No-Arbitrage dynamics for a tractable SABR term structure Libor Model

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1 Introduction

The SABR model (Hagan et al. 2002) has emerged in the last years as a reference stochastic volatility framework for modelling the swaption implied volatility at different strikes. However, many implications of the model behaviour when applied to interest rate derivatives are still overlooked, as remarked for example by Zhou (2007). In the following we summarize some of the main open issues.

The SABR model has been proposed by Hagan et al. (2002) as a stochastic volatility model for a generic underlying forward, under its natural probability measure. Under this measure we know by the first fundamental theorem of asset pricing (see for example Delbaen and Schachermayer (1994)) that the underlying is a martingale. Hagan et al. (2002) suppose that also the stochastic volatility factor is a martingale under this natural measure. When the model is applied to financial payoffs depending on a single underlying, this is sufficient for arbitrage-free valuation, and in fact SABR is commonly used for caps, swaptions and simple CMS (Constant Maturity Swap) products which are priced through replication techniques starting from a continuum of swaption prices.

However, when we apply the model to more general term structure payoffs, there are no-arbitrage issues that arise. Term structure payoffs are characterized by high correlation between the underlying and the discount factor, both depending on the same sources of interest rate risk. Therefore even the simplest interest rate derivatives (caps and swaptions) are evaluated under particular pricing measures that allow to simplify pricing by immunization of the payoff from the volatility of the discount factor.

The issues of no-arbitrage modelling of market rates are solved in the modelling framework usually known as BGM or Libor/Swap Market Models of Brace et al. (1997) and Jamshidian (2007). There it is shown that the joint dynamics of term-structure-dependent quantities changes on the pricing measure chosen, and that it is characterized by dependence of the dynamics of single quantities from the dynamics of other parts of the term structure, a fundamental requirement for arbitrage-free modelling of the term structure. A real world payoff often depends on a plurality of interest rates that need to be modelled jointly, therefore a term structure model needs to provide joint dynamics of all term-structure-dependent quantities in the model.

Extending SABR to be a stochastic volatility term structure model means first of all to provide these joint dynamics. In stochastic volatility Market Models the state variables are interest rates (libor or swap forward rates) and their stochastic volatilities. If the stochastic volatility factors are assumed to be correlated to the term structure of rates (as in Hagan et al. (2002) the underlying is correlated to its stochastic volatility) also the stochastic differential equation governing volatility will be different under different swap and forward measures. In market implementation, a number of volatility factors lower than the number of state variables is often chosen, in particular one single volatility factor (see for example Piterbarg (2005a), Andersen and Andreasen (2002), Wu and Zhang (2006)). In such a case, the issue of change of dynamics affects also the pricing of simple derivatives with one single underlying variable, such as caplets. In fact, each caplet is priced under a different forward measure, and therefore it should be priced using a different volatility dynamics[1].

While the computation of no-arbitrage drifts is trivial for the dynamics of forward rates, this is not the case for the stochastic volatility dynamics. Change of dynamics for stochastic volatility in Market Models has not been object of large research. A preeminent exception is Wu and Zhang (2006), who are working with a volatility process of Heston (1994) type, radically different from the SABR form of stochastic volatility dynamics. Additionally, Wu and Zhang (2006) show that, modulo a standard drift approximation, a volatility dynamics of Heston type is not altered by change between forward/swap measures. Only the parameters in the dynamics change, not the distribution of the volatility. On the contrary, when we start from SABR lognormal assumption in the volatility, change of measure leads to non-trivial changes in distribution. This fact makes it difficult to keep the SABR popular formula for the pricing of European Interest Rate options. Since the SABR model is used in the market through the Hagan et al. (2002) formula, which is particularly appreciated for being intuitive and efficient, loosing compatibility with this formula would harm the usefulness of the Libor Model extension. Therefore we devise approximations still allowing for implementation with the Hagan et al. (2002) formula. We then test numerically their accuracy and empirically their behaviour on market data.

In Section 2 we present some preliminary results and considerations on the modelling framework we

[1]Notice that in Piterbarg (2005a) and Andersen and Andreasen (2002) the interest rates and their stochastic volatility are constrained to be uncorrelated. In this very special case the no-arbitrage drifts for volatility are null.

set up. In Section 3 we compute the no-arbitrage dynamics for a Libor and Swap market models built starting from SABR assumptions under a reference measure. In Section 4 we show how this dynamics can be approximated so as to reach a model where no-arbitrage change of dynamics is captured by a simple modification in the parameters of the SABR formula. In Section 5 these approximations are tested under numerical methods and then applied to market data. We show, also with out-of-sample tests, that the approximated no-arbitrage constraints give results more consistent with market patterns than a model neglecting the issue.

We remark that compliance with arbitrage constraints is a fundamental issue not only in theoretical terms, but also in practice, particularly in the interest rate derivatives market. Since the market is very liquid and completed by reliable information on volatility and correlations, the computation of precise no arbitrage corrections is possible and its importance is magnified by the large notional amounts typical of interest rate transactions. The importance of this issue for practical pricing is confirmed by the fact that the two theoretical advances that in recent years had the strongest effect on everyday pricing (leaving aside the specific issue of smile modelling, common to all markets) have been the introduction of BGM dynamics and the introduction of convexity adjustments. Both advances mainly added a correct dealing with no-arbitrage drifts and related adjustments.

The issue of reconciliating the SABR dynamics with the Libor Market Model framework is also the core of Rebonato (2007). However, Rebonato (2007) takes a different route from ours. We design a Libor market Model starting from the reference SABR dynamics, with the purpose of preserving the SABR closed form formula. Instead Rebonato adapts a Libor Market Model so as to obtain results as close as possible to the SABR result.

The main difference is that we concentrate our analysis on computing the non-trivial modification to stochastic differential equations which are due no arbitrage constraints, then approximating them for preserving tractability and testing their practical effect in market valuation. We prefer to show these results while keeping the most synthetic and automatic extension of Hagan et al. (2002) hypotheses, namely constant parameters and a single stochastic volatility factor.

