Currency Covariance Contracting

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Abstract

We show how contracts paying the realized covariance between two currencies can be constructed by combining static positions in a continuum of options with continuous trading in underlying futures or forward contracts. The construction is general in that the volatilities and correlations are arbitrary.

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I Introduction

Volatility swaps have recently emerged on several over-the-counter markets (see [5] for example). These contracts pay the difference between the realized volatility over a specified time interval and a constant\(^1\) agreed upon at the outset of the contract. The motivation for contracts whose payoffs are tied to volatility has been discussed by several authors. For example, Gastineau[13] and Galai[11] propose the development of option indices which can be used as the underlying for derivative contracts. Brenner and Galai[2] propose the development of realized volatility indices and the development of futures and options contracts on these indices. Similarly, Fleming, Ostdiek and Whaley[10] describe the construction of an implied equity volatility index (the VIX), while Whaley[22] proposes derivative contracts written on this index. Grunbichler and Longstaff[15] develop a valuation model for options on volatility assuming a mean reverting volatility process.

In response to this hue and cry, some volatility contracts have been listed. For example, the OMLX, which is the London based subsidiary of the Swedish exchange OM, has launched volatility futures at the beginning of 1997. At this writing, the Deutsche Terminborse (DTB) recently launched its own futures based on its already established implied volatility index. Thus far, the volume in these contracts has been disappointing.

One possible explanation for this outcome is that variance can already be traded by combining static positions in options on price with dynamic trading in the underlying. Neuberger[18] showed that by delta-hedging a contract paying the log of the price, the hedging error accumulates to the difference between the realized variance and the fixed variance used in the delta-hedge. The contract paying the log of the price can be created with a static position in options as shown in Breeden and Litzenberger[1]. Independently of Neuberger, Dupire[8] showed that a calendar spread of two such log contracts has a payoff which depends on the realized variance between the two maturities. Dupire[9] and Derman, et. al.[7, 6] showed how to create “local” volatility contracts which pay the realized volatility at a particular time and asset price. Carr

\(^1\)This constant is often the implied volatility of an-the-money option.
and Madan[4] synthesize these developments in their review of volatility contracting. An important aspect of these developments is that the pricing and hedging of these variance contracts is completely insensitive to the choice of the volatility process. Just as the creation of a forward contract on a stock is independent of the stock price process, the creation of a contract paying the realized variance is independent of the stochastic volatility process.

Although variance contracts represent an important advance, one of the many issues bedeviling international investment in both financial and real assets is that both the variances and covariances of different currencies are generally unstable over time. This instability can hinder international expansion and can render mean variance analysis impractical for foreign investment. The objective of this paper is to demonstrate the creation of a covariance swap. This contract pays the excess of the realized covariance between two currencies over a constant specified at the outset of the contract. Such a contract would serve as a useful complement for the variance contracts which presently trade over-the-counter on several currencies. By combining variance and covariance swaps, the realized variance of return on a portfolio of currencies can be locked in. Like the variance swap, the creation of a covariance swap is independent of the stochastic volatility process, as well as the stochastic correlation process.

Since the creation of the fixed leg of a covariance swap is trivial, we focus our attention on the creation of the floating leg. The development of a contract paying the realized covariance is a necessary pre-cursor for the development of further derivatives written on covariance. Clearly, a calendar spread of two covariance contracts pays off the realized covariance between two future dates. In principle, the two future dates can be made arbitrarily close and an instantaneous forward price for covariance can be determined. As a consequence, one can develop a term structure of forward covariances and model its evolution through time. If we further assume constant interest rates, then as with any forward price, the risk-neutral drift of this term structure would be zero. By dynamic trading in foreign exchange (FX) options and their underlying currencies, contracts with payoffs which are nonlinear in covariance can be constructed.

