Funding beyond discounting: collateral agreements and derivatives pricing

Standard theory assumes traders can lend and borrow at a risk-free rate, ignoring the intricacies of the repo and collateralisation markets. Here, Vladimir Piterbarg shows that these force adjustments to discounting, forward prices and implied volatilities, depending on the particulars of collateral posting

Standard derivatives pricing theory (see, for example, Hull, 2006) relies on the assumption that one can borrow and lend at a unique risk-free rate. The realities of being a derivatives desk are, however, rather different these days, as historically stable relationships between bank funding rates, government rates, Libor rates, etc, have broken down.

The practicalities of funding, that is, how dealers borrow and lend money, are of central importance to derivatives pricing, because replicating naturally involves borrowing and lending money and other assets. In this article, we establish derivatives valuation formulas in the presence of such complications starting from first principles, and study the impact of market features such as stochastic funding and collateral posting rules on values of fundamental derivatives contracts, including forwards and options.

Simplifying considerably, we can describe a derivatives desk's activities as selling derivatives securities to clients while hedging them with other dealers. Should the desk default, a client would join the queue of the bank's creditors. The situation is a bit different for trading among dealers where, to reduce credit risk, agreements have been put in place to collateralise mutual exposures.

Such agreements are based on the so-called credit support annex (CSA) to the International Swaps and Derivatives Association master agreement, so we often refer to collateralised trades as CSA trades. As collateral is used to offset liabilities in case of a default, it could be thought of as an essentially risk-free investment, so the rate on collateral is usually set to be a proxy of a risk-free rate such as the fed funds rate for dollar transactions, Eonia for euro, etc. Often, purchased assets are posted as collateral against the funds used to buy them, such as in the 'repo' market for shares used in delta hedging.

Secured borrowing will normally attract a better rate than unsecured borrowing. In a bank, funding functions are often centralised within a treasury desk. The unsecured rates that the treasury desk provides to the trading desks are generally linked to the unsecured funding rate at which the bank itself can borrow/lend, a rate typically based on the bank credit rating, that is, its perceived probability of default.

The money that a derivatives desk uses in its operations comes from a multitude of sources, from the collateral posted by counterparties to funds secured by various types of assets. We show in this article how to aggregate these rates to come up with the value of a derivatives security given the rules for collateral posting and repo rates available for the underlying. Note that some desks may be required to borrow at rates different from those that they can lend at – a complication we avoid in this article as our formalism does not extend readily to the non-linear partial differential equations that such a set-up would require.

Having derived an appropriate extension to the standard no-arbitrage result, we then look carefully at the differences in value of CSA (that is, collateralised) and non-CSA (not collateralised) versions of the same derivatives security. This is important as dealers often calibrate their models to market-observed prices of derivatives, which typically reflect CSA-based valuations, yet they also trade a large volume of non-CSA over-the-counter derivatives. We demonstrate that a number of often significant adjustments are required to reflect the difference between CSA and non-CSA trades.

The first adjustment is to use different discounting rates for CSA and non-CSA versions of the same derivative. The second adjustment is a convexity, or quanto, adjustment and affects forward curves – such as equity forwards or Libor forward rates – as they turn out to depend on collateralisation used. This is a consequence of the stochastic funding spread and, in particular, of the correlation between the bank funding spread and the underlying assets. The third adjustment that may be required is to volatility information used for options – in particular, the volatility smile changes depending on collateral. We show some numerical results for these effects.

Preliminaries

We start with the risk-free curve for lending, a curve that corresponds to the safest available collateral (cash). We denote the corresponding short rate at time t by $r_C(t)$; 'C' here stands for 'CSA', as we assume this is the agreed overnight rate paid on collateral among dealers under CSA. It is convenient to parameterise term curves in terms of discount factors; we denote corresponding risk-free discount factors by $P_C(t, T)$, $0 \le t \le T < \infty$. Standard Heath-Jarrow-Morton theory applies, and we specify the following dynamics for the yield curve:

$$dP_C(t,T)/P_C(t,T) = r_C(t)dt - \sigma_C(t,T)^{\mathsf{T}} dW_C(t) \qquad (1)$$

where $W_C(t)$ is a d-dimensional Brownian motion under the risk-neutral measure P and σ_C is a vector-valued (dimension d) stochastic process.