On the other hand, Rebonato (2007) does not consider this issue but performs an in-depth analysis of the most convenient parameterization allowing good and stable (time-homogenous) fit to market data, and extends part of its analysis to non-trivial results applying to the case of a plurality of stochastic volatility factors. Under this point of view, the two works appear complementary.

2 The Data. Preliminary Analysis

Most Libor Market Models introduced so far for dealing with interest rate smile through stochastic volatility allow for only one stochastic volatility factor applied to all forward rates[2]. This is the case for the main examples in the literature: Andersen and Andreasen (2002), Wu and Zhang (2006) and Piterbarg (2005a). Piterbarg (2005c), while outlining some limitations of this single-volatility approach, claims that LMM with multiple volatility factors is complicated, not considered in the literature and there is "no market intelligence that someone is using it".

Accordingly, for the most part of the paper we keep the standard single volatility approach, applied here to the SABR dynamics. As an immediate, simple analysis of what we are loosing in terms of fit, we see how the SABR fitting changes when using the SABR functional form while setting some constraints on the parameterization. We calibrate to the swaption smile for maturities $1y, 5y, 10y$ and tenors $2y, 5y, 10y$, on June 08, 2006. The range of strikes considered is the one provided by the market, namely $S_{a,b}(0)$ plus the elements of vector Y, where, in basis points

$$Y = [-200, -100, -50, -25, 0, 25, 50, 100, 200],$$

while $S_{a,b}(0)$ is the current value of the forward swap rate with first reset in T_a and last payment in T_b. In the Table below we present the average discrepancy, in term of implied volatility errors, of the SABR functional form

[2] Moreover they usually consider one single parameter for the local volatility of the model (one single β for all rates, equivalent to assume the same type of "backbone dynamics" for all rates). We also follow this approach that appears sufficient to guarantee good consistency with market quotation in the SABR model. An exception is Piterbarg (2003), where however there is no correlation between stochastic volatility and forward rates.

	Individual β's	Common $\beta = 1$
Individual VolVol's	0.32 bps (0.00%)	4.53 bps (0.05%)
Common VolVol	15.59 bps (0.16%)	17.06 bps (0.17%)

There are many reasons why market participants may be willing to abandon the virtually perfect fit of the SABR functional form to move to a very good but not perfect fit. First of all, when moving to a model rather than many separate functional forms, a full set of parameters for each rate may lead to overparameterization, with a risk of instability, considering also the presence of many cross-correlations between stochastic volatilities not easy to determine based on market quotes. Then the model will be computationally burdensome when using Monte Carlo, because of the high number of factors.

Even more importantly, a model with one single stochastic volatility disturbance can capture better market regularities on the movements of the term structure, while the important common factors driving the market could be missed when each rate is calibrated independently of the others. As an example, we see in the table below results on the pricing of Constant Maturity Swaps with length $n = 5, 10$ (payments up to five or ten years), $c = 5, 10$ (the CMS leg pays 5/10 year length swap rates) and $\Delta = 0.25$ (yearly swap rates are paid every three months).

	$X_{5,5}$	$X_{10,5}$	$X_{5,10}$	$X_{10,10}$
Market CMS Spread	**47.7**	**44.7**	**70.9**	**64.75**
Individual VolVol's	42.05	40.5	66.4	63.3
Common VolVol	42.1	39.4	66.5	61.3

The model which is best calibrated and involves more parameters does not necessarily price CMS better. As shown for example by Mercurio and Pallavicini (2006), recovery of market CMS spreads is not guaranteed when a model has been calibrated only to swaptions. The swaption smile does not include extreme strikes, and consequently gives little information on the second moment of the underlying, that in turn is fundamental to compute CMS spreads. Since for many exotics out of sample model implications are more relevant than extremely good fitting, a more efficiently parameterized model, able to capture general regularities on the volatility structure making use of data of different maturities and tenors can provide analogous or even better results.

Finally, there are very few quotes that appear to responsible for an important part of the fitting error that we have in the case of synthetic model. For example, in the above tests, the lowest strike (-200 bps) swaptions appear to have a vega as low as 8 bps, so that a very small price error is magnified when expressed in terms of implied volatility (in this specific test a mispricing of less than 0.1 basis point, turns into an implied volatility difference of 145 basis points). So the implied volatility error appears to be non-significant. Since the quotation is also particularly illiquid, particularly for short maturities, we drop it from the calibration set. This leads to an average error dropping from 17.06 bps to 8.38 bps, confirming that this quotation can alter the results in a unnatural way. The latter error is lower than 10bps, the average swaption calibration error considered acceptable by Piterbarg (2005b). Therefore even a model with one single volatility factor and one single local volatility parameter β allows to obtain a satisfactory fit to the swaption market, which is in fact characterized by a quite regular structure, with skew increasing with maturity and tenor larger than one year.

Since in the following we concentrate on the change of dynamics due to no-arbitrage constraints, we aim at keeping the model as simple as possible to concentrate on the analysis of no-arbitrage dynamics. The assumption of one single volatility factor with one single volatility of volatility parameter and a common local volatility backbone allows us to have a model consistent with market standard and to set up a no arbitrage SABR Libor Model in a simpler context. The relationships linking swaption and cap markets remain simple, while Rebonato (2007) shows that in the many-volatilities case specific modifications must be introduced to avoid financially illogical results. According to these considerations and to the results of the above simple tests we stick to this implementation in the following, but we move from the trivial use of the SABR functional form to building a real no-arbitrage term structure model.

However, the reader should be aware that a model assuming a mean reverting dynamics of the stochastic volatility would be more consistent with the swaption smile structure. We point out that the no-arbitrage Wu and Zhang (2006) model can perform the above calibration test with an error as low as 4.01 bps, although at the cost of a much higher computational burden. Again Rebonato (2007) shows precisely which realistic assumptions in terms of behaviour of the volatility of volatility allow to replicate best the market patterns.

