) depends on time \( t \) and thus the optimal solution also depends on \( t \). Once again investment is chosen so that the last dollar invested in each state increases expected utility by the same amount.

Solving for \( w^*(x,t) \) yields the optimal wealth response function:

\[ w^*(t, x) = \ln \left[ \left( J_W \right)^{-1} \left( J_W (t, W) \frac{k_Q(x)}{k_P(x)} \right) \right] - \ln(W). \] (30)

Substituting (30) in (25) results in an i.d.e. for \( J \) which must be solved subject to the boundary condition (27). The solution of this initial value problem for a particular specification of preferences, beliefs, and price processes is taken up in the next section.
4 Optimal Consumption and Wealth Response for HARA Investors in VG Economies

We now restrict preferences, beliefs, and price processes with a view to obtaining explicit solutions to programs A and B. For preference restrictions, we consider the HARA class, which is the only class for which closed form solutions for the original Merton problem are available. The HARA utility function is defined by:

\[ u[c] = \frac{\gamma}{1 - \gamma} \left( \frac{\alpha}{\gamma} c - A \right)^{1 - \gamma}. \]  

(31)

For HARA utility, absolute risk aversion is hyperbolic in consumption:

\[ -\frac{u''(c)}{u'(c)} = \frac{1}{\gamma} \frac{A}{c - A}. \]  

(32)

Alternatively, risk tolerance is linear in consumption with the slope parameter (cautiousness) being $1/\gamma$. The utility function is only defined for values of $c > \gamma A/\alpha$.

For the statistical and risk-neutral price process, we restrict attention to the VG class. Since Clark[6], it has been well-known that the excess kurtosis observed in historical returns and in risk-neutral densities (butterfly spreads) can be generated from standard Brownian motion by randomizing its clock. When a gamma process is used as the subordinator for standard Brownian motion, the resulting stochastic process is known as a symmetric VG process. This process has no skewness which is consistent with the empirical evidence for historical returns as presented in Madan, Carr, and Chang[16] (henceforth MCC). Thus, we will assume that under the statistical measure $P$, the log of the price is driven by a symmetric VG process:

\[ \ln S_t = \ln S_0 + \alpha t + sW(G(t, \kappa)), \]

where $\alpha$ is the drift in the log, $s$ is the volatility, and $G(t, \kappa)$ is a gamma process with a mean rate of unity and a variance rate of $\kappa$. Recall from (4) that $\alpha$ differs from the mean rate of return $\mu$ by \( \int_{-\infty}^{\infty} (1 - e^x) k_P(x) dx \). For the symmetric VG process, $sW(G(t, \kappa))$, MCC show that the Lévy density is given by the symmetric function:

\[ k_P(x) = \frac{1}{\kappa |x|} \exp \left( -\sqrt{\frac{2}{\kappa}} \frac{|x|}{s} \right). \]  

(33)
Hence, the statistical price process is:

\[ S_t = S_0 \exp \left[ \mu t + \frac{t}{\kappa} \ln(1 - s^2\kappa/2) + sW(G(t, \kappa)) \right], \tag{34} \]

since:

\[ t \int_{-\infty}^{\infty} (1 - e^x) k_P(x) dx = \frac{t}{\kappa} \ln(1 - s^2\kappa/2). \tag{35} \]

While historical returns display negligible skewness, the risk-neutral probability distributions implied by index option prices typically display appreciable negative skewness. To generate a skewed risk-neutral process for the log of the stock price, the driver can be amended to be Brownian motion with drift, evaluated under a gamma time change. MCC show that if the drift parameter \( \theta \) is assumed to be negative, then the resulting (asymmetric) VG process will have negative skewness. Thus, suppose we assume that the accumulated jumps in the log of the price are given by \( X_t = \theta G(t, \nu) + \sigma W(G(t, \nu)) \). For the risk-neutral VG process, MCC show that the Lévy density generalizes to:

\[ k_Q(x) = \frac{\exp(\theta x/\sigma^2)}{\nu |x|} \exp \left( -\sqrt{\frac{2}{\nu} + \frac{\theta^2 \sigma^2}{\nu^2}} \right). \tag{36} \]

