Deriving Derivatives of Derivative Securities

Abstract

We use various techniques to simplify the derivations of “greeks” of path-independent claims in the Black-Merton-Scholes model. We first interpret delta, gamma, speed, and other higher order spatial derivatives of these claims as the values of certain quantized contingent claims. We then show that all partial derivatives of such claims can be represented in terms of these spatial derivatives. These observations permit the rapid deployment of high order Taylor series expansions, which we illustrate for European options.

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I Introduction

In spite of increasing evidence against it, the Black-Merton-Scholes (BMS) model remains the lingua franca of option pricing. Widely used terms such as implied volatility and the volatility smile are only defined in terms of this model. Standard definitions of the so-called greeks (eg. delta, gamma, rho) also rely on the model. When other models are used in practice, the outputs of such models are routinely translated into standard BMS outputs such as implied volatility.

Given this state of affairs, a deep understanding of the BMS model is a prerequisite for meaningful interactions with practitioners. Since the mechanism by which arbitrage-free values are obtained in this model is well-understood, this paper examines greeks, which enjoy multiple applications. It is well-known that greeks are frequently used for hedging, market risk measurement, and for profit and loss attribution. They are also used in model risk assessment, optimal contract design, and to imply out parameters. While symbolic math programs can derive arbitrary greeks, they cannot replace an intuitive understanding of the role, genesis, and relationships among all the various greeks.

This paper develops these relationships for path-independent claims such as European calls and puts. For such claims, we show that in the BMS model, delta, gamma, speed, and higher order price derivatives can always be interpreted as the value of a certain quantoed contingent claim. This interpretation allows one to transfer intuitions regarding values to these greeks and to apply any valuation methodology to determine them. We illustrate this result by deriving a completely explicit formula for the price derivatives of an option.

For path-independent claims in the BMS model, this paper generalizes the well known result that theta can be expressed in terms of the first three price derivatives, i.e. value, delta, and gamma. In particular, we relate any partial derivative of such claims to the spatial derivatives. These relationships permit rapid deployment of Taylor series expansions, which can sometimes be faster than recomputing values. Besides providing computational advantages, these relationships allow understanding of the behavior of one greek (eg. gamma) to be transferred to other greeks (eg. vega).

The structure of this paper is as follows. The next section reviews previous literature on determining
griks and reviews the BMS model of contingent claim valuation. The third section shows how delta, gamma, speed, and higher order price derivatives can be interpreted as the values of certain quantoed contingent claims. The following section uses operator calculus to relate first partials w.r.t. the dividend yield, the riskless rate, and the volatility rate to the claim's value, delta, and gamma. The fifth section generalizes these results by expressing an arbitrary greek in terms of spatial derivatives. The sixth section conducts Taylor series expansions of a call in all of its independent variables and explores the limitations of this commonly used approach. The paper concludes with a summary and a description of possible extensions. The appendices contain some technical results.

II  Literature Review and the Black-Merton-Scholes Model

II-A  Literature Review

This subsection reviews the literature on determining the partial derivatives of contingent claims values. Many textbooks (e.g., Hull[20] ch. 14) contain short descriptions of the primary greeks, i.e. delta, gamma, vega, theta, rho, and phi (the dividend yield derivative). Pelsser and Vorst[26] discuss the determination of these greeks in the context of the binomial model (see Cox and Rubinstein[7]). Garman[13] christens three more partial derivatives with the names speed ($\frac{\partial^3 \text{price}}{\partial \text{speed}^3}$), charm ($\frac{\partial^2 \text{price}}{\partial \text{speed}^2 \partial \text{time}}$), and color ($\frac{\partial^3 \text{price}}{\partial \text{speed}^2 \partial \text{time}^2}$). In [12], he defines the duration of option portfolios, while in [14], he defines volatility immunization and gamma duration. Similarly, Haug[18] discusses the aggregation of vegas of options of different maturities. Hull and White[21] compare delta hedging, delta+gamma hedging, and delta+vega hedging of written FX options and conclude that the latter works best. Willard[29] calculates sensitivities for path-independent derivative securities in multi-factor models, while Ross[27] calculates sensitivities for multi-asset European options.


1Vega is sometimes renamed kappa since it is not a greek letter.
2This paper finds the solution of his recursion.
There is a substantial literature on durations of bonds, which this literature survey ignores in the interests of brevity. However, Hull and White[22] examine delta,gamma, and vega of interest rate derivatives. Similarly, using an option pricing context, Ferri, Oberhelman and Goldstein[11] examine yield sensitivities of short-term securities, while Ogden[25] examines yield sensitivities of corporate bonds.

In a very general context, Breeden and Litzenberger[4]) show that the second derivative with respect to an option’s strike price can be used to imply out state contingent prices. Similarly, Schroder[28] shows that the first derivative w.r.t. strike of an American option yields the risk-neutral probability of exercise. He also interprets deltas of American options. In this paper, we do not consider derivatives with respect to strike price, since our focus is on general statements for European-style contingent claims, rather than on option prices per se.

II-B The Black-Merton-Scholes Model

The BMS model assumes frictionless security markets and that over the contingent claim’s life \([0, T]\), there is a constant riskless rate \(r\) and a constant continuous dividend yield \(q\) from the underlying security, whose price \(S\) obeys geometric Brownian motion:

\[
\frac{dS_t}{S_t} = \alpha_t \ dt + \sigma \ dB_t, \quad t \in [0, T].
\]

As usual, the process \(\alpha_t\) is the expected growth rate in the underlying security price, \(\sigma > 0\) is the security’s constant volatility rate, and \(\{B_t; t \geq 0\}\) is a standard Brownian motion defined on a probability space \((\Omega, \mathcal{F}, P)\).

Consider a path-independent claim whose final payoff \(f(S)\) is a known function of \(S\). Let \(V(\tau, S)\) be a function relating the claim’s arbitrage-free value, \(V\), to the claim’s time to maturity, \(\tau = T - t\) and to the underlying security price, \(S\). It is well-known that \(V(S, \tau)\) solves the following initial value problem:

\[
\frac{\partial V(\tau, S)}{\partial \tau} = \frac{\sigma^2 S^2}{2} \frac{\partial^2 V(\tau, S)}{\partial S^2} + (r - q) S \frac{\partial V(\tau, S)}{\partial S} - r V(\tau, S), \quad S > 0, \tau \in (0, T),
\]

subject to: \(V(0, S) = f(S)\).

It is also well known that there exists a unique measure \(Q^{(0)}\) under which the “risk-neutral” stock price process is:

\[
\frac{dS_t}{S_t} = (r - q)dt + \sigma dB_t^{(0)}, \quad t \in (0, T).
\]
where \( \{B_t^{(0)}; t \in (0, T)\} \) is a \( Q^{(0)} \) standard Brownian motion. Under the measure \( Q^{(0)} \), the forward price of the underlying and the forward price of any path-independent claim \( e^{r(T-t)} V(T-t, S_t) \) are both martingales. Consequently, the solution of (2) and (3) can be expressed as:

\[
V(\tau, S) = e^{-r\tau} E^{(0)}[f(S_T)|S_t = S], \quad S > 0, \tau \in (0, T).
\]

(5)

For certain payoff functions (eg. \( f(S) = \max[0, S - K] \)), the integral implicit in (5) can be expressed in terms of special functions (eg. the normal distribution function). However, for a general payoff function (eg. the payoff function is a rational function of \( S \)), the integral implicit in (5) must be done numerically. In such cases, the computation of higher order partial derivatives can be numerically intensive and so there are computational motivations for exploring relationships among these greeks.