3 No-arbitrage dynamics for the SABR Libor Market model

In the context of a term structure model, where the dynamics of the state variables have to be used under different pricing measures, we have to choose a reference initial measure for giving a reference volatility dynamics. From this reference dynamics, making use of change of numeraire theory and formulas, we will be able to compute how the dynamics changes when we move to different pricing measures.

3.1 The reference dynamics for Rates and Volatility

The standard reference measure for stochastic volatility Libor models is the spot measure Q, associated to the discretely rebalanced bank-account numeraire $B_d(t)$. The bank-account starts at 1 and is rebalanced only at the times that appear in the LMM discrete tenor structure. Thus, calling $T_{\gamma(s)}$ the first tenor structure date after time s, the current value of the account is

$$B_d(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{j=0}^{\gamma(t)} P(T_{j-1}, T_j)} = P(t, T_{\gamma(t)}) \prod_{j=0}^{\gamma(t)} (1 + \tau_j F_j(T_{j-1})).$$

This measure, which, unlike forward measures, is not associated to a specific forward rate but to the entire tenor structure of expiry and maturity dates, is chosen as a reference by Andersen and Andreasen (2002) and Piterbarg (2005a). Under the reference spot measure we assume state variables to follow a SABR-like dynamics

$$dF_k(t) = \mu_k^Q dt + \sigma_k V(t) F_k(t)^\beta dZ_k^Q(t)$$

where μ_k^Q is the spot Libor drift of forward rates (see for example Brigo and Mercurio (2006), where it is also shown that this leads to a zero drift under the associated natural forward measure Q^k), σ_k is a deterministic (constant) instantaneous volatility coefficient, $\beta \in [0, 1]$ is the local volatility coefficient, $Z^Q(t)$ is a vector of standard Wiener processes with instantaneous correlation

$$\mathbb{E}\left[dZ^Q(t) dZ^Q(t)'\right] = \rho dt.$$

The stochastic volatility factor $V(t)$ has dynamics

$$\begin{aligned} dV(t) &= vV(t)\, dW^Q(t) \qquad V(0) = \alpha, \\ \mathbb{E}\left[dW^Q(t) dZ_k^Q(t)\right] &= \rho_{k,V} dt \quad \forall k. \end{aligned} \tag{1}$$

where the vector $\rho^V = [\rho_{1,V}, \ldots, \rho_{M,V}]'$ expresses the rate-volatility instantaneous correlation. Since the volatility process is lognormal, one can set $\alpha = 1$ with no loss of generality, since any different initial value can be embedded in the model by adjusting the deterministic coefficients σ_k. This is the choice we adopt in the following. However, this will have some consequences on the dynamics we obtain under different measures, as shown in the next sections.

3.2 Change of numeraire: Forward and Swap Measures

There are standard results for the change of diffusions when we move from a measure $\mathbb{Q}_1$ associated to numeraire N^1, to a measure $\mathbb{Q}_2$ associated to numeraire N^2. We follow Geman et al. (1995) and apply the change of measure to the SABR diffusion, assuming one single volatility factor. The extension to the case of multiple volatility factors can be performed along the same lines.

Assume that under a measure $\mathbb{Q}_1$ associated with numeraire N^1 the dynamics of the process X is given by

$$dX_t = U_t dt + S_t dW_t^1, \tag{2}$$

where X, U are n-dimensional column vectors and W^1 is an n-dimensional brownian motion under $\mathbb{Q}_1$ with instantaneous correlation matrix Ξ. The matrix S_t is $n \times n$ and diagonal. Then the dynamics under an equivalent measure $\mathbb{Q}_2$ associated with N^2 is given by

$$dW_t^1 = dW_t^2 - \Xi \Sigma_{1,2}(t)'\, dt \tag{3}$$

where $\Sigma_{1,2}(t)$, or DC $\left(\ln \frac{N_t^1}{N_t^2}\right)$, where DC stands for diffusion coefficient, is defined by

$$d\left(\ln \frac{N_t^1}{N_t^2}\right) = \ldots dt + \Sigma_{1,2}(t)\, dW_t^x, \quad x = 1, 2.$$

We compute how the drifts change when we move from the reference spot measure Q to a generic forward measure Q^k, which is the pricing measure In this case

$$\begin{aligned} N_t^1 &= \frac{P(t, T_{\gamma(t)})}{\prod_{j=0}^{\gamma(t)} P(T_{j-1}, T_j)}, \\ N_t^2 &= P(t, T_k) \end{aligned}$$

with $k > \gamma(t)$. The vector process we consider is

$$X_t = [V(t), F_1(t), \ldots, F_M(t)],$$

and we use vector notation for the dynamics, as done for the process X_t in (2). In this notation, for example, the diffusion coefficient for $F_j(t)$ is $\sigma_j V(t) F_j(t)^\beta u^{j+1}$, where u^{j+1} is the$(j+1)$-th elementary $(M+1)$-dimensional row vector $[0, \ldots, 1, \ldots, 0]$, with 1 in the $(j+1)$-th position.

Computing the change to be made to the stochastic shocks for performing this change of measure requires, according to (3), the computation of

$$\begin{aligned} & DC\left[\ln\left(\frac{P(t, T_{\gamma(t)})}{\prod_{j=0}^{\gamma(t)} P(T_{j-1}, T_j) P(t, T_k)}\right)\right] \\ = \; & DC\left[\ln\left(\frac{P(t, T_{\gamma(t)})}{P(t, T_k)}\right)\right] - DC\left[\ln\left(\prod_{j=0}^{\gamma(t)} P(T_{j-1}, T_j)\right)\right] \end{aligned}$$