This Lévy density is graphed in Figure 1 using parameters implied from index option prices. The difference between the drift in the log and the risk-neutral expected stock return \( r \) simplifies to:

\[ t \int_{-\infty}^{\infty} (1 - e^x) k_Q(x) dx = \frac{t}{\nu} \ln(1 - \theta \nu - \sigma^2\nu/2), \tag{37} \]

and so one may write the stock price process as:

\[ S_t = S_0 \exp \left[ rt + \frac{t}{\nu} \ln(1 - \theta \nu - \sigma^2\nu/2) + \theta G(t, \nu) + \sigma W(G(t, \nu)) \right]. \tag{38} \]

The three parameters \( \sigma, \theta, \) and \( \nu \) control the volatility, skewness, and kurtosis of the log price respectively. We note from the Lévy density (36) that when \( \theta < 0 \), the left tail is fatter than the right tail. Meanwhile, increases in \( \nu \) symmetrically increase both tails.

We now assume that all options are priced by the closed form formulas obtained in MCC, and thus by construction, all of the available asset
prices satisfy the i.d.e. (6), subject to the appropriate boundary conditions. Furthermore, the discounted stock and option prices are $Q$ martingales by construction. Under these assumptions on investor preferences and on the statistical and risk-neutral processes, we next explicitly solve for the optimal consumption and wealth response in both the infinite and finite horizon problems.

### 4.1 Solution of Infinite Horizon Problem

The trial solution for the $J$ function which we will show provides a complete solution of Program A is of the form:

$$ J(W) = \frac{\gamma}{1 - \gamma} \left( \frac{\eta}{\gamma} W - B \right)^{1 - \gamma}, $$

for constants $\eta$ and $B$ to be determined.

The first step is to solve for the optimal consumption levels and wealth response functions consistent with (22) and (23). Substituting the assumed form of the $J$ function into the first order condition (20) yields the optimal consumption flow:

$$ c^* = \gamma \left[ \frac{A}{\alpha} - \left( \frac{\alpha}{\eta} \right)^{\frac{1}{\gamma - 1}} B \frac{\eta}{\gamma} \right] + \left( \frac{\alpha}{\eta} \right)^{\frac{1}{\gamma - 1}} W. $$

We observe that as in Merton, the optimal consumption is a linear function of the investor’s wealth. For the optimal wealth response function, we first evaluate the measure change:

$$ \frac{k_Q(x)}{k_P(x)} = \frac{\kappa}{\nu} \exp \left( \zeta x + \lambda \mid x \mid \right), $$

where:

$$ \zeta \equiv \frac{\theta}{\sigma^2} \text{ and } \lambda \equiv \sqrt{\frac{2}{\kappa}} - \frac{\sqrt{\frac{2}{\nu} + \frac{\vartheta^2}{\sigma^2}}}{\sigma}. $$

Substituting the candidate for $J(W)$ into the first order condition (21) yields the optimal wealth response function:

$$ w^*(x) = \ln \left[ \frac{\gamma B}{\eta W} + \left( 1 - \frac{\gamma B}{\eta W} \right) \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} \exp \left( -\frac{\zeta x - \lambda}{\gamma} \mid x \mid \right) \right]. $$

Exponentiating both sides implies that the optimal wealth relative is affine in a power of the stock price relative. We consider further the case where
the investor and the market agree on the time change, i.e. \( \kappa = \nu \). In this case, the optimal return can be written as:

\[
e^{w^*(x)} - 1 = \left(1 - \frac{\gamma B}{\eta W}\right) \left[\exp\left(-\frac{\zeta}{\gamma} x - \frac{\lambda}{\gamma} |x| \right) - 1 \right]. \tag{44}
\]