In what follows, we shall consider derivatives contracts on a particular asset, whose price process we denote by $S(t)$, $t \geq 0$. We denote by $r_R(t)$ the short rate on funding secured by this asset (here 'R' stands for 'repo'). The difference $r_C(t) - r_R(t)$ is sometimes called the stock lending fee. Finally, let us define the short rate for unsecured funding by $r_F(t)$, $t \geq 0$. As a rule, we would expect that $r_C(t) \leq r_R(t) \leq r_F(t)$.

The existence of non-zero spreads between short rates based on different collateral can be recast in the language of credit risk, by introducing joint defaults between the bank and various assets used as collateral for funding. In particular, the funding spread $s_F(t) \quad r_F(t) - r_C(t)$ could be thought of as the (stochastic) intensity of default of the bank. We do not pursue this formalism here (see, for example, Gregory, 2009, or Burgard & Kjaer, 2009), postulating the dynamics of funding curves directly instead. Likewise, we ignore the possibility of a counterparty default, an extension that could be developed easily.

Black-Scholes with collateral

Let us look at how the standard Black-Scholes pricing formula changes in the presence of a CSA. Let $S(t)$ be an asset that follows, in the real world, the following dynamics:

$$dS(t)/S(t) = \mu_S(t)dt + \sigma_S(t)dW(t)$$

Let $V(t, S)$ be a derivatives security on the asset; by Itô's lemma it follows that:

$$dV(t) = (\mathcal{L}V(t))dt + \Delta(t)dS(t)$$

where $\mathcal{L}$ is the standard pricing operator:

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\sigma_S(t)^2 S^2}{2}\frac{\partial^2}{\partial S^2}$$

and Δ is the option's delta:

$$\Delta(t) = \frac{\partial V(t)}{\partial S}$$

Let $C(t)$ be the collateral (cash in the collateral account) held at time t against the derivative. For flexibility, we allow this amount to be different[1] from $V(t)$.

To replicate the derivative, at time t we hold $\Delta(t)$ units of stock and $\gamma(t)$ cash. Then the value of the replication portfolio, which we denote by $\Pi(t)$, is equal to:

$$V(t) = \Pi(t) = \Delta(t)S(t) + \gamma(t) \qquad (2)$$

The cash amount $\gamma(t)$ is split among a number of accounts:

- Amount $C(t)$ is in collateral.
- Amount $V(t) - C(t)$ needs to be borrowed/lent unsecured from the treasury desk.
- Amount $\Delta(t)S(t)$ is borrowed to finance the purchase of $\Delta(t)$ stocks. It is secured by stock purchased.
- Stock is paying dividends at rate r_D.

[1] *In what follows we use (3), (5) with either $C = 0$ or $C = V$. However, these formulas, in their full generality, could be used to obtain, for example, the value of a derivative covered by one-way (asymmetric) CSA agreement, or a more general case where the collateral amount tracks the value only approximately*

The growth of all cash accounts (collateral, unsecured, stock-secured, dividends) is given by:

$$d\gamma(t) = \left[r_C(t)C(t) + r_F(t)(V(t) - C(t)) - r_R(t)\Delta(t)S(t) + r_D(t)\Delta(t)S(t)\right]dt$$

On the other hand, from (2), by the self-financing condition:

$$d\gamma(t) = dV(t) - \Delta(t)dS(t)$$

which is, by Itô's lemma:

$$dV(t) - \Delta(t)dS(t) = (\mathcal{L}V(t))dt = \left(\frac{\partial}{\partial t} + \frac{\sigma_S(t)^2}{2}S^2\frac{\partial^2}{\partial S^2}\right)V(t)dt$$

Thus we have:

$$\left(\frac{\partial}{\partial t} + \frac{\sigma_S(t)^2}{2}S^2\frac{\partial^2}{\partial S^2}\right)V = r_C(t)C(t) + r_F(t)(V(t) - C(t)) + (r_D(t) - r_R(t))\frac{\partial V}{\partial S}S$$

which, after some rearrangement, yields:

$$\frac{\partial V}{\partial t} + (r_R(t) - r_D(t))\frac{\partial V}{\partial S}S + \frac{\sigma_S(t)^2}{2}S^2\frac{\partial^2 V}{\partial S^2} = r_F(t)V(t) - (r_F(t) - r_C(t))C(t)$$