At time t, $\ln\left(\prod_{j=0}^{\gamma(t)} P(T_{j-1}, T_j)\right)$ is not anymore stochastic. We are left with

$$\begin{aligned} DC\left[\ln\left(\frac{P(t, T_{\gamma(t)})}{P(t, T_k)}\right)\right] &= DC\left[\ln\left(\frac{P(t, T_{\gamma(t)})}{P(t, T_{\gamma(t)+1})} \frac{P(t, T_{\gamma(t)+1})}{P(t, T_{\gamma(t)+2})} \cdots \frac{P(t, T_{k-1})}{P(t, T_k)}\right)\right] \\ &= DC\left[\ln \prod_{j=\gamma(t)+1}^{k} (1 + \tau_j F_j(t))\right] = \sum_{j=\gamma(t)+1}^{k} DC\left[\ln(1 + \tau_j F_j(t))\right] \\ &= \sum_{j=\gamma(t)+1}^{k} \frac{\tau_j}{1 + \tau_j F_j(t)} DC\left[F_j(t)\right] = \sum_{j=\gamma(t)+1}^{k} \frac{\tau_j \sigma_j V(t) F_j(t)^\beta u^{j+1}}{1 + \tau_j F_j(t)} \end{aligned}$$

The correlation matrix to consider is

$$\Xi = \begin{bmatrix} 1 & \left(\rho^V\right)' \\ \rho^V & \rho \end{bmatrix}$$

so that

$$\begin{aligned} dW^Q(t) &= dW^k(t) - V(t)\, \mu_t(\gamma(t), k; V)\, dt, \\ dZ_k^Q(t) &= dZ_k^k(t) - V(t)\, \mu_t(\gamma(t), k; k)\, dt, \end{aligned}$$

where

$$\mu_t(\alpha, \beta; x) = \sum_{j=\alpha+1}^{\beta} \frac{\tau_j \rho_{j,x} \sigma_j F_j^\beta(t)}{1 + \tau_j F_j(t)}. \tag{4}$$

The results obtained for a forward rate $F_j(t)$ coincide with standard results for the Libor Market model. They are less standard for the stochastic volatility, since they lead us, following (3), to the following dynamics under the $F_k(t)$ natural forward measure

$$dV(t) = -vV^2(t)\, \mu_t(\gamma(t), k; V)\, dt + vV(t) dW^k(t).$$

When moving to the swap measure $Q^{a,b}$, the associated numeraire is

$$N_t^2 = \sum_{k=a+1}^{b} \tau_k P(t, T_k)$$

so

$$DC\left[\ln\left(\frac{N_t^1}{N_t^2}\right)\right] = DC\left[\ln\left(\frac{P(t, T_{\gamma(t)})}{\prod_{j=0}^{\gamma(t)} P(T_{j-1}, T_j) \sum_{k=a+1}^{b} \tau_k P(t, T_k)}\right)\right]$$

Following the same change of numeraire rules seen above, the final relationship between stochastic shocks is

$$dW^Q(t) = dW^{a,b}(t) - \mu_t^{a,b}(\gamma(t))\, V(t)dt,$$

where

$$\mu_t^{a,b}(\gamma(t)) = \sum_{k=a+1}^{b} w_k(t)\, \mu_t(\gamma(t), k; V).$$

3.3 Swap Rate Dynamics under the Swap Measure

In our model setting, computing a swap model associated to the above Libor model can be done following relatively standard procedures. It is a general feature of Libor Market models that they do not imply for swap rates stochastic dynamics consistent with the one assumed for Libor rates. However, the discrepancy is practically so reduced that often a few simple adjustments (based on fixing some low variability quantities to their initial value) allow a perfect recovery of the Libor dynamics with a negligible approximation error.

We will see that, thanks to the presence of one single stochastic volatility factor associated to all forward rates (with different correlations controlling the different caplet skews), the same single stochastic volatility factor affects also swap rates (with different correlations controlling the different swaption skews). This reduces the problem of finding a swap volatility in terms of Libor volatilities and correlations, to performing computations on the CEV dynamics of forward rates. This procedure has already been tested by Andersen and Andreasen (2000) and Andersen and Brotherton-Ratcliffe (2001).

According to the no-arbitrage relationships,

$$S_{a,b}(t) = \sum_{j=a+1}^{b} \frac{P(t, T_j)\tau_j}{\sum_{i=a+1}^{b} P(t, T_i)\tau_i} F_j(t) = \sum_{j=a+1}^{b} w_j(t)\, F_j(t) \tag{5}$$

by Ito's Lemma we have that under $Q^{a,b}$

$$dS_{a,b}(t) = \sum_{j=a+1}^{b} \frac{\partial S_{a,b}(t)}{\partial F_j(t)} F_j^{\beta}(t)\sigma_j V(t) dZ_j^{a,b}(t). \tag{6}$$

We define

$$\gamma_j(t) = \frac{\partial S_{a,b}(t)}{\partial F_j(t)} \frac{F_j^{\beta}(t)}{S_{a,b}^{\beta}(t)} =: \tilde{w}_j(t) \frac{F_j^{\beta}(t)}{S_{a,b}^{\beta}(t)},$$

$$dS_{a,b}(t) = S_{a,b}^{\beta}(t) V(t) \sum_{j=a+1}^{b} \gamma_j(t)\sigma_j dZ_j^{a,b}(t).$$

The process is still not tractable since the $\gamma_j(t)$ are state dependent. However they have a very low variability, so that it is safe to approximate them with $\gamma_j(0)$. Now we need to express the forward rate volatility functions as one single swap rate volatility function, and the forward rate stochastic drivers as one single swap rate stochastic driver. Thus we set

$$\sigma_{a,b} = \sqrt{\sum_{k=a+1}^{b} \sum_{h=a+1}^{b} \gamma_k(0)\sigma_k \gamma_h(0)\sigma_h \rho_{k,h}}\,, \quad dZ_{a,b}^{a,b}(t) = \frac{\sum_{j=a+1}^{b} \gamma_j(0)\sigma_j\, dZ_j^{a,b}(t)}{\sigma_{a,b}}.$$

Notice that, as desired,

$$\mathbb{E}\left[\sum_{j=a+1}^{b}\gamma_j(0)\sigma_j dZ_j^{a,b}(t)\cdot\sum_{j=a+1}^{b}\gamma_j(0)\sigma_j dZ_j^{a,b}(t)\right] = \sigma_{a,b}^2 dt$$

$$\mathbb{E}\left[dZ_{a,b}^{a,b}(t)dZ_{a,b}^{a,b}(t)\right] = \frac{\sigma_{a,b}^2 dt}{\sigma_{a,b}^2} = dt$$