Having explicitly solved for \( c^* \) and \( w^* \), we now need to solve for \( \eta \) and \( B \). Substituting \( c^* \) and \( w^* \) into the i.d.e. (17) and simplifying yields:

\[
\left(\frac{\eta W - B}{\gamma}\right)^{-\gamma} \left\{ \left[ \frac{B \gamma}{1 - \gamma} \left( \beta - \gamma \left( \frac{\alpha}{\eta} \right) \frac{1}{\gamma} \right) - \frac{\eta \gamma A}{\alpha} \right] + \frac{\eta}{1 - \gamma} \left( r(1 - \gamma) - \left( \beta - \gamma \left( \frac{\alpha}{\eta} \right) \frac{1}{\gamma} \right) \right) \right\} W
\]

\[
+ \left(\frac{\eta W - B}{\gamma}\right)^{1-\gamma} \left\{ \int_{-\infty}^{\infty} \left[ \exp\left(-\frac{\zeta(1 - \gamma)}{\gamma} x - \frac{\lambda(1 - \gamma)}{\gamma} |x| \right) - 1 \right] k_P(x) dx 
- \gamma \int_{-\infty}^{\infty} \left[ \exp\left(-\frac{\zeta x - \frac{\lambda}{\gamma} |x| \right) - 1 \right] k_Q(x) dx \right\} = 0. \tag{45}
\]

Both integrals can be done analytically:

\[
c_1 \equiv \int_{-\infty}^{\infty} \left(\exp\left(-\frac{\zeta(1 - \gamma)}{\gamma} x - \frac{\lambda(1 - \gamma)}{\gamma} |x| \right) - 1 \right] k_P(x) dx \tag{46}
\]

\[
= -\frac{1}{\kappa} \ln \left\{ \left[ 1 + \frac{(1 - \gamma)s(\lambda + \zeta)/\gamma}{\sqrt{2/\kappa}} \right] \left[ 1 + \frac{(1 - \gamma)s(\lambda - \zeta)/\gamma}{\sqrt{2/\kappa}} \right] \right\}
\]

\[
c_2 \equiv \int_{-\infty}^{\infty} \left[ \exp\left(-\frac{\zeta x - \frac{\lambda}{\gamma} |x| \right) - 1 \right] k_Q(x) dx \tag{47}
\]

\[
= -\frac{1}{\nu} \ln \left\{ \left[ 1 + \frac{(\lambda + \zeta)/\gamma}{\sqrt{2/\nu + \theta^2/\sigma^2 - \theta/\sigma^2}} \right] \left[ 1 + \frac{(\lambda - \zeta)/\gamma}{\sqrt{2/\nu + \theta^2/\sigma^2 + \theta/\sigma^2}} \right] \right\}.
\]

Substituting these expressions into (45) and simplifying yields:

\[
\left(\frac{\eta W - B}{\gamma}\right)^{-\gamma} \left\{ \left[ \frac{B \gamma}{1 - \gamma} \left( \beta - \gamma \left( \frac{\alpha}{\eta} \right) \frac{1}{\gamma} \right) - \frac{\eta \gamma A}{\alpha} - B \left( \frac{\gamma}{1 - \gamma} c_1 - \gamma c_2 \right) \right] + \frac{\eta}{1 - \gamma} \left( r(1 - \gamma) - \left( \beta - \gamma \left( \frac{\alpha}{\eta} \right) \frac{1}{\gamma} \right) \right) \right\} W
\]

\[
= 0.
\]
It follows that we must have the constant and linear terms in braces, both equal to zero. These may be solved explicitly for \( \eta \) and \( B \):

\[
\eta = \alpha \left[ \frac{1 - \gamma}{\gamma} (c_2 - \nu) + \frac{\beta - c_1}{\gamma} \right]^{-\frac{1}{1 - \gamma}},
\]

(48)

and

\[
B = \frac{\eta \gamma A}{\alpha} \left\{ \frac{\gamma}{1 - \gamma} \left[ \beta - \gamma \left( \frac{\alpha}{\eta} \right)^{\frac{1}{\gamma} - 1} \right] - \frac{\gamma}{1 - \gamma} c_1 + \gamma c_2 \right\}^{-1}.
\]

(49)

This completes the solution of program A for the case \( \kappa = \nu \).