III  Stock Price Derivatives and Quantoing

This section develops expressions for delta, gamma, speed, and higher order derivatives with respect to the stock price. While only the first derivative is needed to perfectly hedge a contingent claim in the model, it can be shown that gamma governs the hedging error when hedging at the wrong volatility. Furthermore, gamma, speed, and other higher order derivatives govern the hedging error when prices jump and/or when rebalancing is discrete. All of these price greeks are needed to do a Taylor series expansion of the value \( V \) in \( S \). This section shows that every price derivative can be regarded as the arbitrage-free value of a certain quantoed contingent claim. This interpretation allows intuition regarding values to be transferred to these greeks. It also allows any methodology for determining values to be applied to determining these greeks.

To simplify calculations, we first transform the BMS p.d.e. (2) by expressing the value \( V \) in terms of the log of the stock price. Let \( U(\tau, x) \equiv V(\tau, S) \) where \( x \equiv \ln S \). Then \( \frac{\partial V(\tau, S)}{\partial \tau} \) on the LHS of (2) can be replaced by \( \frac{\partial U(\tau, x)}{\partial \tau} \). For the RHS, we use the following general relationship between a stock price derivative and log price derivatives:

\[
S^{\ell_s} D_{x}^{\ell_s} = \sum_{i_s=1}^{\ell_s} \mathcal{S}_1(\ell_s, i_s) D_{x}^{i_s}, \quad \ell_s = 1, 2, \ldots
\]

(6)

where \( \mathcal{S}_1(\ell_s, i_s) \) denotes a Stirling number of the first kind\(^3\). Using (6) for \( \ell_s = 0, 1, 2 \) implies that:

\[
V(\tau, S) = U(\tau, x)
\]

(7)

\(^3\)These numbers satisfy a simple recursion and are given by a complicated closed form solution in Appendix 4.
\[ S \frac{\partial V(\tau, S)}{\partial S} = \frac{\partial U(\tau, x)}{\partial x}, \quad (8) \]
\[ S^2 \frac{\partial^2 V(\tau, S)}{\partial S^2} = -\frac{\partial U(\tau, x)}{\partial x} + \frac{\partial^2 U(\tau, x)}{\partial x^2}. \quad (9) \]

Substituting into the BMS p.d.e. (2) yields a p.d.e. with constant coefficients:

\[ \frac{\partial U(\tau, x)}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 U(\tau, x)}{\partial x^2} + \mu \frac{\partial U(\tau, x)}{\partial x} - rU(\tau, x), \quad x \in \mathbb{R}, \tau \in (0, T), \quad (10) \]

where \( \mu \equiv r - q - \frac{\sigma^2}{2} \). Under (7), the initial condition (3) transforms to:

\[ U(0, x) = f(e^x) \equiv \phi(x). \quad (11) \]

Let \( D_x^\ell \) denote the \( \ell \)-th derivative w.r.t \( x \). Differentiating (10) w.r.t. \( x \) \( \ell \) times implies that \( D_x^\ell U(\tau, x) \) satisfies the same p.d.e. as \( U(\tau, x) \). Consequently, the process \( e^{(T-t)} D_x^\ell U(T-t, X_t) \) is a \( Q^{(0)} \) martingale for \( \ell = 0, 1, 2, \ldots \), where \( X_t \equiv \ln S_t \). It follows that \( D_x^\ell U(T-t, X_t) \) can be interpreted as the value at \( t \) of a contingent claim with the single payoff \( \phi^{(\ell)}(X_T) \) occurring at \( T \).

For example, when \( \ell = 1 \), \( U_x(T-t, X_t) = S_t V_s(T-t, S_t) \) is the value at \( t \) of a claim paying \( \phi'(X_T) = S_T f'(S_T) \) at \( T \). In the case of a call, \( f(S) = \max[0, S-K] \), and so the payoff associated with \( U_x \) is that of a gap call \( S_T 1(S_T > K) \). Clearly, \( U_x(T-t, X_t) = S_t V_s(T-t, S_t) \) can also be interpreted as the dollar amount invested in the stock at time \( t \) when dynamically replicating the payoff \( f(S_T) \) occurring at \( T \). For \( \ell = 2 \), (8) and (9) imply that:

\[ U_{xx}(T-t, X_t) = S_t^2 V_{ss}(T-t, S_t) + S_t V_s(T-t, S_t), \quad (12) \]

and so \( U_{xx}(T-t, X_t) \) is the value at \( t \) of a claim paying \( \phi''(X_T) = S_T^2 f''(S_T) + S_T f'(S_T) \) at \( T \). Since \( S_t V_s(T-t, S_t) \) is also a claim price process, (12) implies that \( S_T^2 V_{ss}(T-t, S_t) \) is yet another claim price process, with payoff \( S_T^2 f''(S_T) \) at \( T \). Furthermore, the process \( U_{xx}(T-t, X_t) \) can also be interpreted as the dollar amount invested in the stock at time \( t \) when dynamically replicating the payoff \( {\phi^{(2)}}(X_T) \) occurring at \( T \). Thus for the call example, \( U_{xx}(T-t, X_t) \) is the dollar amount invested in the stock at \( t \) when dynamically replicating the gap call payoff occurring at \( T \).

The inverse to (6) giving the general relationship between a log price derivative and stock price derivatives

\[ {\phi^{(2)}}(X_T) \]
is given by:

\[ D_x^{\ell_x} = \sum_{i_x} S_2(\ell_x, i_x) S_i^x D_x^{i_x}, \]  

where \( S_2(\ell_x, i_x) \) denotes a Stirling number of the second kind\(^5\). Using this expression, it is not hard to prove the following lemma:

**Lemma 1:** For each \( \ell = 0, 1, 2, \ldots \), the process \( \{ S_1^\ell D_t V(S_t, T - t), t \in [0, T] \} \) is the arbitrage-free value at \( t \) of a claim with payoff \( S_T f^{(\ell)}(S_T) \) at \( T \).

We can use this lemma to determine the process followed by the delta of a contingent claim, \( V_\tau(S, S) \), which is defined as the first partial derivative of the claim's value function, \( V(\tau, S) \), w.r.t. the price, \( S \). Since \( S_1 V_\tau(T - t, S_t) \) is the price of a claim paying \( S_T f'(S_T) \) at \( T \), it follows that delta can be interpreted as the price in shares of a claim paying \( f'(S_T) \) shares at \( T \). Thus for the call example, delta is the price in shares of a claim paying one share if \( S_T > K \) at \( T \). Since the function \( V_\tau(\tau, S) \) relates the price in shares to the price of the stock *in dollars*, we are in the same situation as when the payoff of an option is defined in terms of a different currency than the price of the underlying. This option is said to be quantoed and it is well-known that a so-called “quanto correction” is required in specifying the risk-neutral process. When the currency being quantoed into is a share, the quanto correction for the stock price results in the following modification of the “risk-neutral” stock price process (4):

\[ \frac{dS_t}{S_t} = (r - q + \sigma^2)dt + \sigma dB_t^{(1)}, \quad t \in [0, T], \]  

where \( \{ B_t^{(1)}; t \in [0, T] \} \) is a \( Q^{(1)} \) standard Brownian motion. The appropriate discount rate for discounting share-denominated payoffs is the dividend yield \( q \). Thus, delta can be represented as:

\[ V_\tau(S, S) = e^{-qT} E^{(1)}[ f^{(1)}(S_T) | S_T = S], \quad S > 0, \tau \in (0, T), \]  

where the operator \( E^{(1)} \) indicates that the expectations are calculated using (14).

