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Multi-curves: Variations on a Theme

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Abstract

The multi-curves framework is often implemented in a way to recycle to one curve formulas; there is no fundamental reasons behind that choice. Here we present different approaches to the multi-curves framework. They vary by the choice of building blocks instruments (Ibor coupon or futures) and the definition of curve (pseudo-discount factors or direct forward rate). The features of the different approaches are described.

Contents

1 Introduction

Up to 2008, the standard approach to pricing Ibor[1]-related derivatives was to use a unique interest rate curve, supposed to be risk-free, to both discount the cash-flows and estimated the Ibor-related payments.

With the recent crisis it became more apparent that the hypothesis was not realistic and that a different approach was necessary. That necessity was already indicated well before the crisis in some literature and by some practitioners. Earlier developments like Tuckman and Porfirio (2003) and Boenkost and Schmidt (2004) had pointed to the weakness of the then framework but without providing a theoretically sound alternative. A first step, splitting (risk-free) discounting and Ibor fixing, was proposed in a simplified set-up in Henrard (2007). The framework was later extended to a more flexible set-up and is now the base for most of the multi-curves developments. It is in particular described in Henrard (2010). Further developments have been done in different directions, in particular in Kijima et al. (2009), Ametrano and Bianchetti (2009), Chibane and Sheldon (2009), Mercurio (2009), Morini (2009), Bianchetti (2010), Piterbarg (2010), Moreni and Pallavicini (2010), Pallavicini and Tarenghi (2010), and Mercurio (2010a).

One of the starting points of the multi-curves framework is the existence of a set of assets which are not directly related to the discount bonds. The usual choice is the Ibor coupons and the coupons are priced using pseudo-discount factors linked to forward curves created to reproduce market instrument prices.

This is why we refer here to that framework as the *coupon discount factor multi-curves framework*. From the existence hypothesis, one *defines* pseudo-discount factors in such a way that the usual project-and-discount formulas previously used in swap pricing are still valid. This way to proceed is purely conventional. It is also quite practical as all standard instruments can be priced with formulas similar to the one we are used to. Nevertheless the approach is purely based on that definition and selected to used the good old formulas; there is no deeper fundamental reason behind it.

There are potentially multiple other coherent approaches to multi-curves discounting / estimation frameworks. In this note we review several alternative approaches. They are based on different ways to implement the hypothesis on existence of Ibor related products. In the presentation we restrict ourselves to a single currency description; the extension to the multi-currency case can be done like in the coupon discount factor case.

The second approach is to model the forward rates directly, without the pseudo-discount factor intermediary in the forward curve. We refer to that framework as the *coupon forward rate multi-curves framework*. The advantages of such approach are that it uses formula very similar to the current one and at the same time models directly the forward rates on which one may have more intuition. The *saw-tooth* effect that appears with linear interpolation of rates in the coupon discount factor framework disappears (at least is lessen) even if the same linear interpolation scheme is used[2].

The third approach proposed is based on the price of interest rate futures and not of coupons and was first described in Henrard (2012). It is called here the *futures discount factor multi-curves framework*. The fundamental hypothesis on the existence of Ibor coupons is replaced by an

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[0]First version: 18 August 2012; this version: 19 October 2012.

[1]By Ibor we mean any reference rate which is fixed in a way similar to Libor and in particular Euribor, Tibor, Cibor, BBSW, etc. The description of the different indexes and their conventions can be found in Quantitative Research (2012).

[2]To our knowledge, it is the first time this approach is formally described. Some people mentioned to the author closely related approaches as potentially attractive, but at the time they had not been implemented.

hypothesis on the existence of STIR futures linked to Ibor indexes.

In the one curve approach, the pricing of interest rate futures, one of the most liquid interest rate products, has attracted a lot of attention. As a small list of related literature, we indicate: Kirikos and Novak (1997), Cakici and Zhu (2001), Piterbarg and Renedo (2004), Henrard (2005), Jäckel and Kawai (2005). Obviously one would like to price the futures also in the multi-curves framework. This was done for the one-factor Gaussian HJM model in Henrard (2010) using a deterministic hypotheses on the discounting/forward spread and in Mercurio (2010b) in the LMM with stochastic basis.

Here the problem is somehow reverse. From a given future price, we try to obtain the price of the (non-margined) coupons. The techniques used are similar.

At first reading, part of this note may seems quite axiomatic and distant from practical considerations. It is our believe that those questions are fundamental and in some circumstances may simplify the implementation and modelling of interest rate products.

> Cette histoire est vraie puisque je l'ai inventée...

> *Boris Vian*

It is the *artistic freedom* of the mathematician to invent his own axioms, hypothesis and definitions[3]. I have used that freedom to propose several axiomatic approaches to multi-curves.

2   Discounting

The starting point of the multi-curves framework is the discounting of known derivatives cash-flows; this is the first hypothesis:

D The instrument paying one unit in u is an asset for each u. It s value in t is denoted $P^D(t, u)$. The value is continuous in t.

With this curve we are able to value fixed cash-flows.

3   Coupon discount factor multi-curves framework

The framework described in this section is adapted from Henrard (2010).