The solution, obtained by essentially following the steps that lead to the Feynman-Kac formula (see, for example, Karatzas & Shreve, 1997, theorem 4.4.2), is given by:

$$V(t) = E_t\left(e^{-\int_t^T r_F(u)du}V(T) + \int_t^T e^{-\int_t^u r_F(v)dv}(r_F(u) - r_C(u))C(u)du\right) \qquad (3)$$

in the measure in which the stock grows at rate $r_R(t) - r_D(t)$, that is:

$$dS(t)/S(t) = (r_R(t) - r_D(t))dt + \sigma_S(t)dW_S(t) \qquad (4)$$

Note that if our probability space is rich enough, we can take it to be the same risk-neutral measure P as used in (1). We note that this derivation validates the view of Barden (2009) (who also cites Hull, 2006) that the repo rate $r_R(t)$ is the right 'risk-free' rate to use when valuing assets on $S(t)$.

By rearranging terms in (3), we obtain another useful formula for the value of the derivative:

$$V(t) = E_t\left(e^{-\int_t^T r_C(u)du}V(T)\right) - E_t\left(\int_t^T e^{-\int_t^u r_C(v)dv}(r_F(u) - r_C(u))(V(u) - C(u))du\right) \qquad (5)$$

We note that:

$$E_t(dV(t)) = (r_F(t)V(t) - (r_F(t) - r_C(t))C(t))dt = (r_F(t)V(t) - s_F(t)C(t))dt \qquad (6)$$

So, the rate of growth in the derivatives security is the funding spread $r_F(t)$ applied to its value minus the credit spread $s_F(t)$ applied to the collateral. In particular, if the collateral is equal to the value V then:

$$E_t\left(dV(t)\right) = r_C(t)V(t)dt, \qquad V(t) = E_t\left(e^{-\int_t^T r_C(u)du}V(T)\right) \quad (7)$$

and the derivative grows at the risk-free rate. The final value is the only payment that appears in the discounted expression as the other payments net out given the assumption of full collateralisation. This is consistent with the drift in (1) as $P_C(t, T)$ corresponds to deposits secured by cash collateral. On the other hand, if the collateral is zero, then:

$$E_t\left(dV(t)\right) = r_F(t)V(t)dt \quad (8)$$

and the rate of growth is equal to the bank's unsecured funding rate or, using credit risk language, adjusted for the possibility of the bank default. We show later that the case $C = V$ could be handled by using a measure that corresponds to the risk-free bond $P_C(t, T) = E_t(e^{-\int_t^T r_C(u)du})$ as a numéraire and, likewise, the case $C = 0$ could be handled by using a measure that corresponds to the risky bond $P_F(t, T) = E_t(e^{-\int_t^T r_F(u)du})$ as a numéraire.

Before we proceed with valuing derivatives securities in our set-up, let us comment on the portfolio effects of the collateral. When two dealers are trading with each other, the collateral is applied to the overall value of the portfolio of derivatives between them, with positive exposures on some trades offsetting negative exposures on other trades (so-called netting). Hence, potentially, valuation of individual trades should take into account the collateral position on the whole portfolio. Fortunately, in the simple case of the collateral requirement being a linear function of the exact value of the portfolio (the case that includes both the no-collateral case $C = 0$ and the full collateral case $C = V$), the value of the portfolio is just the sum of values of individual trades (with collateral attributed to trades by the same linear function). This easily follows from the linearity of the pricing formula (3) in V and C.