Our approach for creating a covariance contract assumes that we have a triangular market consisting of three exchange rates and European options written on each of the three rates. Margrabe[16] studies
triangular arbitrage in the FX market and Walter and Lopez[21] imply out covariances using the Garman Kohlhagen[12] (GK) model. Of course, the GK model assumes that volatilities and correlations are constant, so the use of implied GK volatilities and correlations is theoretically unsound. In this paper, we show how to develop an alternative covariance forecast which uses option prices, but does not assume that volatilities and correlations are constant.

The outline of this paper is as follows. The next section reviews the theory of static replication using options. The following section reviews how static replication can be combined with the standard theory of dynamic replication using futures or forwards to create contracts written on realized variance. The fourth section extends these results to the creation of covariance contracts. The final section summarizes and suggests some avenues for future research.

II Review of Static Hedging Using Options

This section reviews the theory of static replication using options first developed in Ross[19] and Breeden and Litzenberger[1]. Consider a single period setting in which investments are made at time 0 with all payoffs received at time T. In contrast to the standard intertemporal model, we assume that there are no trading opportunities other than at times 0 and T. We consider the replication of payoffs at T which are a function of the futures FX rate or the forward FX rate for delivery at T' ≥ T. We let F denote both the futures FX rate and the forward FX rate, although we recognize that the two can differ when interest rates are stochastic as presently assumed. We also assume that markets exist for European-style options written on this futures or forward price. This assumption appears restrictive since listed futures options are generally American-style and options written on forward prices are unusual. However, we note that by setting T' = T, European options on the forward or futures exchange rate become European options on the spot exchange rate, which are available in both the listed and the over-the-counter markets. In order to obtain exact results, we assume that there exists an entire continuum of positive strike rates. While the assumption of a continuum of strikes is far from standard, it is essentially the analog of the standard
assumption of continuous trading. Just as the latter assumption is frequently made as a reasonable approximation to an environment where investors can trade frequently, our assumption is a reasonable approximation when there are a large but finite number of option strikes.

It is widely recognized that this market structure allows investors to create any smooth function \( f(F_T) \) of the terminal futures or forward FX rate \( F_T \) by taking a static position at time 0 in FX options\(^2\). When this theory is used to create variance and covariance contracts, this paper will show that it is only the second derivative of the payoff which is relevant. Consequently, we will always choose \( f \) so that its value and slope vanish at the initial futures or forward FX rate \( F_0 \) (i.e. \( f(F_0) = f'(F_0) = 0 \)). Appendix 1 shows that for this special case\(^3\), any twice differentiable payoff can be spanned by the following position in out-of-the-money options:

\[
f(F_T) = \int_0^{F_0} f''(K)(K - F_T)^+ dK + \int_{F_0}^\infty f''(K)(F_T - K)^+ dK. \tag{1}
\]

In words, to create a twice differentiable payoff \( f(\cdot) \) with value and slope vanishing at \( F_0 \), buy \( f''(K)dK \) puts at all strikes less than \( F_0 \) and buy \( f''(K)dK \) calls at all strikes greater than \( F_0 \).

In the absence of arbitrage, a decomposition similar to (1) must prevail among the initial values. Specifically, if we let \( V_0^f \), \( P_0(K) \), and \( C_0(K) \) denote the initial prices of the payoff \( f(\cdot) \), the put, and the call respectively, then the no arbitrage condition requires that:

\[
V_0^f = \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^\infty f''(K)C_0(K)dK. \tag{2}
\]

Thus, the value of an arbitrary payoff can be obtained from the option prices. Note that no assumptions were made regarding interest rates or the stochastic process governing the futures or forward FX rate.

\(^2\)This observation was first noted in Breeden and Litzenberger[1] and established formally in Green and Jarrow[14] and Nachman[17].