Our goal is to price Ibor-related derivatives, in particular IRSs. We need an hypothesis saying that those instruments exist in the framework we are describing. We call a j-Ibor floating coupon a financial instrument which pays at the end of a period the Ibor rate set at the start of the period. The details of the instrument are as follow. The rate is set or fixed at a date t_0 for the period $[t_1, t_2](0 \leq t_0 \leq t_1 < t_2)$, at the end date t_2 the amount paid is the Ibor fixing multiplied by the conventional accrual factor. The lag between t_0 and t_1 is the *spot lag*. The difference between t_2 and t_1 is a j period. All periods and accrual factors should be calculated according to the day count, business day convention, calendar and end-of-month rule appropriate to the relevant Ibor indexes.

As the period addition, $t +$ period j is used often we adopt the notation $t + j$ for that date, without clarifying in which unit the j is; it is usually clear from the context.

Our existence hypothesis for the Ibor coupons reads as

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[3]Mathematicians should be allowed as much legal control on their axioms and definitions than fiction writers have on their characters. Mathematics requires as much imagination than fiction writing.

$\mathbf{I^{CPN}}$ [4] The value of a j-Ibor floating coupon is an asset for each tenor and each fixing date. Its value is a continuous function of time.

This hypothesis is implicit in most of the literature mentioned in the introduction. It is important to state it explicitly as this is not a consequence of the existence of the discounting curve. Once we have assumed that the instrument is an asset, we can give its value a name. We do it indirectly through the curves $P^{\mathrm{CDF},j}$.

Definition 1 (Curve) *The forward curve $P^{\mathrm{CDF},j}$[5] is the continuous function such that, $P^{\mathrm{CDF},j}(t,t) = 1$, $P^{\mathrm{CDF},j}(t,s)$ is an arbitrary strictly positive function for $t \leq s < \mathrm{Spot}(t) + j$, and for $t_0 \geq t$, $t_1 = \mathrm{Spot}(t_0)$ and $t_2 = t_1 + j$*

$$P^D(t,t_2)\left(\frac{P^{\mathrm{CDF},j}(t,t_1)}{P^{\mathrm{CDF},j}(t,t_2)} - 1\right) \tag{1}$$

is the value in t of the j-Ibor floating coupon with fixing date t_0 on the period $[t_1, t_2]$.

At this stage the only link between the curves and market rates is that the Ibor rate fixing in t_0 for the period j, denoted $I_{t_0}^j$, is

$$I_{t_0}^j = \frac{1}{\delta}\left(\frac{P^{\mathrm{CDF},j}(t_0, \mathrm{Spot}(t_0))}{P^{\mathrm{CDF},j}(t_0, \mathrm{Spot}(t_0) + j)} - 1\right) \tag{2}$$

where δ is the fixing period year fraction. To obtain this equality the value time-continuity was used.

3.1   Existence and arbitrariness

Note that Definition 1, which contains an arbitrary function, is itself arbitrary. One could fix any j period (not only the first one) and deduce the rest of the curve from there. Or one could even take an arbitrary decomposition of the period j interval in sub-intervals and distribute those sub-intervals arbitrarily on the real axis in such a way that, modulo j period, they recompose the initial j period. One could also change the value of $P^j(t,t)$ to any value as only the ratios are used, never a value on its own.

We should also add a couple of remarks on the dates. We use the notation $t + j$ as if the time displacements were a real addition. This is not the case. There is no inverse because due to non-business dates, several dates t_1 can lead to the same $t_1 + j$. This is not an exceptional case; three days a week, in the standard following rule, have a $t_1 + j$ ending on the same Monday. Moreover, we set our notation with the t_2 used in P^D and the one used in P^j the same. Again due to non-business day adjustments, this will not always be the case in FRAs and IRSs. The *payment date* (t_2 in P^D) can be several days before the *end of fixing period* date (t_2 in $P^{\mathrm{CDF},j}$). The difference is often one or two days but can be up to six. We will not make that distinction here.

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[4]$^{\mathrm{CPN}}$stands for CouPoN.
[5]CDF stands for Coupon Discount Factor.

3.2 Interest Rate Swap

With hypothesis ($\mathbf{I}^{\mathrm{CPN}}$) and the related definition, the computation of the present value of vanilla interest rate swaps is straightforward. The definition was selected for that reason. An IRS is described by a set of fixed coupons or cash flows c_i at dates $\tilde{t}_i$ ($1 \leq i \leq \tilde{n}$). For those flows, the discounting curve is used. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ ($1 \leq i \leq n$). The accrual factors for the periods $[t_{i-1}, t_i]$ are denoted δ_i. The value of a (fixed rate) receiver IRS is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i) \left(\frac{P^{\mathrm{CDF},j}(t, t_{i-1})}{P^{\mathrm{CDF},j}(t, t_i)} - 1 \right). \tag{3}$$

In the one curve pricing approach, the IRS are usually priced through either the *Ibor forward* approach or the *cash flow equivalent* approach. The *Ibor forward* approach consists in estimating the forward Ibor rate from the discount factors and discounting the result from payment date to today. To keep that intuition, we define the Ibor forward rate in our framework as the figure we have to use to keep the same formula.