Zero-strike call option

Probably the simplest derivatives contract on an asset is a promise to deliver this asset at a given future time T. The contract could be seen as a zero-strike call option with expiry T. In the standard theory, of course, the value of this derivative is equal to the value of the asset itself (in the absence of dividends). Let us see what the situation is in our case. The payout of the derivative is given by $V(T) = S(T)$ and the value, at time t, assuming no CSA, is given by:

$$V_{zsc}(t) = E_t\left(e^{-\int_t^T r_F(u)du}S(T)\right)$$

On the other hand, if $r_D(t) = 0$, then:

$$S(t) = E_t\left(e^{-\int_t^T r_R(u)du}S(T)\right)$$

as follows from (4) and, clearly, $S(t) \neq V_{zsc}(t)$. The difference in values between the derivative and the asset are now easily understood, as the zero-strike call option carries the credit risk of the bank, while the asset $S(\cdot)$ does not. Or, in our language of funding, the asset $S(\cdot)$ can be used to secure funding – which is reflected in the discount rate applied – while V_{zsc} cannot be used for such a purpose.

Forward contract

We now consider a forward contract on $S(\cdot)$, where at time t the bank agrees to deliver the asset at time T, against a cash payment at time T.

■ **Without CSA.** A no-CSA forward contract could be seen as a derivative with the payout $S(T) - F_{noCSA}(t, T)$ at time T, where $F_{noCSA}(t, T)$ is the forward price at t for delivery at T. As the forward contract is cost-free, we have by (3) that:

$$0 = E_t\left(e^{-\int_t^T r_F(u)du}\left(S(T) - F_{noCSA}(t,T)\right)\right)$$

so we get:

$$F_{noCSA}(t,T) = \frac{E_t\left(e^{-\int_t^T r_F(u)du}S(T)\right)}{E_t\left(e^{-\int_t^T r_F(u)du}\right)} \quad (9)$$

Going back to (9), let us define:

$$P_F(t,T) \triangleq E_t\left(e^{-\int_t^T r_F(u)du}\right)$$

Note that this is essentially a credit-risky bond issued by the bank. Then we can rewrite (9) as:

$$F_{noCSA}(t,T) = \tilde{E}_t^T\left(S(T)\right)$$

where the measure $\tilde{P}_T$ is defined by the numeraire $P_F(t, T)$ as:

$$e^{-\int_0^t r_F(u)du}P_F(t,T) = E_t\left(e^{-\int_0^T r_F(u)du}\right)$$

is a P-martingale. Finally we see that $F_{noCSA}(t, T)$ is a $\tilde{P}_T$-martingale.

We note that the value of an asset under no CSA at time t with payout $V(T)$ is given, by (8), to be:

$$V(t) = E_t\left(e^{-\int_t^T r_F(u)du}V(T)\right) = P_F(t,T)\tilde{E}_t^T\left(V(T)\right)$$

so it could be calculated by simply taking the expected value of the payout in the risky T-forward measure.

■ **With CSA.** Now let us consider a forward contract covered by CSA, where we assume that the collateral posted C is always equal to the value of the contract V. Let the CSA forward price $F_{CSA}(t, T)$ be fixed at t, then the value, from (5), is given by:

$$0 = V(t) = E_t\left(e^{-\int_t^T r_C(u)du}V(T)\right)$$
$$= E_t\left(e^{-\int_t^T r_C(u)du}\left(S(T) - F_{CSA}(t,T)\right)\right)$$

so that:

$$F_{CSA}(t,T) = \frac{E_t\left(e^{-\int_t^T r_C(u)du}S(T)\right)}{E_t\left(e^{-\int_t^T r_C(u)du}\right)} \quad (10)$$

Comparing this with (9), we see that in general:

$$F_{CSA}(t,T) \neq F_{noCSA}(t,T)$$

By the arguments similar to the no-CSA case, we obtain:

$$F_{CSA}(t,T) = E_t^T\left(S(T)\right)$$

where the measure P^T is the standard T-forward measure, that is, a measure defined by $P_C(t, T) = E_t(e^{-\int_t^T r_C(u)du})$ as a numeraire.

We note that the value of an asset under CSA at time t with payout $V(T)$ is given, by (7), to be:

$$V(t) = E_t\left(e^{-\int_t^T r_C(u)du}V(T)\right) = P_C(t,T)E_t^T\left(V(T)\right)$$

so it could be calculated by simply taking the expected value of the payout in the (risk-free) T-forward measure.