\(^3\)See Carr and Madan[3] for the corresponding result when the value and slope are unrestricted.
III Variance Contracting

In order to create contracts on the realized variance of returns from investing in a currency, we now assume that interest rates are constant and that the underlying forward or futures price process is continuous. We further assume that investors can trade continuously and that FX futures contracts mark-to-market continuously. The process followed by the futures or forward FX rate is described by the following stochastic differential equation:

$$dF_t = \alpha_t F_t dt + \sigma_t F_t dW_t, \quad t \in [0, T],$$

(3)

where $F_0 > 0$ and $\alpha_t$ and $\sigma_t$ are arbitrary processes denoting the relative drift and relative diffusion coefficients respectively. Thus, $\sigma_t^2$ is the variance rate of relative changes in the futures/forward FX rate and like $\alpha_t$, it is an arbitrary\(^4\) unknown stochastic process. While one could specify a stochastic process for this variance rate and develop the correct delta-hedge in such a model, such an approach is subject to significant model risk since one is unlikely to guess the correct process. Furthermore, such models generally require dynamic trading in options which is costly in practice. Consequently, in what follows we leave the process for the variance rate unspecified and restrict dynamic strategies to forward or futures contracts alone.

Let $f(F)$ be a twice differentiable function of the futures or forward FX rate $F$. By Itô’s lemma:

$$f(F_T) = f(F_0) + \int_0^T f'(F_t) dF_t + \frac{1}{2} \int_0^T f''(F_t) \sigma_t^2 F_t^2 dt.$$  

(4)

Assuming that $f(F_0) = f'(F_0) = 0$ and solving for the last term gives:

$$\frac{1}{2} \int_0^T f''(F_t) \sigma_t^2 F_t^2 dt = f(F_T) - \int_0^T f'(F_t) dF_t.$$  

(5)

The left hand side is a payoff at $T$ which is based on the paths of both the realized instantaneous volatility $\sigma_t^2$ and the FX rate $F_t$. In contrast, the right side of (5) depends only on the FX rate path. It

\(^4\)We still require the existence of a unique non-negative weak solution to (3). If $\alpha_t F_t = a(F_t, t)$ and $\sigma_t F_t = b(F_t, t)$, then sufficient conditions for the existence of a unique weak solution are given in Stroock and Varadhan\cite{20}. These are that the function $a(F, t)$ be measurable and bounded and that the function $b(F, t)$ be continuous, bounded, and non-negative. To rule out negative futures/forward FX rates, we impose absorption at the origin i.e. $a(0, t) = b(0, t) = 0, t \in [0, T]$. 

results from combining the payoff $f(F_T)$ obtained from a static position in options maturing at $T$ with the payoff from a dynamic strategy in either futures or forwards. If futures contracts are used, then the last term in (5) arises from holding $-e^{r(T-t)}f'(F_t)$ futures contracts over the time interval $[0, T)$ (assuming continuous marking-to-market and that the margin account balance earns interest at the riskfree rate). If zero-cost forward contracts are used, then the last term in (5) arises from continuously rolling over $-f'(F_t)$ $T$-maturity forward contracts over the time interval $[0, T)$. In either case, the dynamic strategy can be interpreted as an attempt to create the payoff $-f(F_T)$ at $T$, conducted under the false assumption of zero volatility. Since realized volatility will be positive, an error arises, and the magnitude of this error is given by $\int_0^T \frac{F_T}{2} f''(F_t) \sigma^2 dt$, which is the left side of (5).

Now consider the following payoff function $\phi(F)$ (see Figure 1):

$$\phi(F_T) = 2 \left[ \frac{F_T}{F_0} - 1 - \ln \left( \frac{F_T}{F_0} \right) \right].$$

This payoff is twice the difference between the discretely compounded return $\frac{F_T}{F_0} - 1$ and the continuously compounded return $\ln \left( \frac{F_T}{F_0} \right)$. The first derivative is given by:

$$\phi'(F) = 2 \left[ \frac{1}{F_0} - \frac{1}{F} \right].$$

Thus, the value and slope both vanish at $F = F_0$ as required. The second derivative of $\phi$ is simply:

$$\phi''(F) = \frac{2}{F^2}.$$
Substituting (6) to (8) into (5) results in a relationship between a contract paying the realized variance over the time interval \((0, T]\) and two payoffs based on the FX rate:

\[
\int_0^T \sigma_t^2 dt = 2 \left[ \frac{F_T}{F_0} - 1 - \ln \left( \frac{F_T}{F_0} \right) \right] + 2 \int_0^T \left[ \frac{1}{F_t} - \frac{1}{F_0} \right] dF_t.
\]  