Definition 2 (Forward rate) *The* Ibor forward *rate over the period* $[t_1, t_2]$ *is given at time t by*

$$F_t^{\mathrm{CDF},j}(t_1, t_2) = \frac{1}{\delta} \left(\frac{P^{\mathrm{CDF},j}(t, t_1)}{P^{\mathrm{CDF},j}(t, t_2)} - 1 \right). \tag{4}$$

With that definition the IRS present value is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i) \delta_i F_t^{\mathrm{CDF},j}(t_{i-1}, t_i).$$

Note the fundamental difference between $I_{t_0}^j$ and $F_t^{\mathrm{CDF},j}$. The object I^j is, by hypothesis $\mathbf{I}^{\mathrm{CPN}}$, a fundamental element of our economy; the $F^{\mathrm{CDF},j}$ is purely a definition. The definitions of F and I coincide on the fixing date t_0:

$$I_{t_0}^j = F_{t_0}^{\mathrm{CDF},j}(\mathrm{Spot}(t_0), \mathrm{Spot}(t_0) + j).$$

The *cash flow equivalent* approach in textbook formulas consists in replacing the (receiving) floating leg by receiving the notional at the period start and paying the notional at the period end. We would like to have a similar result in our new framework. To this end we define:

Definition 3 (Spread) *The* spread *between a forward curve and the discounting curve is*

$$\beta_t^{\mathrm{CDF},j}(u, u+j) = \frac{P^{\mathrm{CDF},j}(t, u)}{P^{\mathrm{CDF},j}(t, u+j)} \frac{P^D(t, u+j)}{P^D(t, u)}. \tag{5}$$

Obviously the value of this variable is constant at 1 if $P^D = P^{\mathrm{CDF},j}$. With that definition, a floating coupon price is

$$\begin{aligned}
P^D(t, t_i) \left(\frac{P^{\mathrm{CDF},j}(t, t_{i-1})}{P^{\mathrm{CDF},j}(t, t_i)} - 1 \right) &= P^D(t, t_i) \left(\beta_t^{\mathrm{CDF},j}(t_{i-1}, t_i) \frac{P^D(t, t_{i-1})}{P^D(t, t_i)} - 1 \right) \\
&= \beta_t^{\mathrm{CDF},j}(t_{i-1}, t_i) P^D(t, t_{i-1}) - P^D(t, t_i).
\end{aligned}$$

This last value is equal to the value of receiving $\beta_t^{\mathrm{CDF},j}$ notional at the period start and paying the notional at the period end.

A consequence of hypothesis $\mathbf{I^{CPN}}$ and the definition of $\beta_t^{\mathrm{CDF},j}$ is that $\beta_\cdot^{\mathrm{CDF},j}(t_{i-1},t_i)$ is a martingale in the $P^D(.,t_{i-1})$ numeraire. The Ibor coupon value is $\beta_t^j(t_{i-1},t_i)P^D(t,t_{i-1}) - P^D(t,t_i)$. The coupon is an asset due to $\mathbf{I^{CPN}}$ and so its value divided by the numeraire $P^D(t,t_{i-1})$ is a martingale. The second term, a zero-coupon, is also an asset, hence its rebased value is also a martingale. The rebased first term is thus also a martingale and its value is $\beta_t^{\mathrm{CDF},j}P^D(t,t_{i-1})/P^D(t,t_{i-1}) = \beta_t^{\mathrm{CDF},j}$. This proves that $\beta_\cdot^{\mathrm{CDF},j}(t_{i-1},t_i)$ is a martingale under the $P^D(.,t_{i-1})$-measure.

3.3 Libor Futures

A general pricing formula for interest rate futures in the one-factor Gaussian HJM model in the one curve framework was proposed in Henrard (2005). The formula extended a previous result proposed in Kirikos and Novak (1997). The formula was extended to the multi-curves framework in Henrard (2007).

The goal is to obtain a relatively simple, coherent and practical approach to Ibor derivatives pricing. To achieve the simplicity, our next hypotheses are related to the spreads between the curves, as defined through the quantities $\beta_t^{\mathrm{CDF},j}$.

S0$^{\mathbf{CDF}}$ The multiplicative coefficients between discount factor ratios, $\beta_t^{\mathrm{CDF},j}(u,u+j)$, as defined in Equation (5), are constant[6] through time: $\beta_t^{\mathrm{CDF},j}(u,u+j) = \beta_0^{\mathrm{CDF},j}(u,u+j)$ for all t and u.

We describe the pricing of futures under the hypotheses $\mathbf{I^{CPN}}$ and $\mathbf{S0^{CDF}}$ in a multi-curves one-factor Gaussian HJM model. The pricing of the futures in the LMM with stochastic basis is proposed in Mercurio (2010b).

The future *fixing* or *last trading date* is denoted t_0. The fixing is on the Ibor rate between $t_1 = \mathrm{Spot}(t_0)$ and $t_2 = t_1 + j$. The fixing accrual factor for the period $[t_1, t_2]$ is δ. The fixing is linked to the yield curve by (2).

The futures price in t is denoted $\Phi_t^j(t_1)$. On the fixing date, the relation between the price and the rate is

$$\Phi_{t_0}^j(t_1) = 1 - I_{t_0}^j.$$

The futures margining is done on the futures price (multiplied by the notional and the futures accrual factor).