The gamma of a contingent claim, \( V_{\tau S}(\tau, S) \) is defined as the second partial derivative of the claim’s value, \( V(\tau, S) \), w.r.t. the price, \( S \). Since Lemma 1 implies that \( S_1^2 V_{\tau S}(T - t, S_t) \) is the price of a claim paying \( S_T^2 f''(S_T) \) at \( T \), it follows that gamma can be interpreted as the price in “squares” of a claim paying \( f''(S_T) \) squares at \( T \). By a square, we mean a dividend-paying claim whose value is \( S_T^2 \) for all \( t \in [0, T] \). To determine

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\(^5\)These numbers satisfy a simple recursion and are given by a simple closed form solution in Appendix 4.
this dividend, one replaces $V(\tau, S)$ with $S^2$ in the RHS of (2). The resulting dividend is $(r - 2q + \sigma^2)S^2$ for a constant dividend yield of $r - 2q + \sigma^2$. Thus for the call example, gamma is the price in squares of a claim paying $f''(S) = \delta(S_T - K)$ squares at $T$, where $\delta(\cdot)$ is the Dirac delta function.

The appropriate discount rate for discounting square denominated payoffs is the dividend yield $r - 2q + \sigma^2$. It follows that the gamma of a claim can be represented as:

$$V_{ss}(\tau, S) = e^{(r - 2q + \sigma^2)\tau}E^{(2)}[f''(S_T)|S_t = S], \quad S > 0, \tau \in (0, T), \quad (16)$$

where the operator $E^{(2)}$ indicates that the expectation of the final gamma, $f''(S_T)$, is calculated from the following geometric Brownian motion:

$$\frac{dS_t}{S_t} = (r - q + 2\sigma^2) \, dt + \sigma \, dB_t^{(2)}, \quad t \in [0, T],$$

where $\{B_t^{(2)}; t \in [0, T]\}$ is a $Q^{(2)}$ standard Brownian motion.

Higher order derivatives with respect to the underlying security price can be obtained analogously. Appendices 1 and 2 prove the following general theorem:

**Theorem 1:** The value, delta, gamma, and higher order derivatives of path-independent claims in the BMS model are given by:

$$\frac{\partial^j V(\tau, S)}{\partial S^j} = e^{(\mu - 1)r - jq + (\mu - 1)j\sigma^2/2r}E^{(j)}[f^{(j)}(S_T)|S_t = S], \quad j = 0, 1, \ldots, \quad S > 0, \tau \in (0, T), \quad (17)$$

where the operator $E^{(j)}$ indicates that the expectation of the final $j$-th derivative, $f^{(j)}(S_T)$, is calculated from the following geometric Brownian motion:

$$\frac{dS_t}{S_t} = (r - q + j\sigma^2) \, dt + \sigma \, dB_t^{(j)}, \quad t \in [0, T],$$

and where $\{B_t^{(j)}; t \in [0, T]\}$ is a $Q^{(j)}$ standard Brownian motion.

The discount rate $(\mu - 1)r - jq + (\mu - 1)j\sigma^2/2$ in (17) is the dividend yield on a “power claim” whose value is $S_t^j$ for all $t \in [0, T]$. The measure $Q^{(j)}$ describes prices of Arrow Debreu securities in terms of these power claims. The payoffs of these Arrow Debreu securities are indexed over paths and also pay out in power.

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6The Dirac delta function is a generalized function characterized by two properties:

1. $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$
2. $\int_{-\infty}^{\infty} \delta(x) dx = 1$.
claims. Since the stock price $S$ generating the paths is still denominated in dollars, a quanto correction is needed, which involves adding $j\sigma^2$ to the proportional drift in (18).

III-A European Options

To illustrate higher order price derivatives in the case of an option, let $c$ be a call/put indicator:

$$c = \begin{cases} 
1 & \text{if call} \\
-1 & \text{if put.} 
\end{cases}$$

Recalling that $x = \ln S$, the BMS formula for European option value is:

$$e_{o}(x) = c[e^{x-\sigma^2}N(c_{1}) - Ke^{-rT}N(c_{2})],$$

where:

$$d_{2} = \frac{x - \ln(K) + \mu T}{\sigma \sqrt{T}}, \quad d_{1} = d_{2} + \sigma \sqrt{T},$$

and $N(d) = \int_{-\infty}^{d} \frac{e^{-z^{2}/2}}{\sqrt{2\pi}}dz$.

**Theorem 2:** For $\ell_{x} = 1, 2, \ldots$, the $\ell_{x}$-th derivative w.r.t $x$ of a European option is:

$$D^{\ell_{x}}_{x}e_{o}(x) = ce^{x-\sigma^2}N(c_{1}) + Ke^{-rT}N'(c_{2}) \frac{\ell_{x}}{\sigma \sqrt{T}} \sum_{i_{x}=0}^{\ell_{x}-2} \frac{H_{i_{x}}(d_{2})}{(-\sigma \sqrt{T})^{i_{x}}}, \quad \ell_{x} = 1, 2, \ldots, \quad (19)$$

where $H_{i}(d), i = 0, 1, 2,$ are the Hermite polynomials.

The Hermite polynomials satisfy the recursion:

$$H_{i+1}(d) = dH_{i}(d) - iH_{i-1}(d), \text{with } H_{0}(d) = 1, H_{1}(d) = d.$$

The closed form formula for the $i$-th Hermite polynomial is well known to be:

$$H_{i}(d) = \sum_{g=0}^{[\frac{i}{2}]} \frac{d^{i-2g}}{(i-2g)!g!(2)^{g}},$$

To obtain higher order price derivatives of an option, substitute (19) in (6). The details are left to the reader.

IV Rate Derivatives and Operator Calculus

This section uses operator calculus to express derivatives with respect to the dividend yield, riskfree rate, and volatility in terms of the stock price derivatives derived in the last section. These derivatives are used to approximate the model risk arising from assuming that these parameters are constant over time. The volatility derivative (vega) is also often used to calculate implied volatility numerically.
The initial value problem (2) and (3) governing $V(\tau, S)$ can be rewritten as:

\[
\frac{\partial V(\tau, S)}{\partial \tau} = \mathcal{L}[V(\tau, S)], \quad \tau > 0, \tag{20}
\]

subject to: $V(0, S) = f(S), \quad S > 0$. \tag{21}

where $\mathcal{L}$ is the following linear operator:

\[
\mathcal{L} = \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q)S \frac{\partial}{\partial S} - \tau I, \quad S > 0. \tag{22}
\]

Operator calculus treats (20) as an ordinary differential equation in $\tau$, by treating $\mathcal{L}$ as a constant and $S$ as a fixed parameter. The formal solution of (20) subject to (21) is then given as:

\[
V(\tau, S) = \exp\{\tau \cdot \mathcal{L}\} f(S), \quad S > 0, \tau \in (0, T), \tag{23}
\]

where:

\[
\exp\{\tau \cdot \mathcal{L}\} \equiv \sum_{j=0}^{\infty} \frac{(\tau \mathcal{L})^j}{j!}. \tag{24}
\]

To justify this representation, consider a Taylor series expansion of $V(\tau, S)$ in $\tau$ about $\tau = 0$:

\[
V(\tau, S) = V(0, S) + \tau \frac{\partial V(0, S)}{\partial \tau} + \frac{\tau^2}{2} \frac{\partial^2 V(0, S)}{\partial \tau^2} + \ldots + \frac{\tau^j}{j!} \frac{\partial^j V(0, S)}{\partial \tau^j} + \ldots
\]

\[
= \left[ 1 + \tau \mathcal{L} + \frac{\tau^2}{2} \mathcal{L}^2 + \ldots + \frac{\tau^j}{j!} \mathcal{L}^j + \ldots \right] V(0, S), \quad S > 0, \tau > 0, \tag{25}
\]

where $\mathcal{L}^j$ is the j-fold composite of $\mathcal{L}$, i.e. $\mathcal{L}^j[V(0, S)] = \mathcal{L} \circ \mathcal{L} \circ \ldots \circ \mathcal{L}[V(0, S)]$, Substituting (21) and (24) in (25) yields (23).