When \( \kappa \) differs from \( \nu \), we observe from (43) that at \( x = 0 \):

\[
w^*(0) = \ln \left[ \frac{\gamma B}{\eta W} + \left( 1 - \frac{\gamma B}{\eta W} \right) \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} \exp \left( -\frac{\zeta}{\gamma} x - \frac{\lambda}{\gamma} |x| \right) \right],
\]

which is positive for \( \kappa < \nu \) and negative otherwise. Since there are an infinite number of jumps whose absolute size is arbitrarily small, such payoffs have infinite value and must be excluded. Thus, for this case we suppose that derivatives are available only to enable investors to alter positions for values of \( x \) where \( |x| > a \) for some small value of \( a \). Hence, we restrict the optimal wealth response function \( w^*(x) \) to be of the form:

\[
w^*(x) = \left. b(x) \right|_{|x| > a},
\]

(50)

and we solve for \( b(x) \). The first order condition (21) defining \( w^*(x) \) is now applied to just the case \( |x| > a \) and defines:

\[
b(x) = \ln \left[ \frac{\gamma B}{\eta W} + \left( 1 - \frac{\gamma B}{\eta W} \right) \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} \exp \left( -\frac{\zeta}{\gamma} x - \frac{\lambda}{\gamma} |x| \right) \right].
\]

For \( |x| < a \), \( w(x) = 0 \), and the terms involving integration over \( x \) with respect to \( k_P \) or \( k_Q \) in (17) are altered to have zero integrands for \( |x| < a \), while for \( |x| > a \), the integrands are altered to revise the definitions of \( c_1 \) and \( c_2 \) to:

\[
c'_1 \equiv \int_{|x| > a} \left[ \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} \exp \left( -\frac{\zeta(1 - \gamma)}{\gamma} x - \frac{\lambda(1 - \gamma)}{\gamma} |x| \right) - 1 \right] k_P(x) dx
\]

and:

\[
c'_2 \equiv \int_{|x| > a} \left[ \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} \exp \left( -\frac{\zeta}{\gamma} x - \frac{\lambda}{\gamma} |x| \right) - 1 \right] k_Q(x) dx
\]

21
respectively. These integrations may be performed in terms of the exponential integral function and yield:

\[ c_1' = \left( \frac{\kappa}{\nu} \right)^{-\frac{1-\gamma}{\gamma}} (d_{1p} + d_{1n}) - 2e_p \]  

\[ c_2' = \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} (d_{2p} + d_{2n}) - (e_{2p} + e_{2n}) , \]  

where:

\[ d_{1p} \equiv \frac{1}{\kappa} \text{ExpInt} \left[ \left( \frac{(\zeta + \lambda)(1 - \gamma)}{\gamma} + \frac{\sqrt{2}}{s\sqrt{\kappa}} \right) a \right] , \]

\[ d_{1n} \equiv \frac{1}{\kappa} \text{ExpInt} \left[ \left( \frac{(\zeta - \lambda)(1 - \gamma)}{\gamma} + \frac{\sqrt{2}}{s\sqrt{\kappa}} \right) a \right] , \]

\[ d_{2p} \equiv \frac{1}{\nu} \text{ExpInt} \left[ \left( \frac{\zeta + \lambda + \lambda_p}{\gamma} \right) a \right] , \quad d_{2n} \equiv \frac{1}{\nu} \text{ExpInt} \left[ \left( \frac{\zeta - \lambda + \lambda_n}{\gamma} \right) a \right] , \]

\[ e_{2p} \equiv \frac{1}{\nu} \text{ExpInt} \left( \lambda_p a \right) , \quad e_{2n} \equiv \frac{1}{\nu} \text{ExpInt} \left( \lambda_n a \right) , \]

\[ \lambda_p \equiv \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2\gamma}{\sigma^2\nu} - \frac{\theta}{\sigma^2}} , \quad \lambda_n \equiv \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2\gamma}{\sigma^2\nu} + \frac{\theta}{\sigma^2}} , \]

and \( e_p \equiv \frac{1}{\kappa} \text{ExpInt} \left( \frac{\sqrt{2}}{s\sqrt{\kappa}} a \right) . \)

4.2 Solution of Finite Horizon Problem

The trial solution for the \( J \) function which we will show provides a complete solution of Program B is of the form:

\[ J(t, W) = \frac{\gamma}{1 - \gamma} \left[ \frac{\eta(t)}{\gamma} W - B(t) \right]^{1-\gamma} , \]  

where \( \eta(t) \) and \( B(t) \) are functions of time to be determined subject to the boundary conditions:

\[ \eta(T) = \alpha \text{ and } B(T) = A. \]  