The exact notation for the HJM one-factor model used here is that in Henrard (2005). When the discount curve $P^D(t,.)$ is absolutely continuous (which is something that is always the case in practice as the curve is constructed by some kind of interpolation) there exists $f(t,u)$ such that

$$P^D(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{6}$$

The short rate associated with the curve is $(r_t)_{0 \leq t \leq T}$ with $r_t = f(t,t)$. The cash-account numeraire is $N_t = \exp(\int_0^t r_s ds)$.

In the HJM framework, the equations in the cash-account numeraire measure associated with N_t are

$$df(t,u) = \sigma(t,u)\nu(t,u)dt + \sigma(t,u)dW_t.$$

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[6]In this framework, constant spread is equivalent to deterministic spread due to the martingale property of β.

where $\nu(t,u) = \int_t^u \sigma(t,s)ds$. The model is one-factor Gaussian if W_t is a one-factor Brownian motion and σ is a deterministic function.

Theorem 1 (Futures price) *Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the one-factor Gaussian HJM model on the discounting curve under the hypotheses $\boldsymbol{D}$, $\boldsymbol{I^{CPN}}$ and $\boldsymbol{S0^{CDF}}$, the price of the futures fixing in t_0 for the period $[t_1, t_2]$ with accrual factor δ is given by*

$$
\begin{aligned}
\Phi_t^j &= 1 - \frac{1}{\delta}\left(\frac{P^{\mathrm{CDF},j}(t,t_1)}{P^{\mathrm{CDF},j}(t,t_2)}\gamma(t) - 1\right) \tag{7}\\
&= 1 - \gamma(t)F_t^{\mathrm{CDF},j} + \frac{1}{\delta}(1-\gamma(t))
\end{aligned}
$$

where

$$
\gamma(t) = \exp\left(\int_t^{t_0} \nu(s,t_2)(\nu(s,t_2) - \nu(s,t_1))ds\right).
$$

Proof: Using the generic pricing future price process theorem (Hunt and Kennedy, 2004, Theorem 12.6),

$$
\Phi_t^j(t_1) = \mathrm{E}^{\mathbb{N}}\left[1 - I_{t_0}^j \,\Big|\, \mathcal{F}_t\right]
$$

where $\mathrm{E}^{\mathbb{N}}[\,]$ is the cash account numeraire expectation.

In $I_{t_0}^j$, the only non-constant part is the ratio of j-pseudo-discount factors which is, up to $\beta_{t_0}^j$, the ratio of discount factors. Using (Henrard, 2005, Lemma 1) twice, we obtain

$$
\begin{aligned}
\frac{P^D(t_0,t_1)}{P^D(t_0,t_2)} &= \frac{P^D(t,t_1)}{P^D(t,t_2)}\exp\left(-\frac{1}{2}\int_t^{t_0}\nu^2(s,t_1)-\nu^2(s,t_2)ds\right.\\
&\quad\left. + \int_t^{t_0}\nu(s,t_1)-\nu(s,t_2)dW_s\right).
\end{aligned}
$$

Only the second integral contains a stochastic part. This integral is normally distributed with variance $\int_t^{t_0}(\nu(s,t_1)-\nu(s,t_2))^2 ds$. The expected discount factors ratio is reduced to

$$
\frac{P^D(t,t_1)}{P^D(t,t_2)}\quad\exp\left(-\frac{1}{2}\int_t^{t_0}\nu^2(s,t_1)-\nu^2(s,t_2)ds+\int_t^{t_0}(\nu(s,t_1)-\nu(s,t_2))^2ds.\right)
$$

By hypothesis $\mathbf{SI^{CDF}}$, the coefficient $\beta_{t_0}^{\mathrm{CDF},j}$ is constant, and so we have obtained the announced result. $\qquad\square$

3.4 Curve building

A relative standard way to calibrate the curves P^D and $P^{\mathrm{CDF},j}$ is to select a set of market instruments for which the present value is known and an equal number of node points. An interpolation scheme is selected and the rates on the node points are calibrates to reproduce the market prices. The market forward rates $F_0^{\mathrm{CDF},j}(t_1, t_1 + j)$ can be computed from that curve. A typical forward rate curve is displayed[7] in Figure 1. The swap data used to build the curve are the one used in (Andersen and Piterbarg, 2010, Section 6.2) and the interpolation scheme is linear on (continuously compounded) rates. We suppose that the swap rates are fixed versus three months Ibor and that the discounting curve has flat market rates at 4%.

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[7]All the numbers in the figures of this note have been produced using OpenGamma OG-Analytics library. The library is open source and available at http://www.opengamma.com.
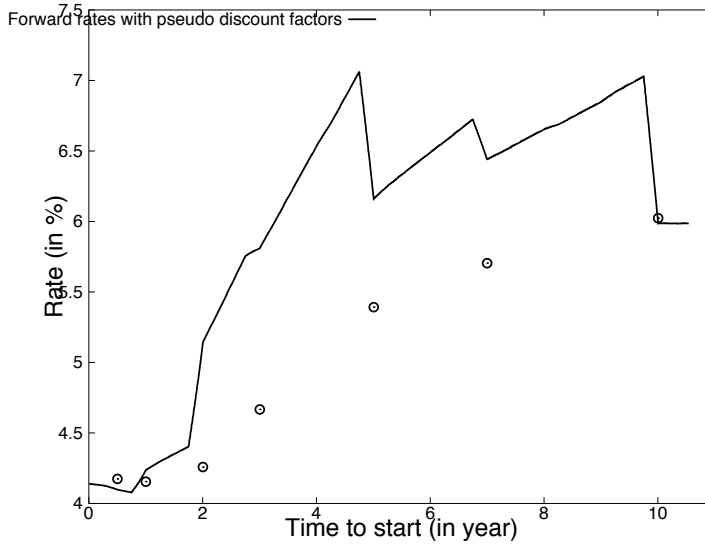
Figure 1: Forward rates computed using pseudo-discount factors. The circles indicate the zero-coupon rate associated to the different nodes.