Treating $\mathcal{L}$ as a constant, differentiating (23) with respect to $\tau$ recovers (20). If we analogously differentiate the formal solution (23) with respect to the dividend yield $q$, the riskless rate $r$, or the volatility $\sigma$, we rapidly obtain useful and intuitive representations of partial derivatives w.r.t. these variables:

\[
\frac{\partial V(\tau, S)}{\partial q} = \tau \frac{\partial \mathcal{L}}{\partial q} \exp\{\tau \cdot \mathcal{L}\} f(S) = -\tau S \frac{\partial}{\partial S} V(\tau, S) \tag{26}
\]

\[
\frac{\partial V(\tau, S)}{\partial r} = \tau \frac{\partial \mathcal{L}}{\partial r} \exp\{\tau \cdot \mathcal{L}\} f(S) = -\tau \left[ V(\tau, S) - S \frac{\partial}{\partial S} V(\tau, S) \right] \tag{27}
\]

\[
\frac{\partial V(\tau, S)}{\partial \sigma} = \tau \frac{\partial \mathcal{L}}{\partial \sigma} \exp\{\tau \cdot \mathcal{L}\} f(S) = \sigma \tau S \frac{\partial^2}{\partial S^2} V(\tau, S), \quad S > 0, \tau > 0. \tag{28}
\]

Appendix 3 proves that these manipulations are correct.

\footnote{For a call struck at $K$, all time derivatives become unbounded as $\tau \downarrow 0$ at $S = K$. Nonetheless, the RHS of (23) is well-defined (see Hirschman and Widder [19] page 5).}
The three results (26) to (28) are easily justified using risk-neutral valuation. To understand the result (26) for the dividend yield derivative (phi), note that a small shift upward in the dividend yield lowers the risk-neutral mean of the terminal stock price by an amount which increases with the time to maturity (tenor). The effect of this small upward shift is thus similar to the effect of a small proportional shift downward in the initial stock price, S. However, since the effect of a given downward shift on the risk-neutral mean is independent of tenor, the mimicking downward shift must be multiplied by time to maturity. Hence up to sign, the dividend yield derivative \( \frac{\partial V(\tau, S)}{\partial \phi} \) is the product of the tenor \( \tau \) and the dollar investment \( S \frac{\partial V(\tau, S)}{\partial S} \) in the underlying security in the replicating portfolio. The smaller is this tenor and/or investment, the lower is the sensitivity to dividend yield. Thus, risk managers contemplating the design of a contingent claim can only lower phi by shortening the tenor or by lowering the expected \(^{8}\) payoff delta, which will lower the initial delta. A delta-neutral portfolio of equal maturity claims is thus automatically immunized against shifts in the dividend yield. However, a short call hedged by delta shares is not phi-neutral, since the dividend-paying stock should be considered as a portfolio of claims with many maturities.

Similarly, to understand the result (27) for the riskfree rate derivative (rho), note that a small shift upward in the riskfree rate raises the risk-neutral mean of the terminal stock price by an amount which increases with tenor. In analogy with dividend yields, this effect can be mimicked by alternatively raising the initial stock price by a proportional amount, which increases with time to maturity. Since the riskfree rate is also used to discount the expected claim payoff, an upward shift in the riskfree rate also lowers the claim value by the duration \( \tau \). Hence up to sign, a claim’s rho \( \frac{\partial V(\tau, S)}{\partial r} \) is the product of the tenor \( \tau \) and the dollar investment \( V(\tau, S) - S \frac{\partial V(\tau, S)}{\partial S} \) in the riskless asset in the replicating portfolio. The smaller is this tenor and/or investment, the lower is the sensitivity to the riskless rate. Thus, risk managers contemplating the design of a contingent claim can only lower this sensitivity by shortening the tenor or by lowering the expected fixed component of the terminal payoff, which will lower the implicit initial investment in the riskless asset. A costless delta-neutral portfolio of equal maturity claims is immunized against shifts in the (assumed flat) term structure of interest rates. For example, a short call hedged by being long \( e^{-\rho \tau} N(d_1) \) shares and short \( KN(d_2) \) pure discount bonds is immunized, since the shares have no rho.

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\(^{8}\) The results of the previous section implies that \( Q^{(1)} \) is the measure used in computing expectations.
Finally, to understand the result (28) for the volatility derivative (vega), note that a small shift upward in the volatility rate $\sigma$ raises the standard deviation of the terminal stock price by an amount which increases with tenor. Focussing on convex payoffs, this effect can be mimicked by increasing the expected terminal gamma of the payoff, which raises the initial gamma. The larger the tenor, the more the gamma must be raised. The result (28) implies that for a given spot price and volatility, vega depends only on the product of gamma and tenor. The smaller is the tenor and/or gamma, the lower is the sensitivity to volatility. Risk managers contemplating the design of contingent claims can only lower vega by shortening the tenor or lowering convexity. A gamma-neutral portfolio of equal maturity claims must have zero vega. For example, a collar which is initially gamma-neutral will also be vega-neutral.

V Arbitrary Greeks

Just as phi, rho, and vega are all simple functions of a claim's value, delta, and gamma, the BMS p.d.e. (2) implies that a claim's time derivative (theta) can also be expressed in terms of the first three spatial derivatives. This section generalizes these results by expressing an arbitrary greek in terms of its spatial derivatives. By an arbitrary greek, we mean a partial derivative of the form $D_q^\ell_q D_r^\ell_r D_t^\ell_t D_\sigma^\ell_\sigma D_s^\ell_s V$, where $D_q, D_r, D_t, D_\sigma$, and $D_s$ denote the first order derivative operators w.r.t. $q, r, t, \sigma$, and $s$ respectively, and where $V(q, r, t, \sigma, s)$ denotes the BMS value of a path-independent claim when considered as a function of these 5 variables. By a spatial derivative, we mean a partial derivative w.r.t. the log stock price. If a representation in terms of stock price derivatives is desired, then (13) can be used with the results that follow. However, the representation in terms of derivatives w.r.t. log stock prices is convenient if finite differences are used to approximate the solution to the initial value problem (10) and (11).

**Theorem 3:** For $S = e^r$ and $\ell_q, \ell_r, \ell_t, \ell_\sigma, \ell_s$ positive integers, the following operators are equivalent when applied to solutions of the initial value problem (10) and (11):

$$D_q^\ell_q D_r^\ell_r D_t^\ell_t D_\sigma^\ell_\sigma D_s^\ell_s V$$

$$= (-\tau)^{\ell_q+\ell_r-\ell_t} \frac{\ell_r! \ell_\sigma!}{(2\sigma)^{\ell_\sigma} S^{\ell_\sigma}} \sum_{i_r=0}^{\ell_r} \frac{(-1)^{i_r}}{i_r! (\ell_r - i_r)!} \sum_{i_\sigma=0}^{\ell_\sigma} \frac{(-\tau)^i_\sigma}{(\ell_\sigma - i_\sigma)! (2i_\sigma - \ell_\sigma)!} \left( \frac{-2\sigma^2 \tau}{(\ell_\sigma - i_\sigma)!} \right)^i_\sigma \sum_{i_s=0}^{\ell_s} \mathcal{S}_1(\ell_s, i_s) \times$$

$$\sum_{\ell_s - \max\{\ell_r - \ell_t - \ell_q - i_r, i_s\} = \ell_s - \ell_t - \ell_q - i_r}^{\ell_s} \frac{(\ell_q + \ell_\sigma + i_\sigma)}{(\ell_\sigma - i_\sigma)! (2i_\sigma - \ell_\sigma)!} \left( \frac{\ell_\sigma + i_\sigma}{\ell_q + i_\sigma} \right)^{\ell_\sigma - i_\sigma} \sum_{i_r=0}^{\ell_r} \mathcal{S}_1(\ell_r, i_r) \times$$

$$\sum_{i_\sigma=0}^{\ell_\sigma} \frac{(-1)^{i_\sigma}}{i_\sigma! (\ell_\sigma - i_\sigma)!} \frac{(-2\sigma^2 \tau)^i_\sigma}{(\ell_\sigma - i_\sigma)! (2i_\sigma - \ell_\sigma)!} \right) \left( \frac{-\tau}{\ell_\sigma - i_\sigma} \right)^{i_\sigma} \sum_{i_s=0}^{\ell_s} \mathcal{S}_1(\ell_s, i_s) \times$$