(54)
As in the infinite horizon case, the first order condition (29) may be solved for the optimal wealth response:

\[
w^*(x, t) = \ln \left[ \frac{\gamma B(t)}{\eta(t) W} + \left( 1 - \frac{\gamma}{\eta(t)} \frac{B(t)}{W} \right) \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} \exp \left( -\frac{\zeta}{\gamma} x - \frac{\lambda}{\gamma} | x | \right) \right],
\]

and once again for the case \( \kappa = \nu \), we have that the optimal return is:

\[
e^{w^*(x, t)} - 1 = \left( 1 - \frac{\gamma}{\eta(t)} \frac{B(t)}{W} \right) \left[ \exp \left( -\frac{\zeta}{\gamma} x - \frac{\lambda}{\gamma} | x | \right) - 1 \right]. \tag{55}
\]

Substituting (55) into (25) yields an ordinary differential equation (o.d.e.) in \( J \):

\[
J_t = - \left[ \frac{\eta(t)}{\gamma} W - B(t) \right] \gamma \left[ (r - c_2 + \frac{c_1}{1 - \gamma}) \eta(t) W + (\gamma c_2 - \gamma c_1) B(t) \right]. \tag{56}
\]

Differentiating the trial solution (53) for the \( J \) function yields:

\[
J_t = \left[ \frac{\eta(t)}{\gamma} W - B(t) \right] \gamma \left[ W \eta'(t) - \gamma B'(t) \right]. \tag{57}
\]

Equating (57) and (56) yields two separate o.d.e.’s in \( \eta \) and \( B \):

\[
\eta'(t) = - \left( r - c_2 + \frac{c_1}{1 - \gamma} \right) \eta(t), \tag{58}
\]

\[
B'(t) = \left( c_2 - \frac{c_1}{1 - \gamma} \right) B(t). \tag{59}
\]

Solving these o.d.e.’s subject to the boundary conditions yields,

\[
\eta(t) = \alpha \exp \left[ \left( r - c_2 + \frac{c_1}{1 - \gamma} \right) (T - t) \right], \tag{60}
\]

\[
B(t) = A \exp \left[ - \left( c_2 - \frac{c_1}{1 - \gamma} \right) (T - t) \right]. \tag{61}
\]

Finally, substituting (60) and (61) in (53) completes the description of the \( J \) function. This completes the solution of program B in the case \( \kappa = \nu \).

For the case when \( \kappa \neq \nu \), we follow the same strategy as in the infinite horizon case and define \( w(x) \) to be zero for \( |x| < a \), where \( a \) is a small positive number. It follows that the solution is similar to that obtained for the case \( \kappa = \nu \) except that we replace \( c_1 \) and \( c_2 \) by \( c'_1 \) and \( c'_2 \) as defined by (51) and (52).
5 Discussion of Solutions

For both an infinite and a finite horizon, the optimal wealth response is a function of the price relative $R = e^x$, where $x$ is the jump in the log of the stock price. The specific function of interest to a HARA investor is:

$$f(R) = 1_{R>f^a} \left[ \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} R^{\frac{\zeta + \lambda}{\gamma}} - 1 \right] + 1_{R<f^a} \left[ \left( \frac{\kappa}{\nu} \right)^{-\frac{1}{\gamma}} R^{\frac{\nu - \lambda}{\gamma}} - 1 \right], \quad (62)$$

and the position taken in this derivative is at the level of the investor’s risk capital defined by:

$$RC = W - \frac{\gamma}{\eta} B,$$

where the wealth is time-dependent as are $B$ and $\eta$ when we have a finite horizon. The infinite horizon problem has a fixed floor for wealth, with the excess over this floor being the invested risk capital. The finite horizon problem has a rising floor, that rises at the interest rate as may be observed from the solutions (60) and (61). The final level of the finite horizon floor is that of the terminal utility function $A/\alpha$. This rise in the level of the floor reflects a decline in risk tolerance as we approach the terminal date. All capital is invested in the risky optimal derivative if the horizon is far away.