The familiar sawtooth pattern can be seen. There is two angles in the curve for each node point. One when the fixing period end date is on one node and one when the start date is on the node.

One of the reasons of this unpleasant shape is probably that we have an intuition on a market quantity (forward rate) but model it indirectly through a ratio of discount factors where our intuition is diluted.

4   Coupon forward rate multi-curves framework

We introduce a different framework still based on coupons. The forward rates are modelled directly and not through pseudo-discount factors. For this reason we refer to it as the *coupon forward rate multi-curves framework*. From a pure theoretical point of view it is equivalent to the previous framework as there is a bijection between the pseudo-discount factors (once the arbitrary part is selected) and the forward rate. From a practical point of view they are different as the description and interpolation schemes will be applied on the discount factors or directly on the forward rates and give different results. This is in some sense similar to the HJM/LMM duality. One is technically easier but hte other refers to market quantities.

For this framework, the same existence hypothesis $\mathbf{I^{CPN}}$ is used. The associated definition is now:

Definition 4 (Forward rate) *The forward curve* $F^{\mathrm{CFWD},j}$[8] *is the continuous function such that,*

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[8]CFWD stands for Coupon ForWarD rate.

$$P^D(t, t_2)\delta F_t^{\mathrm{CFWD}, j}(t_1) \tag{8}$$

is the price in t of the j-Ibor coupon with start date t_1 and maturity date t_2 ($t \leq t_0 \leq t_1 = \mathrm{Spot}(t_0) < t_2$).

Note that in this framework, the *Ibor discounting* is impossible as there is no discount factor associated to the Ibor curves.

The link between the curves and market rates is

$$I_{t_0}^j = F_{t_0}^{\mathrm{CFWD}, j}. \tag{9}$$

There is no arbitrary part anymore to the curve. The curve is defined unambiguously (as long as the corresponding market instruments exist) for all $t_1 \geq \mathrm{Spot}(0)$.

With hypothesis ($\mathbf{I^{CPN}}$) and Definition 4, the computation of the present value of vanilla interest rate swaps is straightforward. The definition was selected for that reason. The IRS description is the same as in the previous section. The value of a (fixed rate) receiver IRS is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i)\delta_i F_t^{\mathrm{CFWD}, j}(t_{i-1}). \tag{10}$$

Definition 5 (Spread) *The spread between a forward curve and the discounting curve is*

$$\beta_t^{\mathrm{CFWD}, j}(u, u+j) = (1 + \delta F_u^{\mathrm{CFWD}, j})\frac{P^D(t, u+j)}{P^D(t, u)}. \tag{11}$$

With that definition, a floating coupon price is

$$
\begin{aligned}
P^D(t, t_i)\delta F_t^{\mathrm{CFWD}, j}(t_1) &= P^D(t, t_i)\left(\beta_t^{\mathrm{CFWD}, j}(t_{i-1}, t_i)\frac{P^D(t, t_{i-1})}{P^D(t, t_i)} - 1\right) \\
&= \beta_t^{\mathrm{CFWD}, j}(t_{i-1}, t_i)P^D(t, t_{i-1}) - P^D(t, t_i).
\end{aligned}
$$

This last value is equal to the value of receiving $\beta_t^{\mathrm{CFWD}, j}$ notional at the period start and paying the notional at the period end.

4.1 Libor Futures

The goal is to obtain a relatively simple, coherent and practical approach to Ibor derivatives pricing. To achieve the simplicity, our next hypothesis is related to the spreads between the curves, as defined through the quantities $\beta_t^{\mathrm{CFWD}, j}$.

$\mathbf{S0^{CFWD}}$ The spreads $\beta_t^{\mathrm{CFWD}, j}(u, u+j)$, as defined in Equation (11), are constant through time: $\beta_t^{\mathrm{CFWD}, j}(u, u+j) = \beta_0^{\mathrm{CFWD}, j}(u, u+j)$ for all t and u.

We describe the pricing of futures under the hypotheses $\mathbf{I^{CPN}}$ and $\mathbf{S0^{CFWD}}$ in a multi-curves one-factor Gaussian HJM model. The notation is the same as in the previous section.