$\mathcal{S}_1(\ell_s, i_s)$...
\[
\sum_{h_0}^{i_x} \left( \frac{-\mu}{\sigma} \right)^{h_0} \sum_{h_0}^{i_x} \frac{(-1)^{h_0}}{h_0!} \sum_{g_0=0}^{i_x-h_0} \left( \frac{-\sigma^2}{2\sigma} \right)^{g_0} \frac{g_0!}{g_0!} D_{h_0+g_0}^{i_x-g_0+i_x-h_0+2g_0}. \tag{29}
\]

In this formula, \([x]\) denotes the largest integer less than or equal to \(x\). Letting \(m\) and \(n\) denote nonnegative integers, the notation \(m^{\underline{n}}\) denotes the falling factorial i.e.:

\[
m^{\underline{n}} = m(m-1) \cdots (m-n+1). \]

To relate an arbitrary greek to a stock price derivative, one substitutes (13) in (29)\(^9\).

**VI Taylor Series**

By standard calculus, the Taylor series expansion of \(V(q, r, t, \sigma, S)\) about the point \((q_0, r_0, t_0, \sigma_0, S_0)\) is:

\[
V(q, r, t, \sigma, S) = V(q_0, r_0, t_0, \sigma_0, S_0) + \sum_{m=1}^{\infty} \sum_{\ell_0=0}^{m} \sum_{r_0=0}^{m} \sum_{\sigma_0=0}^{m} \sum_{S_0=0}^{m} \frac{m! D_{\ell_0}^{m} D_{r_0}^{m} D_{\sigma_0}^{m} D_{S_0}^{m}}{(\ell_0! r_0! \sigma_0! S_0!)^m} (\Delta q)^{\ell_0} (\Delta r)^{r_0} (\Delta \sigma)^{\sigma_0} (\Delta S)^{S_0} V(q_0, r_0, t_0, \sigma_0, S_0),
\]

where \(\ell_0 = m - \ell_q - \ell_r - \ell_t - \ell_\sigma\) and \(\Delta q \equiv q - q_0, \Delta r \equiv r - r_0, \Delta t \equiv t - t_0, \Delta \sigma \equiv \sigma - \sigma_0\) and \(\Delta S \equiv S - S_0\).

Substituting (29) in (30) and simplifying, the resulting expression gives the Taylor series expansion of a path-independent claim value in all 5 variables:

\[
V(q, r, t, \sigma, S) = V(q_0, r_0, t_0, \sigma_0, S_0) + \sum_{m=1}^{\infty} \sum_{\ell_0=0}^{m} \sum_{r_0=0}^{m} \sum_{\sigma_0=0}^{m} \sum_{S_0=0}^{m} \frac{m! D_{\ell_0}^{m} D_{r_0}^{m} D_{\sigma_0}^{m} D_{S_0}^{m}}{(\ell_0! r_0! \sigma_0! S_0!)^m} (\Delta q)^{\ell_0} (\Delta r)^{r_0} (\Delta \sigma)^{\sigma_0} (\Delta S)^{S_0} V(q_0, r_0, t_0, \sigma_0, S_0),
\]

where \(U(q, r, t, \sigma, x) \equiv V(q, r, t, \sigma, S)\).

**VI-A European Options**

Substituting (19) in (31) yields the Taylor series expansion of the European option value \(EO(q, r, t, \sigma, S) \equiv e^{\sigma \ln S}\) about the point \((q_0, r_0, t_0, \sigma_0, S_0)\). This formula simplifies upon observing that \(\sum_{i_x=0}^{\ell_\ell} S_1(\ell_\ell, i_x) = 0\) for

\[
| S_1(\ell_\ell, i_x) | S_2(\ell_\ell, i_x) | (-1)^{\ell_\ell-i_x} = \delta_{i_x-h_0},
\]

where \(\delta\) is the Kronecker delta function.
$l_s \geq 2$. The upper limit $\ell_s$ in the last sum in (31) equals one only when $m = 1$ and $\ell_q = \ell_r = \ell_t = \ell_\sigma = 0$.

In this case, the nested sum in (31) simplifies to

$$d_{10} = d_{20} + \sigma_0 \sqrt{\tau_0}, \quad d_{20} = \frac{\ln(S_0/K) + \mu_0 \tau_0}{\sigma_0 \sqrt{\tau_0}}, \quad \mu_0 \equiv r_0 - q_0 - \frac{\sigma_0^2}{2}.$$

Thus, the final result for the Taylor series expansion of the European option value is:

$$EO(q, r, t, \sigma, S) = EO(q_0, r_0, t_0, \sigma_0, S_0) + c e^{-\tau_0 q_0} N(c d_{10}) \Delta S$$

$$+ K e^{-\tau_0 r_0} \frac{N'(d_{20})}{\sigma_0 \sqrt{\tau_0}} \sum_{m=1}^{\infty} \sum_{\ell=-0}^{m-\ell_2} (-\tau_0 \Delta q) \ell_q \sum_{\ell_r=0}^{\ell_r} (-\tau_0 \Delta r) \ell_r \sum_{\ell_i=0}^{\ell_i} \frac{(-1)^{i_r}}{i_r!} \frac{m-\ell_2}{i_i} \sum_{i_t=0}^{\ell_t} \frac{(-1)^{i_t}}{(i_t)!} \frac{m-\ell_r}{i_r} \sum_{i_t=0}^{\ell_t} \frac{(-1)^{i_t}}{(i_t)!} \frac{m-\ell_r}{i_r} \sum_{i_t=0}^{\ell_t} \frac{(-1)^{i_t}}{(i_t)!}$$

$$\times \sum_{i_t=0}^{\ell_t} \sum_{g_t=0}^{\ell_t} \frac{(\Delta t)^{g_t}}{g_t!} \left( \frac{2\sigma_0}{\sigma_0^2} \right)^{\ell_2} \sum_{i_i=0}^{\ell_i} \sum_{g_i=0}^{\ell_i} \frac{(\Delta \sigma)^{g_i}}{g_i!} \left( \frac{\Delta S}{\sigma_0} \right)^{\ell_2} \sum_{\ell_1=0}^{\ell_1} \frac{(-2\sigma_0^2 \tau_0)^{i_1}}{(\ell_1 - i_1)!} \left( \frac{H_1(d_{20})}{\sqrt{1 - \tau_0}} \right)^{\ell_2}.$$

Figure 1 shows a well-behaved expansion of a European call value at a point $(q_1, r_1, t_1, \sigma_1, S_1) = (0.05, 0.01, 0.25, 0.25, 110)$ about $(q_0, r_0, t_0, \sigma_0, S_0) = (0.02, 0.06, 0, 2, 100)$ for $(K, T) = (100, 1)$. As the order is increased from 1 to 4, the

![Figure 1: Taylor Series Expansion of European Call Value in all Variables](image-url)

truncated Taylor series gets closer to the correct value until the specified tolerance of .01 is achieved. The
example illustrates that high order truncated expansions are sometimes needed to achieve the desired result.