(9)

The first term on the right side arises from a static position in options. Substituting (8) into (2) implies that the required position is given by:

\[
2 \left[ \frac{F_T}{F_0} - 1 - \ln \left( \frac{F_T}{F_0} \right) \right] = \int_0^{F_0} 2 \frac{K}{K^2} (K - F_T)^+ dK + \int_0^\infty \frac{2}{K^2} (F_T - K)^+ dK,
\]

(10)

Thus, to create the contract paying \(\int_0^T \sigma_t^2 dt\) at \(T\), at \(t = 0\), the investor should buy \(\frac{2}{K} dK\) units of each out-of-the-money option maturing at \(T\). The initial cost of this position is given by:

\[
\int_0^{F_0} \frac{2}{K^2} P_0(K, T) dK + \int_0^\infty \frac{2}{K^2} C_0(K, T) dK.
\]

(11)

The investor should also start a dynamic strategy in futures or forward contracts. If futures are used, the investor holds \(2e^{-r(T-t)} \left[ \frac{1}{F_t} - \frac{1}{F_0} \right]\) futures contracts for each \(t \in [0, T]\). If zero-cost forward contracts are used, the investor continuously rolls over \(2 \left[ \frac{1}{F_t} - \frac{1}{F_0} \right]\) T-maturity forward contracts for each \(t \in [0, T]\).

The net payoff at \(T\) is:

\[
2 \left[ \frac{F_T}{F_0} - 1 - \ln \left( \frac{F_T}{F_0} \right) \right] + 2 \int_0^T \left[ \frac{1}{F_t} - \frac{1}{F_0} \right] dF_t = \int_0^T \sigma_t^2 dt,
\]

as required. Since the initial cost of achieving this payoff is given by (11), an interesting forecast \(\hat{\sigma}_{0,T}^2\) of the variance over \((0, T]\) is given by the future value of this cost:

\[
\hat{\sigma}_{0,T}^2 = \int_0^{F_0} \frac{2e^{rT}}{K^2} P_0(K, T) dK + \int_0^\infty \frac{2e^{rT}}{K^2} C_0(K, T) dK.
\]

In contrast to implied volatility, this forecast does not use a model in which volatility is assumed to be constant. However, in common with any forward price, this forecast is a reflection of both statistical expected value and risk aversion. Consequently, by comparing this forecast with the ex-post outcome, the market price of volatility risk can be inferred. This forecast is also the fixed rate in a variance swap implied by the absence of arbitrage.
IV Covariance Contracting

The objective of this section is to create a contract paying the realized covariance of returns between two exchange rates, where both rates have the same base currency. We denote the two exchange rates used in the covariance calculation by $F_{a/b}^t$ and $F_{c/b}^t$. We assume there exist European-style FX options written on the terminal futures or forward exchange rate for all three possible pairings of the three currencies, $F_{T}^{a/c}$, $F_{T}^{a/b}$, and $F_{T}^{c/b}$. Absence of triangular arbitrage forces the following relationship at all times $t$ prior to maturity:

$$F_{t}^{a/c} = F_{t}^{a/b} / F_{t}^{c/b}, t \in [0, T].$$

Setting $t = T$ and taking logs implies:

$$\ln F_{T}^{a/c} = \ln F_{T}^{a/b} - \ln F_{T}^{c/b}.$$  \hspace{1cm} (12)

Similarly, setting $t = 0$ in (12) and taking logs implies:

$$\ln F_{0}^{a/c} = \ln F_{0}^{a/b} - \ln F_{0}^{c/b}.$$  \hspace{1cm} (13)