Theorem 2 *Let $0 \le t \le t_0 \le t_1 \le t_2$. In the one-factor Gaussian HJM model on the discounting curve under the hypotheses $\boldsymbol{D}$, $\boldsymbol{I^{CPN}}$ and $\boldsymbol{S0^{CFWD}}$, the price of the futures fixing in t_0 for the period $[t_1, t_2]$ with accrual factor δ is given by*

$$\Phi_t^j = 1 - \gamma(t)F_t^{\mathrm{CFWD},j} + \frac{1}{\delta}(1 - \gamma(t))$$

where

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s, t_2)(\nu(s, t_2) - \nu(s, t_1))ds\right).$$

Proof: Using the generic pricing future price process theorem (Hunt and Kennedy, 2004, Theorem 12.6),

$$\Phi_t^j = \mathrm{E}^{\mathbb{N}}\left[1 - I_{t_0}^j \,\Big|\, \mathcal{F}_t\right].$$

The value $I_{t_0}^j$ when written in term of $\beta_{t_0}^j$ depends on the ratio of discount factors. Using (Henrard, 2005, Lemma 1) twice, we obtain

$$
\begin{aligned}
\frac{P^D(t_0, t_1)}{P^D(t_0, t_2)} &= \frac{P^D(t, t_1)}{P^D(t, t_2)} \exp\left(-\frac{1}{2}\int_t^{t_0} \nu^2(s, t_1) - \nu^2(s, t_2)ds\right. \\
&\quad \left. + \int_t^{t_0} \nu(s, t_1) - \nu(s, t_2)dW_s\right).
\end{aligned}
$$

Only the second integral contains a stochastic part. This integral is normally distributed with variance $\int_t^{t_0}(\nu(s, t_1) - \nu(s, t_2))^2 ds$. The expected discount factors ratio is reduced to

$$\frac{P^D(t, t_1)}{P^D(t, t_2)} \quad \exp\left(-\frac{1}{2}\int_t^{t_0} \nu^2(s, t_1) - \nu^2(s, t_2)ds + \int_t^{t_0}(\nu(s, t_1) - \nu(s, t_2))^2 ds.\right)$$

By hypothesis $\boldsymbol{S0^{CFWD}}$, the coefficient $\beta_{t_0}^{\mathrm{CFWD},j}$ is constant, and so we have obtained the announced result. $\qquad\square$

4.2 Curve building

The advantages of the approach is that the market rates on which we have some intuition are modelled directly. In some sense, and borrowing a well known name, it could be called the *Libor Market Model* of curve description (not of curve dynamic as its namesake).

There is no requirement anymore of an arbitrary part like in Definition 1 of the discount factor approach. The interpolation and constraints can be imposed directly on the market quantities. Figure 2 presents the forward rate using the same data as Figure 1 and the same linear interpolation scheme (even if applied to a different quantity).

The comparison between the two approaches is done in Figure 3(a). It is to each market maker or risk manager to decide which one he prefers. With the reported data, they market rate curves display less "zig-zag" with the direct rate approach. With some other market rates, the picture can be different.

In Figure 3(b), we zoomed on a part of the curve. Beyond the angles at date for which there is no data in the pseudo-discount factor framework, one can also see the waves due to the week-end effects which varies with the months lengths.
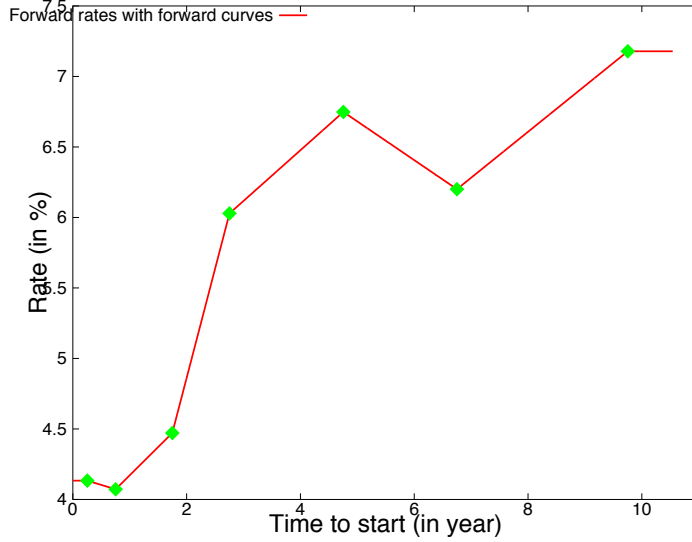
Figure 2: Forward rates computed using direct forward rate curve.

5 Futures discount factor multi-curves framework

The framework described this section was first presented in Henrard (2012).

Our existence hypothesis replaces the hypothesis $\mathbf{I^{CPN}}$. It links Ibor futures to martingale futures price processes in the sense of (Hunt and Kennedy, 2004, Section 12.4).

$\mathbf{I^{FUT}}$ The prices of the (j-Ibor) futures are martingale futures price processes for each fixing date.

Once we have assumed that the instrument exists in our economy, we can give its price a name. We do it indirectly through the curves $P^{\mathrm{FDF},j}$[9]. The notations concerning futures are the same as in the previous sections.