Figure 2 focuses on a univariate Taylor series expansion in volatility, holding \((q, r, t, S, K, T)\) constant at \((.02, 0.0, 100, 100, 1)\). The left panel shows that even an 8th order truncated Taylor series expansion is insufficient if the volatility changes by a sufficient magnitude. However, the right panel shows that for the small change in volatility from .2 to .25, convergence occurs rapidly.

Since truncated Taylor series are sometimes used in place of a recalculation, it is worth investigating the region of convergence for the 5 independent variables. The following theorem holds for any payoff function.

**Theorem 4:** Let \(\rho_y \geq 0\) denote the radius of convergence in the variable \(y\) of the Taylor series expansion of claim value \(V\) about the point \((q_0, r_0, t_0, \sigma_0, S_0)\). Then \(\rho_r = \infty, \rho_q = \infty, \rho_s = S_0, \rho_t = T - t_0,\) and \(\rho_r = \frac{\sigma_0}{\sqrt{2}}\).

Thus, the radius of convergence is unbounded for Taylor series expansions in the interest rate or dividend yield. For expansions in the stock price, Estrella[10] proved that the radius of convergence is the stock price, so that Taylor series should not be used if the stock price is increased by a factor of 2 or more. Similarly, for expansions in the time variable \(t\), a similar analysis shows that the radius of convergence is the tenor, so that
Taylor series should not be used to move backwards in calendar time by more than the tenor. For expansions in volatility, the radius of convergence is $\frac{1}{\sqrt{2}}$, so that Taylor series should not be used if volatility is raised or lowered by more than about 70%. Since implied volatilities have been known to rise by more than this amount following a crash, it is worth exploring the Taylor series expansion in volatility when this condition is violated. Figure 3 shows such an expansion when volatility doubles from .2 to .4, holding $(q, r, t, S, K, T)$ constant at (.02, .06, 0, 100, 100, 1). Surprisingly, the expansion appears to work until about the 40th term when it begins to diverge. This example shows the danger in increasing the order of a truncated Taylor series in an attempt to improve accuracy when convergence is not guaranteed.

**VII Summary and Extensions**

For path-independent claims in the BMS model, delta, gamma, speed, and higher order price derivatives can all be interpreted as the values of certain quantized contingent claims. This interpretation allows their values to be calculated as a discounted expectation. Any partial derivative w.r.t. $q, r, t, \sigma$, and/or $S$ can be expressed in terms of the security’s spatial derivatives. Since the latter are easily determined, Taylor series
in all 5 variables becomes feasible. However, the efficacy of the truncated versions of these series depends on the magnitude of the change in the variables. For sufficiently large changes in $S$, $t$, or $\sigma$, Taylor series diverge.

The results of this paper realize their greatest practical significance when numerical methods must be employed to value a claim. The same technique used to numerically value the claim can be used to numerically determine spatial derivatives. Given numerical results for these spatial derivatives, the other derivatives can be determined analytically. Thus, computational resources should be spent accurately determining the claim's spatial derivatives, rather than attempting a coarser approximation of all the greeks.

Our results easily extend to contingent claims with intermediate payouts, either discrete or continuous. The extension of our results to multiple state variables or to more complex stochastic processes or payout structures should be explored. In the interests of brevity, these extensions are left for future research.
References


Appendix 1: Functional Analysis Proof of Theorem 1

This appendix proves the following general result for the \( j \)-th order derivative \( D_j^V(\tau, S) \equiv \frac{\partial^j V(\tau, S)}{\partial S^j} \) with respect to the underlying security price:

\[
    D_j^V(\tau, S) = e^{[(j-1)r - j\sigma^2/2] \tau} E^{(j)}[f^{(j)}(S_T)|S_t = S], \quad j = 0, 1, \ldots, \quad S > 0, \tau > 0. \tag{33}
\]

Define a family of linear operators by:

\[
    \mathcal{L}_j = \frac{\sigma^2 S^2}{2} \frac{\partial}{\partial S} + (r - q + j\sigma^2) S \frac{\partial}{\partial S} + [(j - 1)r - jq + (j - 1)\sigma^2/2], \quad j = 0, 1, \ldots \tag{34}
\]

Then the general result (33) is implied by the Feynman-Kac formula (see Duffie[9]) if:

\[
    \frac{\partial D_j^V(\tau, S)}{\partial \tau} = \mathcal{L}_j D_j^V(\tau, S), \quad j = 0, 1, \ldots, \quad S > 0, \tau > 0. \tag{35}
\]

Recall the Black-Scholes p.d.e.:

\[
    \frac{\partial V(\tau, S)}{\partial \tau} = \frac{\sigma^2 S^2}{2} \frac{\partial^2 V(\tau, S)}{\partial S^2} + (r - q) S \frac{\partial V(\tau, S)}{\partial S} - rV(\tau, S), \quad S > 0, \tau \in (0, T). \tag{36}
\]

This p.d.e implies that (35) holds for \( j = 0 \):

\[
    \frac{\partial V(\tau, S)}{\partial \tau} = \mathcal{L}_0 V(\tau, S), \quad S > 0, \tau > 0. \tag{37}
\]

To show that (35) holds for all \( j \), we use induction. Thus, suppose that (35) holds for a particular \( j \). To show that (35) also holds with \( j \) replaced by \( j + 1 \), differentiate (35) with respect to \( S \):

\[
    \frac{\partial D_{j+1}^V(\tau, S)}{\partial \tau} = \mathcal{L}_{j+1} D_{j+1}^V(\tau, S) + \frac{\partial \mathcal{L}_j}{\partial S} D_j^V(\tau, S)
    = \frac{\sigma^2 S^2}{2} \frac{\partial}{\partial S} D_{j+1}^V(\tau, S) + (r - q + j\sigma^2) S \frac{\partial}{\partial S} D_{j+1}^V(\tau, S)
    + [(j - 1)r - jq + (j - 1)\sigma^2/2] D_{j+1}^V(\tau, S) + \frac{\partial^2 D_j^V(\tau, S)}{\partial S^2} + (r - q + j\sigma^2) \frac{\partial D_{j+1}^V(\tau, S)}{\partial S}
    = \frac{\sigma^2 S^2}{2} \frac{\partial}{\partial S} D_{j+1}^V(\tau, S) + [r - q + (j + 1)\sigma^2] S \frac{\partial}{\partial S} D_{j+1}^V(\tau, S) + [jr - (j + 1)q + j(j + 1)\sigma^2/2] D_{j+1}^V(\tau, S)
    = \mathcal{L}_{j+1} D_{j+1}^V(\tau, S).
\]