In the market practice, often the computation is further simplified by freezing directly the $w_i(t)$ to their time-zero value in (5). See Jackel and Rebonato (2000) for a sound justification for this passage. In this case we would have

$$\sigma_{a,b} = \frac{1}{S_{a,b}^{\beta}(0)}\sqrt{\sum_{k=a+1}^{b}\sum_{h=a+1}^{b} w_k(0)F_k^{\beta}(0)\sigma_k w_h(0)F_k^{\beta}(0)\sigma_h\rho_{k,h}}.$$

In the context of a stochastic volatility model with general rate/volatility correlation, one also needs the correlation $\rho_{a,b}^V$ between the swap rate and the stochastic volatility factor. The application of the above procedure to correlations yields

$$\begin{aligned}\rho_{a,b}^V dt &= \mathbb{E}\left[dW^{a,b}(t)dZ_{a,b}^{a,b}(t)\right] = \\ &= \frac{\sum_{j=a+1}^{\beta}\gamma_j(0)\sigma_j\mathbb{E}\left[dW^{a,b}(t)dZ_j^{a,b}(t)\right]}{\sigma_{a,b}} = \frac{\sum_{j=a+1}^{\beta}\gamma_j(0)\sigma_j\rho_{j,V}dt}{\sigma_{a,b}}\end{aligned}$$

See again Rebonato and Jackel (1999), Andresen and Andreasen (2000) or Wu and Zhang (2006) for results on the accuracy of these freezing approximations. The literature on the LMM provides various examples about the efficiency of these approximations in general contexts. However, their efficiency in this specific context is implicitly assessed in the tests again Montecarlo simulation which are performed in Section 5 for assessing the goodness of the approximations we introduce below.

4 Approximations for a tractable dynamics consistent with SABR formula

When the Libor SABR model just introduced is implemented though Monte Carlo Simulation, as in our results of Section 5, the above dynamics are sufficient for pricing and hedging a very general range of financial products, using a range of techniques which are standard of Libor models and summarize by Piterbarg (2005b).

4.1 First Approximation: Volatility Drift Freezing

The term $\mu_t(\gamma(t), k; V)$ depends on the dynamics of the entire term structure of forward rates, and this obviously prevents from having any closed-form formula, even for simple products. An analogous result is found in Wu and Zhang (2006) for the LMM version of the celebrated Heston model. Wu and Zhang (2006), consistently with all the literature on the LMM, observe that the variability of the drift correction obtained is negligible compared to the variability of the state variables in the models, so that it is safe to approximate the values $F_j(t)$ that appear in the drift correction with their initial value $F_j(0)$. In our context, this leads to

$$\mu_t(\gamma(t), k; V) \approx \mu_0(\gamma(t), k; V) = \sum_{j=\gamma(t)+1}^{k}\frac{\tau_j\sigma_j\rho_{j,V}F_j(0)^{\beta}}{1+\tau_j F_j(0)}, \tag{7}$$

that for constant volatilities turns out to be a deterministic, piecewise constant drift that will allow us to conserve the tractability of the model for reference options. One may argue that $\mu_t(\gamma(t), k; V)$ tends to be more volatile than the drifts obtained by Wu and Zhang (2006), since they, among other differences, assume $\beta = 1$. However, we have to notice that β is concurrent to $\rho_{j,V}$ in fitting the skewness in market quotations. In case it happened that $\beta \ll 1$, this is associated to values $\rho_{j,V}$ very close to zero, setting both $\mu_t(\gamma(t), k; V)$ and its volatility to extremely low values. This happens in all of our calibration cases, where in any case most of the times (with the obvious exception of the case when we impose $\rho_{j,V} = 0$) the exponent β turns out to be in the range 0.8-1. This may explain why in the numerical comparisons with non-approximated montecarlo simulation shown in Section 5 the approximation (7) does not spoil precision.

The approximation for making tractable the no-arbitrage drift under the swap measure is totally analogous to the Wu and Zhang (2006) case and based on the analysis performed by Rebonato and Jackel (1999), showing the low variability of the weights $w(t)$ that appear in (5) and also in the swap measure volatility drift $\mu_t^{a,b}(\gamma(t))$. Accordingly, we set

$$\begin{aligned} dW^{a,b}(t) &\approx dW^Q(t) + \mu_0^{a,b}(\gamma(t))\, V(t)dt, \\ \mu_0^{a,b}(\gamma(t)) &= \sum_{j=a+1}^{b} w_j(0)\, \mu_0(\gamma(t), j; V). \end{aligned}$$

In the following we take as reference the dynamics under a generic forward measure Q^k, since the extension to the swap measure is trivial.

4.2 Second Approximation: Projecting Volatility Onto Lognormal Dynamics

We mentioned in the introduction that, after the freezing approximation described above, Wu and Zhang (2006), that start from a mean reverting, square root variance dynamics, find a dynamics under Q^k which is again a mean reverting, square root variance dynamics, just with modified parameters. Therefore, if one assumes under a reference measure the standard dynamics which characterizes the celebrated Heston (1994) model, changing to one of the natural pricing measures associated to interest rate options does not alter the type of dynamics we have.

Instead, when starting from the initial assumptions that characterize the SABR model, change of measure leads from reference dynamics

$$dV(t) = vV(t)\, dW^Q(t) \tag{8}$$

to a Q^k dynamics which, starting from the above results and approximating the less-volatile drift correction with a deterministic term, is

$$dV(t) = -vV^2(t)\, \mu_0(\gamma(t), k; V)\, dt + vV(t)dW^k(t) \tag{9}$$

The dynamics is not of the same driftless, geometric brownian motion type we started from. The drift is not proportional to the volatility factor, but to its square. The distribution of model variables has been modified in a non-trivial way. This can cause two problems. First, this new solution could have a less regular behaviour. Secondly, it could prevent us from using again SABR European Option formula, the feature of the SABR model which is responsible for its popularity in the market. Our desiderata would be to incorporate this no-arbitrage dynamics into the same SABR formula, with just a correction to the input parameters.