Definition 6 (Futures pseudo-discount curves) *The forward curve $P^{\mathrm{FDF},j}$ is the continuous function such that, $P^{\mathrm{FDF},j}(t,t) = 1$, $P^{\mathrm{FDF},j}(t,s)$ is an arbitrary function for $t \leq s < \mathrm{Spot}(t) + j$, and for $t_0 \geq t$, $t_1 = \mathrm{Spot}(t_0)$ and $t_2 = t_1 + j$*

$$\Phi_t^j(t_1) = 1 - \frac{1}{\delta}\left(\frac{P^{\mathrm{FDF},j}(t,t_1)}{P^{\mathrm{FDF},j}(t,t_2)} - 1\right). \tag{12}$$

The futures price is obtained directly from the pseudo-discount factor curves (or more exactly the curve is obtained directly from the futures prices) without convexity adjustment.

With that definition, the link between $I_{t_0}^j$ and $P^{\mathrm{FDF},j}$ is

$$I_{t_0}^j = \frac{1}{\delta}\left(\frac{P^{\mathrm{FDF},j}(t_0,t_1)}{P^{\mathrm{FDF},j}(t_0,t_2^f)} - 1\right).$$

---

[9]FDF stands for Futures Discount Factor.
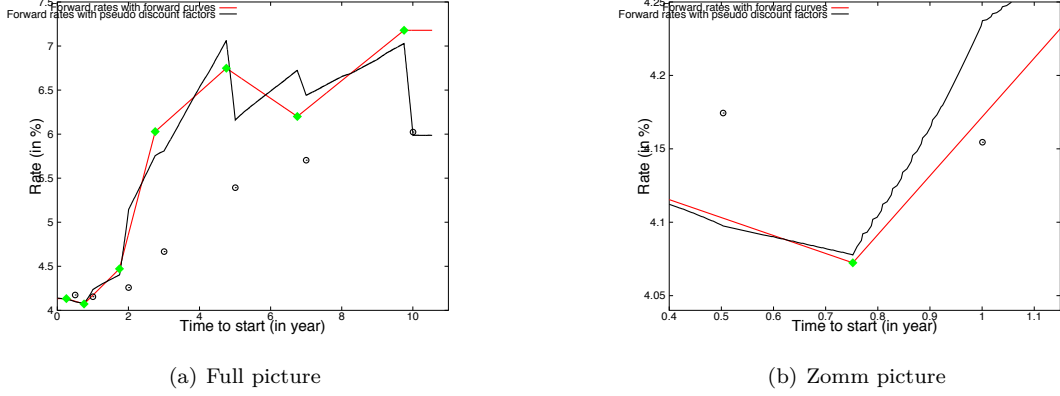
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(a) Full picture

(b) Zomm picture

Figure 3: A comparison of forward rates computed using pseudo-discount factors and using direct forward rate curve.

We will also use the following definition of *futures rate*:

Definition 7 (Futures rate) *The* futures rate *is given by*

$$F_t^{\mathrm{FUT},j}(t_1) = 1 - \Phi_t^j(t_1). \tag{13}$$

In hypothesis $\mathbf{I^{FUT}}$ and Definition 6, we suppose the existence of a continuum of futures with all possible fixing date t_0. Obviously finance is discrete in payment dates, with at most one payment by day, and any real number t exists in practice only on discrete daily points. For futures there is the extra constraint that futures are traded with settlement date only every months on the short part of the curve and quarterly up to 10 years. This may appear as a lot less points than the usual FRAs and swaps. This is not really the case as the FRA are not specially liquid and mainly traded only on the short part with at-best monthly maturities and above two year, there are only swap with annual maturities. The futures curve contains more points in the 2 to 10 years range. The coupon curves contains more points only in theory, not in practice.

Like in the coupon framework, we define a new variable:

Definition 8 (Spread) *The variable $\beta_t^{\mathrm{FDF},j}(t_1, t_2)$ is defined as a ratio of discount factors ratios*

$$\beta_t^{\mathrm{FDF},j}(t_1, t_2) = \frac{P^{\mathrm{FDF},j}(t, t_1)}{P^{\mathrm{FDF},j}(t, t_2^f)} \frac{P^D(t, t_2^p)}{P^D(t, t_1)} = (1 + \delta F_t^{\mathrm{FUT},j}(t_0)) \frac{P^D(t, t_2^f)}{P^D(t, t_1)}. \tag{14}$$

In the coupon framework an hypothesis often used is that the ratios $\beta_t^{\mathrm{FDF},j}$ are constant through time.

We propose to use the next best thing: a deterministic spread hypothesis

$\mathbf{SD^{FDF}}$ The multiplicative coefficients between discount factor ratios, $\beta_t^{\mathrm{FDF},j}(t_1, t_2)$, defined in Equation (14), are deterministic for all t_1.

This is the equivalent to the constant spread hypothesis $\mathbf{S0^{CDF}}$ used in the coupon multi-curves framework.

An Ibor coupon pays the amount $\delta_p I_{t_0}^j$ in t_2. Its today's value is given by the following theorem.

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Theorem 3 (Coupon value) *In the futures multi-curves framework, under the hypothesis $\boldsymbol{I^{FUT}}$ and $\boldsymbol{SD^{FDF}}$, the price of the Ibor coupon fixing in t_0 for the period $[t_1, t_2]$ is given by*

$$P^D(0, t_1)\delta_p \frac{1}{\delta} \beta_{t_0}^{\mathrm{FDF}, j}(t_1, t_2) - P^D(0, t_2).$$

Proof: The proof is immediate. It is enough to use the link between $I_{t_0}^j$ and $P^{\mathrm{FDF}, j}$, use the definition of $\beta_{t_0}^{\mathrm{FDF}, j}$ and take the expectation with $P^D(., t_2)$ as numeraire. $\qquad\square$

The formula is very closed to the one for Ibor coupon in the coupon multi-curves framework. The difference is that here the $\beta^{\mathrm{FDF}, j}$ is taken in t_0, not in 0. We mentioned above that the quantity is not constant, so the two formulas are different and this is to be expected as the definitions of P^j are different.