Subtracting equations gives a relationship among continuously compounded returns:

$$\ln \left( \frac{F_{T}^{a/c}}{F_{0}^{a/c}} \right) = \ln \left( \frac{F_{T}^{a/b}}{F_{0}^{a/b}} \right) - \ln \left( \frac{F_{T}^{c/b}}{F_{0}^{c/b}} \right).$$

Taking the variance of both sides gives:

$$\int_{0}^{T} \sigma_{t,a/c}^{2} dt = \int_{0}^{T} \sigma_{t,a/b}^{2} dt + \int_{0}^{T} \sigma_{t,c/b}^{2} dt - 2 \int_{0}^{T} \gamma_{t,a/b,c/b} dt,$$

where $\gamma_{t,a/b,c/b}$ denotes the realized covariance of returns between FX rates $F_{a/b}^t$ and $F_{c/b}^t$. Solving for the covariance term gives:

$$\int_{0}^{T} \gamma_{t,a/b,c/b} dt = \frac{1}{2} \int_{0}^{T} \sigma_{t,a/b}^{2} dt + \frac{1}{2} \int_{0}^{T} \sigma_{t,c/b}^{2} dt - \frac{1}{2} \int_{0}^{T} \sigma_{t,a/c}^{2} dt.$$

From the results of the previous section, each integral can be created via a static position in options and dynamic trading in the underlying:

$$\int_{0}^{T} \gamma_{t,a/b,c/b} dt, = \frac{F_{T}^{a/b}}{F_{0}^{a/b}} - 1 - \ln \left( \frac{F_{T}^{a/b}}{F_{0}^{a/b}} \right) + \int_{0}^{T} \left[ \frac{1}{F_{t}^{a/b}} - \frac{1}{F_{0}^{a/b}} \right] dF_{t}^{a/b}.$$
\[
\frac{F_T^{c/b}}{F_0^{c/b}} - 1 - \ln \left( \frac{F_T^{c/b}}{F_0^{c/b}} \right) + \int_0^T \left[ \frac{1}{F_T^{a/b}} - \frac{1}{F_0^{a/b}} \right] dF_T^{a/b} \\
- \frac{F_T^{a/c}}{F_0^{a/c}} + 1 + \ln \left( \frac{F_T^{a/c}}{F_0^{a/c}} \right) - \int_0^T \left[ \frac{1}{F_T^{a/c}} - \frac{1}{F_0^{a/c}} \right] dF_T^{a/c}.
\]

(13)

Thus the covariance contract is created via a static position in options on all three futures/forwards and by dynamic trading in all three underlying futures/forwards.

The initial cost of the static options position is given by:

\[
\int_0^{F_0^{a/b}} \frac{1}{K^2} P_0^{a/b}(K, T)dK + \int_{F_0^{a/b}}^{\infty} \frac{1}{K^2} C_0^{a/b}(K, T)dK \\
+ \int_0^{F_0^{c/b}} \frac{1}{K^2} P_0^{c/b}(K, T)dK + \int_{F_0^{c/b}}^{\infty} \frac{1}{K^2} C_0^{c/b}(K, T)dK \\
- \int_0^{F_0^{a/c}} \frac{1}{K^2} P_0^{a/c}(K, T)dK - \int_{F_0^{a/c}}^{\infty} \frac{1}{K^2} C_0^{a/c}(K, T)dK.
\]

(14)

The investor should also conduct a dynamic strategy in either futures or forward contracts. If futures contracts are used, the investor holds \(e^{-r(T-t)} \left[ \frac{1}{F_t^{c/b}} - \frac{1}{F_0^{c/b}} \right] \) futures contracts on futures FX rate \( F_t^{a/b} \), \( e^{-r(T-t)} \left[ \frac{1}{F_t^{c/b}} - \frac{1}{F_0^{c/b}} \right] \) futures contracts on futures FX rate \( F_t^{c/b} \), and \(-e^{-r(T-t)} \left[ \frac{1}{F_t^{a/c}} - \frac{1}{F_0^{a/c}} \right] \) futures contracts on futures FX rate \( F_t^{a/c} \) for each \( t \in [0, T) \). If zero-cost forward contracts are used, the number of forwards held is obtained from the corresponding number of futures by suppressing the discount factor \( e^{-r(T-t)} \). In either case, the net payoff at \( T \) is given by either side of (13). Since the initial cost of achieving this payoff is given by (14), an interesting forecast \( \hat{\gamma}_{0,T}^2 \) of the covariance between \( F_t^{a/b} \) and \( F_t^{c/b} \) over \( (0, T] \) is given by the future value of this cost:

\[
\hat{\gamma}_{0,T}^2 = e^{rT} \left\{ \int_0^{F_0^{a/b}} \frac{1}{K^2} P_0^{a/b}(K, T)dK + \int_{F_0^{a/b}}^{\infty} \frac{1}{K^2} C_0^{a/b}(K, T)dK \\
+ \int_0^{F_0^{c/b}} \frac{1}{K^2} P_0^{c/b}(K, T)dK + \int_{F_0^{c/b}}^{\infty} \frac{1}{K^2} C_0^{c/b}(K, T)dK \\
- \int_0^{F_0^{a/c}} \frac{1}{K^2} P_0^{a/c}(K, T)dK - \int_{F_0^{a/c}}^{\infty} \frac{1}{K^2} C_0^{a/c}(K, T)dK \right\}.
\]

(15)

In contrast to implied covariance, this forecast does not use a model in which covariance is assumed to be constant. However, in common with any forward price, this forecast is a reflection of both statistical expected value and risk aversion. Consequently, by comparing this forecast with the ex-post outcome, the
market price of covariance risk can be inferred. This forecast is also the fixed rate in a covariance swap implied by the absence of arbitrage.

**Summary and Suggestions for Future Research**

We showed that by combining static positions in options with dynamic trading in futures or forwards, investors can synthesize contracts paying the realized variance of a currency or paying the realized covariance between two currencies. Importantly, these contracts were created without specifying either a volatility process or a correlation process.

By dynamically trading in at-the-money short-term FX options, one should be able to synthesize the payoffs from options on realized variance or covariance. We conjecture that it would still not be necessary to specify a volatility process or a correlation process. It would also be interesting to consider contracts whose payoffs also depend on the exchange rate. One can also consider using other approaches for creating covariance contracts, such as quantos options or spread options. Finally, it would be interesting to develop contracts on other statistics of the sample path such as the standard deviation, the correlation, the Sharpe ratio, the realized skewness, etc. In the interests of brevity, such inquiries are best left for future research.

**References**


Appendix 1: Spanning with Options

The fundamental theorem of calculus implies that for any fixed $F_0$:

$$f(F) = f(F_0) + 1(F > F_0) \int_{F_0}^{F} f'(u)du - 1(F < F_0) \int_{F}^{F_0} f'(u)du$$

$$= 1(F > F_0) \int_{F_0}^{F} \left[ f'(F_0) + \int_{F_0}^{u} f''(v)dv \right] du$$

$$- 1(F < F_0) \int_{F}^{F_0} \left[ f'(F_0) - \int_{u}^{F_0} f''(v)dv \right] du,$$

since $f(F_0) = 0$. Noting that $f'(F_0)$ does not depend on $u$ and reversing the order of integration yields:

$$f(F) = f'(F_0)(F - F_0) + 1(F > F_0) \int_{F_0}^{F} \int_{u}^{F} f''(v)dvdu + 1(F < F_0) \int_{F}^{F_0} \int_{u}^{v} f''(v)dvdu.$$

Setting $f'(F_0) = 0$ and performing the integral over $u$ yields:

$$f(F) = 1(F > F_0) \int_{F_0}^{F} f''(v)(F - v)dv + 1(F < F_0) \int_{F}^{F_0} f''(v)(v - F)dv,$$

and this may be equivalently re-written as:

$$f(F) = \int_{F_0}^{\infty} f''(v)(F - v)^+dv + \int_{0}^{F_0} f''(v)(v - F)^+dv.$$