Q.E.D.
Appendix 2: Probabilistic Proof of Theorem 1

This appendix derives Theorem 1 using purely probabilistic means. Recall our original assumption that the underlying security price follows geometric Brownian motion:

\[
\frac{dS_t}{S_t} = \alpha_t \, dt + \sigma \, dB_t, \quad t \in [0, T],
\]

where \( \{B_t, t > 0\} \) is a standard Brownian motion on the probability space \((\Omega, \mathcal{F}, P)\). Let \( \{\lambda_t \equiv \frac{\alpha_t + q}{\sigma}, t \in [0, T]\} \) denote the market price of risk process and define the risk-neutral probability measure \(Q^{(0)}\), equivalent to \(P\), by its Radon-Nikodym derivative \(\frac{dQ^{(0)}}{dP} = \exp\{- \int_0^T \frac{\lambda_s^2}{2} \, dt - \int_0^T \lambda_s \, dB_t\}\). Then by Girsanov’s theorem (see Karatzas and Shreve[23] pg. 191), \(B_t^{(0)} = B_t + \int_0^t \lambda_s \, ds, t \in [0, T]\), is a standard Brownian motion on \((\Omega, \mathcal{F}, Q^{(0)})\). Substituting into (38) gives:

\[
\frac{dS_t}{S_t} = \alpha_t \, dt + \sigma \, d\left[B_t^{(0)} - \int_0^t \lambda_s \, ds\right] = (r - q) \, dt + \sigma \, dB_t^{(0)}, \quad t \in [0, T],
\]

with solution:

\[
S_t = S_0 e^{(r-q-\sigma^2/2)t + \sigma B_t^{(0)}}, \quad t \in [0, T].
\]

Let \(E^{(0)}\) denote expectation under the risk-neutral measure \(Q^{(0)}\). Then Harrison and Kreps[16] and Harrison and Pliska[17] show that the value at \(t \in [0, T]\) of a path-independent claim depends on this expectation as follows:

\[
V(\tau, S) = e^{-r\tau} \mathbb{E}^{(0)}[f(S_T)|S_t = S], \quad S > 0, \tau \in [0, T],
\]

where recall \(f(S_T)\) is the final payoff of the claim.

To prove Theorem 1, we develop the following theorem, which is easily proved\(^{10}\):

\[\text{Theorem 5}\]

\(^{10}\)Define the probability measure \(Q^{(j)}\), equivalent to \(Q^{(0)}\), by its Radon-Nikodym derivative \(\frac{dQ^{(j)}}{dQ^{(0)}} = \xi_j\). Then by Girsanov’s theorem, \(B_t^{(j)} = B_t^{(0)} - j \sigma t, t \in [0, T]\), is a standard Brownian motion on \((\Omega, \mathcal{F}, Q^{(j)})\). Substituting into (39) gives:

\[
\frac{dS_t}{S_t} = (r - q) \, dt + \sigma \, dB_t^{(j)} + j \sigma \xi_t\]

\[
= (r - q + j \sigma^2) \, dt + \sigma \, dB_t^{(j)}, \quad t \in [0, T].
\]
Define a process $\xi_t^{(j)}$ by:

$$
\xi_t^{(j)} = e^{-j^2 \sigma^2 T/2 + j \sigma (B_T^{(q)} - B_t^{(q)})}, \quad j = 0, 1, \ldots, t \in (0, T).
$$

(42)

Then for any function $g(\cdot)$:

$$
E^{(0)}[g(S_T)\xi_t^{(j)}] = E^{(j)} g(S_T).
$$

(43)

The operator $E^{(j)}$ indicates that the expectation of $g(S_T)$ is calculated as if the underlying security’s price process is:

$$
\frac{dS_t}{S_t} = (r - q + j \sigma^2) \, dt + \sigma \, dB_t^{(j)}, \quad j = 0, 1, \ldots, t \in [0, T].
$$

(44)

If we differentiate the claim’s value in (41) using the chain rule, we can express the claim’s time $t$ delta, $V_s(T - t, S_0)$, in terms of its final delta, $f^{(1)}(S_T)$:

$$
V_s(\tau, S) = e^{-r \tau} E^{(0)}[f^{(1)}(S_T)e^{(r-q-\sigma^2/2)\tau + \sigma (B_T^{(q)} - B_t^{(q)})}|S_t = S], \text{ from (40)}
$$

(45)

$$
V_s(\tau, S) = e^{-q \tau} E^{(0)}[f^{(1)}(S_T)\xi_t^{(1)}|S_t = S],
$$

(46)

where $\xi_t^{(1)} = e^{-r^2 \tau/2 + \sigma (B_T^{(q)} - B_t^{(q)})}, t \in [0, T]$. Similarly, if we differentiate the claim’s delta in (45) using the chain rule, we can express the claim’s initial gamma, $V_{ss}(T, S_0)$, in terms of its final gamma, $f^{(2)}(S_T)$:

$$
V_{ss}(\tau, S) = e^{-r \tau} E^{(0)}[f^{(2)}(S_T)e^{(r-q-\sigma^2/2)\tau + 2\sigma (B_T^{(q)} - B_t^{(q)})}|S_t = S] = e^{(r-2q+\sigma^2)\tau} E^{(0)}[f^{(2)}(S_T)\xi_t^{(2)}|S_t = S],
$$

(47)

where $\xi_t^{(2)} = e^{-2r^2 \tau + 2\sigma (B_T^{(q)} - B_t^{(q)})}, t \in [0, T]$. By repeated differentiation, the $j$-th derivative of the claim’s value with respect to the underlying security price, $D_j^0 V(\tau, S) = \frac{\partial^j V(\tau, S)}{\partial S^j}$, is:

$$
D_j^0 V(\tau, S) = e^{-r \tau} E^{(0)}[f^{(j)}(S_T)e^{(r-q-\sigma^2/2)\tau + j \sigma (B_T^{(q)} - B_t^{(q)})}|S_t = S] = e^{(r-1)(r-jq+(j-1)j\sigma^2/2)\tau} E^{(0)}[f^{(j)}(S_T)\xi_t^{(j)}|S_t = S],
$$

(48)

where $\xi_t^{(j)} = e^{-r^2 \tau/2 + j \sigma (B_T^{(q)} - B_t^{(q)})}, j = 0, 1, \ldots, t \in [0, T]$.

Applying Theorem 5 with $g(S_T) = f^{(j)}(S_T)$ allows us to calculate the initial $j$-th derivative at $t$, $D_j^0 V(\tau, S)$ by discounting the conditional expected final $j$-th derivative, $E^{(j)}[f^{(j)}(S_T)|S_t = S]$:

$$
D_j^0 V(\tau, S) = e^{(r-1)(r-jq+(j-1)j\sigma^2/2)\tau} E^{(j)}[f^{(j)}(S_T)|S_t = S], \quad j = 0, 1, \ldots, S > 0, \tau \in (0, T).
$$

(49)
The operator $E^{(j)}$ indicates that the expectation of the final $j$-th derivative, $f^{(j)}(S_T)$, is calculated assuming that the terminal security price $S_T$ is given by:

$$S_T = S_0 e^{[r-q+(\frac{\sigma^2}{2})T+\sigma p^{(j)}_T], \, j = 0, 1, \ldots, \quad (50)}$$

where $\{B_t^{(j)}, t \in [0, T]\}$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, Q^{(j)}), \, j = 0, 1, \ldots$.  