■ **Calculating CSA convexity adjustment.** Let us now calculate the difference between CSA and non-CSA forward prices. We have:

$$\begin{aligned} F_{noCSA}(t,T) = \tilde{E}_t^T\left(S(T)\right) &= \frac{E_t\left(e^{-\int_t^T r_F(u)du}S(T)\right)}{P_F(t,T)} \\ &= \frac{E_t\left(e^{-\int_t^T r_C(u)du}e^{-\int_t^T (r_F(u)-r_C(u))du}S(T)\right)}{P_F(t,T)} \\ &= \frac{P_C(t,T)}{P_F(t,T)}E_t^T\left(e^{-\int_t^T s_F(u)du}S(T)\right) \\ &= E_t^T\left(\frac{M(T,T)}{M(t,T)}S(T)\right) \end{aligned} \tag{11}$$

where:

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)}e^{-\int_0^t s_F(u)du} \tag{12}$$

is a P^T-martingale, as:

$$M(t,T) = E_t^T\left(e^{-\int_0^T s_F(u)du}\right)$$

We note that, trivially:

$$E_t^T\left(\frac{M(T,T)}{M(t,T)}\right) = 1$$

so:

$$\begin{aligned} &F_{noCSA}(t,T) - F_{CSA}(t,T) \\ &= E_t^T\left(\left(\frac{M(T,T)}{M(t,T)} - E_t^T\left(\frac{M(T,T)}{M(t,T)}\right)\right)\left(S(T) - F_{CSA}(t,T)\right)\right) \\ &= \frac{1}{M(t,T)}Cov_t^T\left(M(T,T), F_{CSA}(T,T)\right) \end{aligned} \tag{13}$$

To obtain the actual value of the adjustment we would need to postulate joint dynamics of $s_F(u)$ and $S(u)$, $u \geq t$. We present a simple model below where we carry out the calculations.

■ **Relationship with futures contracts.** At first sight, a forward contract with CSA looks rather like a futures contract on the asset. Recall that with futures contracts, the (daily) difference in the futures price gets credited/debited to the margin account. In the same way, as forward prices move, a CSA forward contract also specifies that money exchanges hands. But there is an important difference. Consider the value of a forward contract at $t' > t$, a contract that was entered at time t (so $V(t) = 0$). Then:

$$\begin{aligned} V(t') &= E_{t'}\left(e^{-\int_{t'}^T r_C(u)du}\left(S(T) - F_{CSA}(t,T)\right)\right) \\ &= E_{t'}\left(e^{-\int_{t'}^T r_C(u)du}S(T)\right) - E_{t'}\left(e^{-\int_{t'}^T r_C(u)du}\right)F_{CSA}(t,T) \end{aligned}$$

By (10):

$$V(t') - V(t) = E_{t'}\left(e^{-\int_{t'}^T r_C(u)du}\right)\left(F_{CSA}(t',T) - F_{CSA}(t,T)\right)$$

so the difference in contract values on t' and t that exchanges hands at t' is equal to the discounted (to T) difference in forward prices. For a futures contract, the difference will not be discounted. Therefore, the type of convexity effects we see in futures contracts are different from what we see in CSA versus no-CSA forward contracts, a conclusion different from that reached in Johannes & Sundaresan (2007).

European-style options

Consider now a European-style call option on $S(T)$ with strike K. Depending on the presence or absence of CSA, we get two prices:

$$V_{noCSA}(t) = E_t\left(e^{-\int_t^T r_F(u)du}\left(S(T) - K\right)^+\right)$$

$$V_{CSA}(t) = E_t\left(e^{-\int_t^T r_C(u)du}\left(S(T) - K\right)^+\right)$$

(where for the CSA case we assumed that the collateral posted, C, is always equal to the option value, V_{CSA}). By the same measure-change arguments as in the previous section:

$$V_{noCSA}(t) = P_F(t,T)\tilde{E}_t^T\left(\left(S(T) - K\right)^+\right)$$

$$V_{CSA}(t) = P_C(t,T)E_t^T\left(\left(S(T) - K\right)^+\right)$$

The difference between measures $\tilde{P}_t^T$ and P_t^T not only manifests itself in the mean of $S(T)$ – as already established in the previous section – but also shows up in other characteristics of the distribution of $S(\cdot)$, such as its variance and higher moments. We explore these effects in the next section.