This means first of all performing what we call *Second Approximation*: mapping the above dynamics into a particular lognormal dynamics showing as much as possible a similar behaviour.

4.2.1 Equation-based lognormal approximation by simple freezing

The first, rough approach to obtain the same type of dynamics we started from, is to freeze to their time-zero value all stochastic quantities in the change-of-measure correction, including stochastic volatility itself. This corresponds to setting

$$dW^Q(t) \approx dW^k(t) - V(0)\, \mu_0(\gamma(t), k; V)\, dt$$

which includes freezing $V(t)$ to its initial level $V(0)$. This leads to the approximated lognormal dynamics

$$dV(t) = -v\mu_0\left(\gamma\left(t\right), k; V\right)\ V(0)\ V\left(t\right)dt + vV(t)dW^k\left(t\right), \tag{10}$$

This approximation is derived in a very rough way. So it is useful to investigate into the SDE (9) we found after the change of measure. Solving the equation would allow us to assess the behaviour of (9) and can help us in developing a more well-founded approximation.

4.2.2 Solution for the forward-measure SABR volatility SDE

The drift coefficient in (9) is

$$drift\left(t, X\right) = c_t X^2.$$

where $c_t = -v\mu_0\left(\gamma\left(t\right), k; V\right)$. We can find a semi closed-form solution as follows. Starting from

$$dV(t) = c_t V\left(t\right)^2 + vV(t)dW^k\left(t\right),$$

we first compute by Ito's Lemma the dynamics of $\frac{1}{V(t)}$:

$$d\left(\frac{1}{V(t)}\right) = \left(\frac{v^2}{V(t)} - c_t\right)dt - \frac{v}{V(t)}dW^k\left(t\right).$$

This equation is a standard linear SDE in $\frac{1}{V(t)}$. Following for example Kloeden and Platen (1995), we have

$$V(t) = \tilde{V}\left(t\right)\left[\frac{1}{V\left(0\right)} - \int_0^t c_u \tilde{V}\left(u\right)du\right]^{-1}, \tag{11}$$

where the auxiliary process $\tilde{V}\left(t\right)$ is

$$\tilde{V}\left(t\right) = \exp\left\{-\frac{1}{2}v^2 t + v\int_0^t dW^k\left(s\right)\right\}.$$

Notice $\tilde{V}\left(t\right)$ is the solution of the following SDE:

$$d\tilde{V}\left(t\right) = v\tilde{V}\left(t\right)dW^k\left(t\right); \quad \tilde{V}\left(0\right) = 1.$$

This is lognormal driftless, just the dynamics of stochastic volatility in the standard SABR model under the martingale reference measure, thus the term

$$\varphi_t = \left[\frac{1}{V\left(0\right)} - \int_0^t c_u \tilde{V}\left(u\right)du\right]^{-1}$$

represents exactly the correction made to the stochastic volatility factor to take into account no-arbitrage constraints under a different pricing measure. Thus, a correct account of no-arbitrage leads to a "convexity adjustment" that depends on some average value of the stochastic volatility from now to the option maturity. Notice that, in principle, this solution can have an explosive behaviour, reaching infinity in finite time, depending on the sign and the magnitude of the piecewise constant function c_t. If the correlations $\rho_{j,V}$ between stochastic volatility and forward rates are all positive then $\mu_0\left(\gamma\left(0\right), k; V\right)$ is positive and c_t is negative, guaranteeing that $V(t)$ does not explode at finite time. When they are all zero, then $c_t = 0$. Otherwise c_t can be positive, implying possibility of explosion of volatility when $\int_0^t c_u \tilde{V}\left(u\right)du$ approaches $1/V\left(0\right)$.

However, we point out that (11) applies to the forward measure Q^k, whose numeraire expires at T_k which is usually the maturity of the payoff one needs to evaluate. With market values of model parameters, c_t is extremely low, so that the expected explosion time is beyond any possible financial maturity. This will be pointed out even clearer in the next section, where both expectation and variance of $\int_0^t c_u \tilde{V}\left(u\right)du$ are computed.

While the model remains practically reliable, this behaviour confirms that lognormality of stochastic volatility, typical of the SABR dynamics, can have undesirable implications, in particular if we compare these results with the more regular dynamics obtained starting from a standard Heston-like volatility.

4.2.3 Solution-based lognormal approximation by variability analysis

The "convexity correction" φ_t applied to the lognormal $\tilde{V}(t)$ to take into account no-arbitrage constraints is a stochastic quantity. Therefore $V(t)$ is not lognormal, while we have seen that for using the SABR formula we should first of all find a valid lognormal approximation. This requires approximating the convexity adjustment with a deterministic function, for example replacing some low variability quantities with their expectation, as often done in financial approximations. We have to choose a quantity giving a reduced contribution to the total variability of $V(t)$, and such that if we replace it with its expectation we find a lognormal dynamics.

A good candidate is the integral $\int_0^t c_u \tilde{V}(u)\, du$, considering that the average of a stochastic process tends to be less volatile than the process itself, as is well known from Average Price Asian options. We can be more precise by computing the first and second moments of the stochastic quantity $\int_0^t c_u \tilde{V}(u)\, du$ and checking the magnitude of its variance compared to the magnitude of its expected value, and compared to the variance of $\tilde{V}(t)$, which is the leading term in the solution we have found. Only if the former was negligible compared to the two latter quantities, the approximation of $\int_0^t c_u \tilde{V}(u)\, du$ with its expected value would have some justification.