It was proved in Henrard (2010) that the quantity $\gamma(t)P^D(t, t_1)/P^D(t, t_2)$ is a N-martingale in the one-factor Gaussian HJM model for

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s, t_2)(\nu(s, t_2) - \nu(s, t_1))ds\right)$$

and ν the bond volatility in the one-factor Gaussian HJM model. This is the base of the pricing of futures in the coupon framework. Our next hypothesis is coherent with that observation

HJM1 The quantities $\beta_t^{\mathrm{FDF}, j}(t_1, t_2)$ are such that

$$\beta_t^{\mathrm{FDF}, j} = \beta_0^{\mathrm{FDF}, j} \frac{\gamma(t)}{\gamma(0)}.$$

Under **HJM1** hypothesis, we have the following equalities

$$\frac{P^{\mathrm{FDF}, j}(t, t_1)}{P^{\mathrm{FDF}, j}(t, t_2)} = \frac{P^D(t, t_1)}{P^D(t, t_2)}\beta_t^{\mathrm{FDF}, j} = \frac{P^D(t, t_1)}{P^D(t, t_2)}\gamma(t)\frac{\beta_0^{\mathrm{FDF}, j}}{\gamma(0)}$$

with the first and the last variables N-martingales in the one factor Gaussian HJM model. It may seem a very strong hypothesis to impose a specific model. This is equivalent to the HJM hypothesis to price futures often done in curve construction.

Theorem 4 *In the futures multi-curves framework, under the hypothesis $\boldsymbol{I^{FUT}}$ and $\boldsymbol{HJM1}$, in the one-factor Gaussian HJM model the price of the Ibor coupon fixing in t_0 for the period $[t_1, t_2]$ is given by*

$$P^D(0, t_1)\frac{\beta_0^{\mathrm{FDF}, j}(t_1, t_2)}{\gamma(0)} - P^D(0, t_2) = P^D(0, t_2^p)\delta_p \frac{1}{\gamma(0)}\left(F_0^{\mathrm{FDF}, j} + \frac{1}{\delta}(1 - \gamma(0))\right).$$

The convexity adjustment is now done on the Ibor coupon, not on the futures anymore. Note that the adjustment is obtained by dividing by the coefficient $\gamma(0)$ and not multiplying by it. Should the adjustment be called a *concavity adjustment*?

5.1 Zero rate collateral

In this framework meaningful in practice? There is at least one scenario where it could become the standard framework. There is a push for more standardisation of the products and of the legal

terms (CSA in particular). One particular discussion is around the changes of the CSA terms and the related collateral renumeration. In the current standard terms an overnight rate (Fed Fund; Eonia, etc.) is paid. One potential solution to simplify the term of the CSAs, which has been proposed by several market participants, is to pay zero interest on the collateral. This is equivalent to a futures margining. If that proposal, which simplifies a certain number of practical problems, is put in place, this framework would be the natural one also for the swaps with zero rate collateral.

Suppose that there is a continuous (daily) margining for swaps and the rate paid on the collateral is 0. This is the similar as the margining on futures: the difference of value with the previous valuation is paid and no interest is paid on that amount. The general *futures price process* theory, as described in (Hunt and Kennedy, 2004, Section 12.3), applies in that case. What is the value of the new Ibor coupon with CSA at rate 0?

The coupon *pays* $I_{t_0}^j$ in t_2 and is a futures price process up to that date. According to the general futures price theorem its value in 0 is

$$\mathrm{E}^N\left[I_{t_0}^j\right] = 1 - \mathrm{E}^N\left[1 - I_{t_0}^j\right] = 1 - \Phi_0^j(t_1).$$

Note that the fact that the coupon pays in t_2 has no impact on the valuation. The value is known at the fixing date t_0 and from that date on the require collateral is paid. Note that payments in advance or in arrear have the same value. Maybe there is would be a legal distinction between the amount paid as collateral and the amount paid as coupon, but if we ignore that distinction, from a cash-flow exchange, everything is exchanged as soon as the cash-flow is known.

The general collateral principle is still valid: a promise to pay tomorrow is fulfilled by paying today the (discounted) expected value and adapting the amount up to the final payment. The difference here is the discounting at a zero rate and the fact that no adaptation is required after the last fixing. The no-adaptation after last fixing is also the case for the collateral with deterministic interest.

6 Conclusion

We presented several multi-curves frameworks. They are differentiated by the fundamental market instruments (coupons or futures) and by the way the forward curves are represented (pseudo-discount factors or direct market forward rates). We described three of the four combinations; the extension to the fourth is immediate. The coupons pseudo-discount factor framework is the standardly used combination, most for historical reasons than for deep fundamental reasons. In some circumstances, other combinations can be more efficient.

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