■ **Distribution impact of convexity adjustment.** Let us see how a change of measure affects the distribution of $S(\cdot)$. In the spirit of (11), we have:

$$V_{noCSA}(t) = P_F(t,T)E_t^T\left(\frac{M(T,T)}{M(t,T)}\left(S(T) - K\right)^+\right)$$

where $M(t, T)$ is defined in (12). Then, by conditioning on $S(T)$, we obtain:

$$V_{noCSA}(t) = P_F(t,T)E_t^T\left(\alpha\left(t,T,S(T)\right)\left(S(T) - K\right)^+\right) \tag{14}$$

where the deterministic function $\alpha(t, T, x)$ is given by:

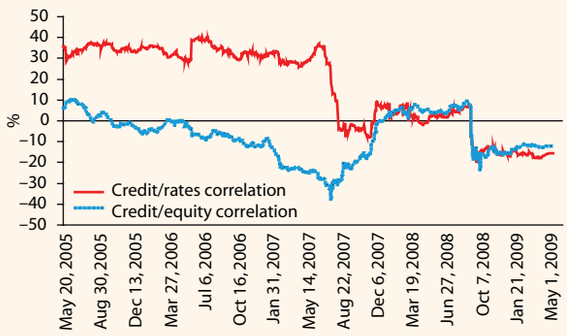
$$\alpha(t,T,x) = E_t^T\left(\frac{M(T,T)}{M(t,T)}\middle| S(T) = x\right)$$

Inspired by Antonov & Arneguy (2009), we approximate the function $\alpha(t, T, x)$ by a linear (in x) function:

$$\alpha(t,T,x) \approx \alpha_0(t,T) + \alpha_1(t,T)x$$

and obtain α_0 and α_1 by minimising the squared difference (while

1 Historical credit spread/interest rates and credit spread/equity correlation calculated with a rolling one-year window



using the fact that $E_t^T(M(T,T)/M(t,T)) = 1$ and $E_t^T(S(T)) = F_{CSA}(t,T)$):

$$\alpha_1(t,T) = \frac{E_t^T\left(\frac{M(T,T)}{M(t,T)}S(T)\right) - F_{CSA}(t,T)}{Var_t^T(S(T))}$$

$$\alpha_0(t,T) = 1 - \alpha_1 F_{CSA}(t,T)$$

We recognise the term:

$$E_t^T\left(\frac{M(T,T)}{M(t,T)}S(T)\right) - F_{CSA}(t,T)$$

as the convexity adjustment of the forward between the no-CSA and CSA versions (see (13)), and rewrite:

$$\alpha_1(t,T) = \frac{F_{noCSA}(t,T) - F_{CSA}(t,T)}{Var_t^T(S(T))}$$

Differentiating (14) with respect to K twice, we obtain the following relationship between the probability density functions (PDFs) of $S(T)$ under the two measures:

$$\tilde{P}_t^T(S(T) \in dK) = (\alpha_0(t,T) + \alpha_1(t,T)K)P_t^T(S(T) \in dK) \quad (15)$$

so the PDF of $S(T)$ under the no-CSA measure is obtained from the density of $S(T)$ under the CSA measure by multiplying it with a linear function. It is not hard to see that the main impact of such a transformation is on the slope of the volatility smile of $S(\cdot)$. We demonstrate this impact numerically below.

Example: stochastic funding model

Let us consider a simple model that we can use to estimate the impact of collateral rules on forwards and options. We start with an asset that follows a lognormal process:

$$dS(t)/S(t) = O(dt) + \sigma_S dW_S(t)$$

and funding spread that follows dynamics inspired by a simple one-factor Gaussian model of interest rates[2]:

$$ds_F(t) = -\aleph_F(\theta - s_F(t))dt + \sigma_F dW_F(t)$$

with $\langle dW_S(t), dW_F(t)\rangle = \rho dt$. Here ρ is the correlation between the asset and the funding spread. We also assume for simplicity that $r_C(t)$, $r_R(t)$ are deterministic, while $r_D(t) = 0$. Then:

$$F_{CSA}(t,T) = E_t(S(T))$$

and:

$$dF_{CSA}(t,T)/F_{CSA}(t,T) = \sigma_S dW_S(t)$$

with $W_S(t)$ being a Brownian motion in the risk-neutral measure P. On the other hand:

$$dP_F(t,T)/P_F(t,T) = O(dt) - \sigma_F b(T-t)dW_F(t)$$

where:

$$b(T-t) = \frac{1 - e^{-\aleph_F(T-t)}}{\aleph_F}$$

As $M(t, T)$ is a martingale under P (since $r_C(t)$ is deterministic, the measures P and P_T coincide), we have from (12) that:

$$dM(t,T)/M(t,T) = -\sigma_F b(T-t)dW_F(t)$$

Also both $M(t, T)$ and $F_{CSA}(t, T)$ are martingales under P. We then have:

$$d(M(t,T)F_{CSA}(t,T))/(M(t,T)F_{CSA}(t,T))$$
$$= \sigma_S\sigma_F b(T-t)\rho dt + O(dW(t))$$

Recall that:

$$F_{noCSA}(0,T) - F_{CSA}(0,T)$$
$$= E\left(\frac{M(T,T)}{M(0,T)}(F_{CSA}(T,T) - F_{CSA}(0,T))\right)$$

so that:

$$F_{noCSA}(0,T) = F_{CSA}(0,T)\exp\left(-\int_0^T \sigma_S\sigma_F b(T-t)\rho dt\right)$$
$$= F_{CSA}(0,T)\exp\left(-\sigma_S\sigma_F\rho\frac{T - b(T)}{\aleph_F}\right) \quad (16)$$

and, in the case $\aleph_F = 0$:

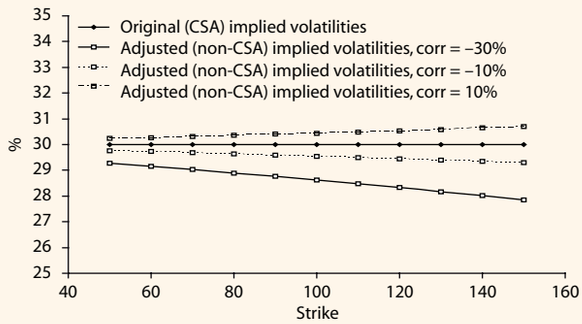
$$F_{noCSA}(0,T) - F_{CSA}(0,T)$$
$$= F_{CSA}(0,T)\left(\exp\left(-\sigma_S\sigma_F\rho T^2/2\right) - 1\right)$$

A. Relative differences between non-CSA and CSA forward prices with $\sigma_S = 30\%$, $\sigma_F = 1.50\%$, $\aleph_F = 5.00\%$

Time/r	–30%	–20%	–10%	0%	10%
1	0.07%	0.04%	0.02%	0.00%	–0.02%
2	0.26%	0.17%	0.09%	0.00%	–0.09%
3	0.58%	0.39%	0.19%	0.00%	–0.19%
4	1.02%	0.68%	0.34%	0.00%	–0.34%
5	1.57%	1.04%	0.52%	0.00%	–0.52%
6	2.23%	1.48%	0.74%	0.00%	–0.73%
7	3.00%	1.99%	0.99%	0.00%	–0.98%
8	3.87%	2.56%	1.27%	0.00%	–1.26%
9	4.85%	3.20%	1.59%	0.00%	–1.56%
10	5.92%	3.91%	1.94%	0.00%	–1.90%

[2] *While a diffusion process for the funding spread may be unrealistic, the impact of more complicated dynamics on the convexity adjustment is likely to be muted*

2 Difference in CSA v. non-CSA implied distribution for European options using (15), expressed in implied vol across strikes, for different levels of correlation ρ



Note: T = 10 years, $F_{CSA}(0, T)$ = 100, σ_F = 1.50%, $\aleph_F$ = 5.00%

We note that the adjustment grows as (roughly) T^2. A similar formula was obtained by Barden (2009) using a model in which funding spread is functionally linked to the value of the asset.