We can first write, using integration by parts,

$$\begin{aligned}
\int_0^t c_u \tilde{V}(u)\, du &= \int_0^t c_u du \tilde{V}(t) - \int_0^t \int_0^u c_s ds d\tilde{V}(u) \\
&= \int_0^t c_u du + \int_0^t \left(\int_0^t c_u du - \int_0^u c_u du \right) d\tilde{V}(u) \\
&= \int_0^t c_u du + \int_0^t \left(\int_u^t c_s ds \right) v\tilde{V}(u) dW^k(u)
\end{aligned}$$

We set

$$I(t) = \int_0^t \left(\int_u^t c_s ds \right) v\tilde{V}(u) dW^k(u),$$

which is martingale, and we want to compute $Var(I(t)) = \mathbb{E}\left[I^2(t)\right]$. Thanks to Ito's Isometry,

$$Var[I(t)] = v^2 \int_0^t \left(\int_u^t c_s ds \right)^2 \exp\left(v^2 u\right) du$$

It can be easily proven by Cauchy-Schwartz inequality that

$$Var[I(t)] \le v^2 c^2 \int_0^t (t-u)^2 \exp\left(v^2 u\right) du$$

when $c = \sqrt{\max\left(c_u^2\right)}$. Since we aim at finding un upper boundary to the variance, for ease of notation we continue the computation with flat c.

Solving the integral, we obtain

$$Var[I(t)] = \frac{2c^2}{v^4} \exp\left(v^2 t\right) - \left[t^2 + \frac{2t}{v^2} + \frac{2}{v^4} \right],$$

so we have the first and second moments of $\int_0^t c\tilde{V}(u)\, du$:

$$\begin{aligned}
\mathbb{E}\left[\int_0^t c\tilde{V}(u)\, du \right] &= ct, \\
Var\left[\int_0^t c\tilde{V}(u)\, du \right] &= c^2 \left\{ \frac{2}{v^4} \exp\left(v^2 t\right) - \left[t^2 + \frac{2t}{v^2} + \frac{2}{v^4} \right] \right\}.
\end{aligned}$$

We know that

$$\frac{2}{v^4} \exp\left(v^2 t\right) = \frac{2}{v^4} \sum_{i=0}^{\infty} \frac{\left(v^2 t\right)^i}{i!}$$

so

$$Var\left[\int_0^t c\tilde{V}(u)\,du\right] = \frac{2c^2}{v^4}\sum_{i=3}^{\infty}\frac{\left(v^2t\right)^i}{i!} = 2c^2\sum_{i=3}^{\infty}\frac{v^{2i-4}t^i}{i!}.$$

This shows that $\int_0^t c\tilde{V}(u)\,du$ is a random variable with a variance which for small t is infinitesimal of order higher than 2 compared to its mean ct, and also compared to the variance of $\tilde{V}(t)$, which is $\exp\left(v^2t\right)-1$. Thus for small t the integral $\int_0^t c(u)\,\tilde{V}(u)\,du$ is almost deterministic, and it appears reasonable to approximate it with its expected value $\int_0^t c(u)\,du$,

$$\begin{aligned}V(t) &= \frac{\tilde{V}(t)}{\frac{1}{V(0)}-\int_0^t c(u)\,\tilde{V}(u)\,du}\\ &\approx \frac{\tilde{V}(t)}{\frac{1}{V(0)}-\mathbb{E}\left[\int_0^t c(u)\,\tilde{V}(u)\,du\right]} = \left[\frac{\tilde{V}(t)}{\frac{1}{V(0)}-\int_0^t c(u)\,du}\right].\end{aligned}$$

The validity of this approximation for long maturities depends on the actual size and sign of

$$c(t) = -v\mu_0(\gamma(t), k; V) = -v\sum_{j=\gamma(t)+1}^{k}\frac{\tau_j\rho_{V,j}\sigma_j F_j^{\beta}(0)}{1+\tau_j F_j(0)}.$$

Using parameters from calibration to the swaption market, the volatility of $\int_0^t c(u)\,\tilde{V}(u)\,du$ remains very low compared to the volatility of $\tilde{V}(t)$ for a very large range of possible maturities. See Figures 1 and 2, comparing the variance of $\tilde{V}(u)$ with the upper boundary of the variance of $\int_0^t c(u)\,\tilde{V}(u)\,du$.

The approximation seems valid also for all maturities of financial significance. This corresponds to a volatility dynamics given by the following Solution-based approximated lognormal dynamics**:**

$$dV(t) = \frac{c(t)}{\frac{1}{V(0)}-\int_0^t c(u)\,du}V(t)dt + vV(t)dW^k(t).$$

4.3 Third Approximation: a non-zero drift into SABR formula

Both forward and swap rates are CEV martingales under their natural measures,

$$\begin{aligned}dS_{a,b}(t) &= V(t)\sigma_{a,b}S_{a,b}^{\beta}(t)dZ_{a,b}^{a,b}(t),\\ dF_k(t) &= V(t)\sigma_k F_k(t)^{\beta}dZ_k^{Q}(t),\end{aligned}$$

and under any swap or forward measure the volatility, after the above approximations, is a process

$$dV(t) = M(t)\,V(t)dt + vV(t)dW(t)$$

where the drift $M(t)$ is measure-dependent but deterministic. For a forward measure Q^k, depending on the approximation chosen, the drift is

Equation-Based Dynamics

$$M(t) = c(t)\,V(0) = -v\mu_0(\gamma(t), k; V)\,V(0). \tag{12}$$

Solution-Based Dynamics

$$M(t) = \frac{c(t)}{\frac{1}{V(0)}-\int_0^t c(s)\,ds} = \frac{-v\mu_0(\gamma(t), k; V)}{\frac{1}{V(0)}+v\int_0^t \mu_0(\gamma(t), k; V)\,ds}. \tag{13}$$

We have a no-arbitrage term structure model where volatility, although not driftless, shares (under all interest-rate derivatives pricing measures) the same kind of (lognormal) distribution as the standard, single asset SABR model. However the SABR popular closed-form formula admits only driftless volatility with flat parameters, while we have a volatility drift $M(t)$ and it is time-dependent.