Consider now the optimal payoff (62). The general shape of the payoff desired in response to market jumps may be determined by restricting attention to the case $\kappa = \nu$. Consider first the case $s = \sigma$. In this case for $\theta < 0$ (negative skewness), we have that $\zeta$ and $\lambda$ are both negative. For positive returns, we observe that risk-averse investors ($\gamma > -\zeta - \lambda$) would be positioned to have increasing but concave payoffs as functions of $R$. These investors buy at-the-money calls and write out-of-the-money calls. For low risk aversion ($\gamma < -\zeta - \lambda$), the payoff is convex in $R$ and these investors in addition to stock, write at-the-money calls and buy out-of-the-money calls.

For negative returns, one may show that $\zeta - \lambda$ is negative and so payoffs decline with returns. Highly risk-averse investors ($\gamma > -\zeta + \lambda$) prefer a concave payoff. Thus, in addition to holding the stock, they buy at-the-money puts and then sell out-of-the-money puts to achieve the concavity. Investors with low risk aversion ($\gamma < -\zeta + \lambda$) prefer a convex payoff, achieved by selling at-the-money puts and buying out-of-the-money puts. Investors with intermediate risk aversion ($-\zeta + \lambda < \gamma < -\zeta - \lambda$) take convex positions on the upside and concave positions on the downside.

In summary, for $s = \sigma$, the most risk-averse investors buy at-the-money options of both types and sell out-of-the-money options, while the less risk
averse do the opposite. Investors with intermediate risk aversion take convex positions on the upside and concave positions on the downside.

Figure 2: Optimal Payoff for Intermediate Risk Aversion.

There is considerable empirical evidence that historical volatilities are below their risk-neutral counterparts, i.e. $s < \sigma$. In this case, $\lambda$ may be positive and we can have $-\zeta - \lambda < \gamma < -\zeta + \lambda$. For $s < \sigma$, investors with such intermediate risk aversion achieve convex payoffs on the downside by buying out-of-the-money puts and create concave payoffs on the upside by writing out-of-the-money calls. These collared payoffs have long been popular for options on price and this analysis suggests that collared payoffs
linked to returns are optimal for investors with intermediate levels of risk aversion in markets with $s < \sigma$.

If investors differ in their ability to dynamically trade short-term options, one would expect that low cost traders such as investment houses would provide the optimal payoffs to others and would then hedge this liability using dynamic trading strategies in options. Thus, it is interesting to observe that derivative security payoffs tied to daily return levels are now emerging in certain over-the-counter markets.

6 Summary and Extensions

We considered the problem of optimal investment in continuous time economies in which dynamic trading strategies in options allow investors to hedge jumps of all sizes. In particular, we studied this problem when the underlying asset price dynamics are given by a pure jump Lévy process with an infinite arrival rate. Both the infinite and finite horizon problems are considered for HARA utility and for price dynamics given by the VG process.

We found that investors in these economies are interested in derivatives written on future price relatives, rather than on future prices. The position of HARA investors in such a derivative varies with their wealth and time horizon. Infinite horizon investors place the excess over a floor in the optimal derivative, while finite horizon investors raise the floor as they approach their horizon.

The optimal payoff for highly risk-averse investors is achieved by buying at-the-money options and selling out-of-the-money options, while low risk aversion investors do the opposite. For the typical case of statistical volatility lower than implied, we find that the optimal financial product is a collar structure on the price relative, with investors financing downside protection by sacrificing upside gain. The resulting position is concave with respect to market up jumps and convex with respect to market drops. We also note that market interest is emerging for precisely such a derivative product.

This research can be extended in a number of directions. For example, there may well be alternative restrictions on preferences, beliefs, and price processes which yield explicit solutions. A random horizon problem may be considered and stochastic labor income may be added. Finally, a difficult open problem concerns the existence of a general equilibrium in which
the risk-neutral price process is a consequence of heterogeneous agents simultaneously optimizing their intertemporal consumption and investment decisions. In the interests of brevity, these questions are left for future research.
References