Let us perform a couple of numerical experiments. We start with an equity-related example. Let us set $\sigma_F = 30\%$, a number roughly in line with implied volatilities of options on the S&P 500 equity index (SPX). We estimate the basis-point volatility of the funding tspread to be σ_F = 1.50% and mean reversion to be $\aleph_F = 5\%$ by looking at historical data of credit spreads on US banks. Figure 1 shows a rolling historical estimate of correlations between credit spreads and the SPX (as well as credit spread and interest rates in the form of a five-year swap rate). From this graph, we estimate a reasonable range for the correlation ρ to be [–30%, 10%]. In table A, we report relative adjustments:

$$\frac{F_{noCSA}(0,T) - F_{CSA}(0,T)}{F_{CSA}(0,T)}$$

for different values of correlations and for different T from one to 10 years. Clearly, the adjustments could be quite significant.

Next we look at the difference in implied volatilities for CSA and non-CSA options. We look at options expiring in 10 years across different strikes, with $F_{CSA}(0, T) = 100$. We assume that the market prices of CSA options are given by the 30% implied volatility (for all strikes), so that the 'CSA distribution' of the asset is lognormal with 30% volatility. Then we express the distribution of the underlying asset for non-CSA options as given by (15) in terms of implied volatilities (using put options and the original value of the forward, 100, to ensure fair comparison). Figure 2 demonstrates the impact – non-CSA options have lower volatility (lower put option values), and the volatility smile has a higher (negative) skew.

Finally, let us look at CSA convexity adjustments to forward Libor rates. Table B presents absolute differences (that is, $F_{noCSA}(0, T) - F_{CSA}(0, T)$) in non-CSA versus CSA forward Libor rates fixing in one to 30 years over a reasonable range of possible correlations. We use the same parameters for the funding spread as above together with recent market-implied caplet volatilities and forward Libor rates. Again, the differences are not negligible, especially for longer-expiry Libor rates.

B. Absolute differences between non-CSA and CSA forward Libor rates, using market-implied caplet volatilities and σF = 1.50%, ℵF = 5.00%

Time/r	–20%	0%	20%	40%
1	0.00%	0.00%	0.00%	0.00%
2	0.01%	0.00%	–0.01%	–0.01%
3	0.01%	0.00%	–0.01%	–0.02%
4	0.02%	0.00%	–0.02%	–0.04%
5	0.03%	0.00%	–0.03%	–0.05%
7	0.05%	0.00%	–0.05%	–0.10%
10	0.09%	0.00%	–0.09%	–0.18%
15	0.18%	0.00%	–0.18%	–0.37%
20	0.30%	0.00%	–0.30%	–0.60%
25	0.42%	0.00%	–0.42%	–0.84%
30	0.54%	0.00%	–0.54%	–1.07%

Conclusions

In this article, we have developed valuation formulas for derivative contracts that incorporate the modern realities of funding and collateral agreements that deviate significantly from the textbook assumptions. We have shown that the pricing of non-collateralised derivatives needs to be adjusted, as compared with the collateralised version, with the adjustment essentially driven by the correlation between market factors for a derivative and the funding spread. Apart from rather obvious differences in discounting rates used for CSA and non-CSA versions of the same derivative, we have exposed the required changes to forward curves and, even, the volatility information used for options. In a simple model with stochastic funding spreads we demonstrated the typical sizes of these adjustments and found them significant. ■

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References

Antonov A and M Arneguy, 2009
Analytical formulas for pricing CMS products in the Libor market model with the stochastic volatility
SSRN eLibrary

Barden P, 2009
Equity forward prices in the presence of funding spreads
ICBI Conference, Rome, April

Burgard C and M Kjaer, 2009
Modelling and successful management of credit-counterparty risk of derivative portfolios
ICBI Conference, Rome, April

Gregory J, 2009
Being two-faced over counterparty credit risk
Risk February, pages 86–90

Hull J, 2006
Options, futures and other derivatives
Prentice Hall

Johannes M and S Sundaresan, 2007
Pricing collateralized swaps
Journal of Finance 62, pages 383–410

Karatzas I and S Shreve, 1997
Brownian motion and stochastic calculus
Springer