Appendix 3

This appendix justifies the manipulations that lead to equations (26) to (28) for the derivatives of claim value with respect to \( r, q, \) and \( \sigma \). We begin by rewriting (10) as:

\[
\frac{\partial U(\tau, x)}{\partial \tau} = LU(\tau, x), \quad x \in (-\infty, \infty), \tau > 0, \tag{51}
\]

where \( L \) is a linear operator defined by \( L \equiv \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} - rI, \mu \equiv r - q - \sigma^2/2 \). Let \( \phi(x) \) be the transformed initial condition:

\[
\phi(x) \equiv f(S). \tag{52}
\]

Then for each fixed \( x \in \mathbb{R} \), the operational solution of the initial value problem (51) and (52) is:

\[
U(\tau, x) = \exp(\tau \cdot L) \phi(x), \quad \tau \in (0, T), \tag{53}
\]

where \( \exp(\tau \cdot L) \) is another operator defined by:

\[
\exp(\tau \cdot L) = \sum_{j=0}^{\infty} \frac{(\tau L)^j}{j!}. \tag{54}
\]

To prove this result, differentiate the proposed solution (53) with respect to \( \tau \):

\[
\frac{\partial U(\tau, x)}{\partial \tau} = \frac{\partial \exp(\tau \cdot L) \phi(x)}{\partial \tau}
\]

\[
= \lim_{\Delta \tau \to 0} \frac{\exp((\tau + \Delta \tau) \cdot L) - \exp(\tau \cdot L) \phi(x)}{\Delta \tau}
\]

\[
= \lim_{\Delta \tau \to 0} \frac{\exp(\Delta \tau \cdot L)}{\Delta \tau} \frac{\exp(\tau \cdot L) \phi(x)}{\exp(\tau \cdot L) \phi(x)} - 1
\]

\[
= L \exp(\tau \cdot L) \phi(x)
\]

\[
= LU(\tau, x), \quad \tau \in (0, T).
\]

To verify equations (26) to (28) for the derivatives w.r.t. the dividend yield, riskless rate, and volatility, we express the linear operator \( L \) as the sum of three linear operators, each dependent on only one parameter:

\[
L = L_q + L_r + L_\sigma, \quad \text{where:}
\]

\[
L_q \equiv -q \frac{\partial}{\partial x}, \quad L_r \equiv r \left( \frac{\partial}{\partial x} - I \right), \quad L_\sigma \equiv \frac{\sigma^2}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right). \tag{55}
\]

It is easily verified that these 3 operators commute i.e.:

\[
L_q L_r = L_r L_q, \quad L_\sigma L_q = L_q L_\sigma, \quad L_\sigma L_r = L_r L_\sigma.
\]

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Consequently\(^{11}\), the exponential operator in (54) can be written as the composition of 3 new exponential operators:

\[
\exp\{\tau \cdot L\} = \exp\{\tau \cdot L_q\} \exp\{\tau \cdot L_r\} \exp\{\tau \cdot L_\sigma\}.
\]  (56)

Since \(L_r\), \(L_q\) and \(L_\sigma\) commute, the 3 exponential operators also commute.

Define 3 new linear operators by differentiating each of the \(L\) operators defined in (55) with respect to its associated parameter:

\[
L_q^{(1)} = -\frac{\partial}{\partial x} L_q^{(1)} = \left(\frac{\partial}{\partial x} - I\right),
\]

\[
L_r^{(1)} = \sigma \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right).
\]

Since each of these 3 new operators commutes with its corresponding operator in (55), each of these 3 new operators also commutes with its corresponding exponential operator in (56). Consequently, the derivatives of the exponential operator defined in (54) with respect to the 3 parameters satisfy:

\[
\frac{\partial}{\partial q}\exp\{\tau \cdot L\} = \tau L_q^{(1)} \exp\{\tau \cdot L\}
\]

\[
\frac{\partial}{\partial r}\exp\{\tau \cdot L\} = \tau L_r^{(1)} \exp\{\tau \cdot L\}
\]

\[
\frac{\partial}{\partial \sigma}\exp\{\tau \cdot L\} = \tau L_\sigma^{(1)} \exp\{\tau \cdot L\}.
\]

The foregoing implies that the following manipulations are justified:

\[
\frac{\partial V(\tau, S)}{\partial q} = \frac{\partial U(\tau, x)}{\partial q} = \frac{\partial}{\partial q}\exp\{\tau \cdot L\} \phi(x) = \tau L_q^{(1)} \exp\{\tau \cdot L\} \phi(x)
\]

\[
= -\tau \frac{\partial}{\partial x} U(\tau, x) = -\tau S \frac{\partial V(\tau, S)}{\partial S}, \text{ from (8)},
\]

\[
\frac{\partial V(\tau, S)}{\partial r} = \frac{\partial U(\tau, x)}{\partial r} = \frac{\partial}{\partial r}\exp\{\tau \cdot L\} \phi(x) = \tau L_r^{(1)} \exp\{\tau \cdot L\} \phi(x)
\]

\[
= \tau \left(\frac{\partial}{\partial x} - I\right) U(\tau, x) = -\tau \left[V(\tau, S) - S \frac{\partial V(\tau, S)}{\partial S}\right], \text{ from (7) and (8)},
\]

\[
\frac{\partial V(\tau, S)}{\partial \sigma} = \frac{\partial U(\tau, x)}{\partial \sigma} = \frac{\partial}{\partial \sigma}\exp\{\tau \cdot L\} \phi(x) = \tau L_\sigma^{(1)} \exp\{\tau \cdot L\} \phi(x)
\]

\[
= \tau \sigma \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) U(\tau, x) = \tau \sigma S \frac{\partial^2 V(\tau, S)}{\partial S^2}, \text{ from (9)}.\]

\(^{11}\)If two operators \(A\) and \(B\) commute, then \(\exp(A + B) = \exp(A)\exp(B)\). To verify this assertion, define an operator \(f(\lambda) \equiv \exp(\lambda A)\exp(\lambda B), \forall \lambda \in \mathbb{R}\). Then:

\[
\frac{df(\lambda)}{d\lambda} = A \exp(\lambda A) \exp(\lambda B) + \exp(\lambda A) B \exp(\lambda B)
\]

\[
= A \exp(\lambda A) \exp(\lambda B) + B \exp(\lambda A) \exp(\lambda B), \text{ since } A \text{ and } B \text{ commute}
\]

\[
= (A + B) f(\lambda).
\]

The solution to this ordinary differential equation in \(\lambda\) is \(f(\lambda) = \exp(\lambda (A + B))\). Setting \(\lambda = 1\) verifies the assertion.
Appendix 4: Stirling Numbers of the First and Second Kind

The Stirling numbers of the first kind satisfy the recursion:

\[ S_1(\ell, i) = \begin{cases} S_1(\ell - 1, i - 1) - (\ell - 1)S_1(\ell - 1, i) & \text{for } \ell = 1, 2, \ldots, i = 1, 2, \ldots, \ell \\ 0 & \text{otherwise, except that} \end{cases} \]

\[ S_1(0, 0) = 1. \] The first few Stirling numbers of the first kind are given in the table below:

<table>
<thead>
<tr>
<th>\ell/i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
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<td>-6</td>
<td>11</td>
<td>-6</td>
<td>1</td>
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</tbody>
</table>

From Comtet[6], the solution of the recursion is given by the following complicated closed form solution:

\[ S_1(\ell, i) = \sum_{j=0}^{\ell-i} \sum_{h=-j}^{\ell-i} (-1)^j h^j \binom{h}{j} \left( \frac{(\ell - 1 + h)^{\ell - i + h}}{(\ell - i + h)!} \right) \left( \frac{(h - j)^{\ell - i + h}}{h!} \right), \quad \ell = 1, 2, \ldots, i = 1, 2, \ldots, \ell. \]

The Stirling numbers of the second kind satisfy the recursion:

\[ S_2(\ell, i) = \begin{cases} S_2(\ell - 1, i - 1) + iS_2(\ell - 1, i) & \text{for } \ell = 1, 2, \ldots, i = 1, 2, \ldots, \ell \\ 0 & \text{otherwise, except that} \end{cases} \]

\[ S_2(0, 0) = 1. \] The first few Stirling numbers of the second kind are given in the table below:

<table>
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<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

From Abramowitz and Stegun[1], the solution of the recursion is given by the following simple closed form solution:

\[ S_2(\ell, i) = \frac{1}{\ell!} \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^\ell, \quad \ell = 1, 2, \ldots, i = 1, 2, \ldots. \]