In our term structure model, with $V(0) = 1$, the volatility $V(T)$ at maturity T is

$$V(T) = e^{\int_0^T M(s)ds}\, e^{-\frac{1}{2}v^2T+vW(T)} \tag{14}$$

while in the reference SABR formula volatility $V^{SABR}(t)$ at maturity T is

$$V^{SABR}(T) = V^{SABR}(0)e^{-\frac{1}{2}v^2T+vW(T)}. \tag{15}$$

Thus the two solutions coincide at time T only when:

$$V^{SABR}(0) = e^{\int_0^T M(s)ds}. \tag{16}$$

We can act on the initial value of volatility in standard flat-parameter SABR formula in order to recover a desired final solution of volatility. However, European Options do not only depend only on the final distribution of volatility, but they depend on the entire path followed by the volatility from 0 to T. Thus, a less rough approximation is to assume that they depend on the average volatility from now to maturity. Accordingly we set $V^{SABR}(0) = \bar{V}_T$, where $\bar{V}_T$ has been chosen so that, approximately, the average values of $V(t)$ and $V^{SABR}(t)$ from 0 to T are matched.

This third approximation can be improved in different ways. One may set $\bar{V}_T$ in such a way that, for example, the root-mean-squared volatilities are approximately matched.[3] One may consider treating the volatility drift as part of a time dependent volatility coefficient $\sigma_k(t)$, and then replace it with a flat volatility by using Piterbarg (2005a) formula for approximating time-dependent parameters with flat parameters in stochastic volatility models. Otherwise, when treating the volatility drift as part of a time dependent volatility coefficient $\sigma_k(t)$ one could compute option prices with no further approximation by using the SABR formula with time dependent parameters, described in Hagan et al. (2002) but not common in the marketplace.

We plan to test these improvements in subsequent research. However, we first would like to understand if it is worth the effort. Our first goal is to obtain a simple way of taking no-arbitrage corrections into account, to see immediately: first, if the procedure, even with a rough final passage, results acceptably precise when compared with exact numerical methods (numerical tests); second, if these corrections are financially relevant (empirical tests). These tests are presented in the next section, and they appear to give a positive answer to both questions.

5 Numerical and Empirical tests for SABR Libor Market Model

We first present a few results on the comparison of our approximations to non-approximated numerical results. Then we test on market data the behaviour of the formulas linking cap and swaption market, and how the approximations we introduce influence the regularity of the parameters. Then we move to out of sample tests to see if having an approximately arbitrage free term structure model, like the one developed above, allows remarkable improvements in terms of consistency with market data, compared to a trivial application of SABR dynamics without no-arbitrage corrections.

5.1 Numerical Tests

When comparing our closed-form results with numerical methods, it is hard to disentangle the effect of the different approximations involved, including the one intrinsic to the standard SABR formula. We may do it as follows. First we want to assess the precision of the simplification made to the dynamics of the of the stochastic volatility to map it into a lognormal dynamics.

In order to test first the effect to simplifying the actual dynamics (9) with the lognormal approximated dynamics (13), we compare montecarlo price with (9) and montecarlo price with (13). Namely, we are testing the error due to First and Second Approximation, no other approximation involved. See the following example on a 5 into 1 swaption.

[3] We have tried this option but it seems that the increased numerical burden is more relevant than improvement in accuracy.

Strikes	No approximation (%)	GBM approximation (%)	Error (bps)
$S_{a,b}(0)$-1.00%	16.55	16.57	-2.13
$S_{a,b}(0)$-0.50%	15.89	15.90	-1.70
$S_{a,b}(0)$-0.25%	15.70	15.71	-1.57
$S_{a,b}(0)$+0.00%	15.59	15.60	-1.50
$S_{a,b}(0)$+0.25%	15.55	15.57	-1.34
$S_{a,b}(0)$+0.50%	15.58	15.59	-1.18
$S_{a,b}(0)$+1.00%	15.76	15.77	-0.83
$S_{a,b}(0)$+2.00%	16.43	16.44	-0.38

The errors are low, but they increase with strike. However even the highest error is less than 3 bps. It seems therefore that the most accurate of the above lognormal approximations of a square drift process is reliable.

In order to test the effect of simplifying the option computation by incorporating the drift $M(t)$ of (13) into a modified initial value $V^{SABR}(0) = \bar{V}_T$, we compare montecarlo price with (13) versus the montecarlo price obtained using zero drift and $V^{SABR}(0) = \bar{V}_T$. Namely, we are testing the error due to Third Approximation. Its error is summed to the errors due to the first and second approximation, already shown above. See the example on a 5×1 swaption:

Strikes	No approximation (%)	GBM frozen approximation (%)	Error (bps)
$S_{a,b}(0)$-1.00%	16.55	16.53	1.95
$S_{a,b}(0)$-0.50%	15.89	15.87	1.73
$S_{a,b}(0)$-0.25%	15.70	15.68	1.57
$S_{a,b}(0)$+0.00%	15.59	15.57	1.49
$S_{a,b}(0)$+0.25%	15.55	15.54	1.57
$S_{a,b}(0)$+0.50%	15.58	15.56	1.63
$S_{a,b}(0)$+1.00%	15.76	15.74	1.81
$S_{a,b}(0)$+2.00%	16.43	16.41	2.16

This third approximation brings about an error which is higher than the one due to our lognormal projection. In fact, it is the roughest among the approximations introduced, and we have already mentioned how it could be improved. However, even with roughest option fot the Third Approximation, the error tendency with strike is opposite to the one due to lognormal projection, keeping the total error under 3 bps.

Since the error in our approximation procedure seems concentrated on the third approximation, results could be improved by using Piterbarg (2005a) approximation or SABR formula with time-dependent parameters. However, we first need to check if the corrections found are financially significant, assessing what is their effect on the fitting, on the regularity of the parameters obtained and on prices of derivatives out of the calibration sample.

5.2 Empirical Tests

We first test the simple formulas introduced in 3.3 for linking swap and libor parameters. We perform calibration to the same data as Section 2 with the Libor and Swap SABR models just introduced. If the formulas using a swap or a libor model should not alter the results in terms of fitting or regularity of the parameters.

If the correlation $\rho_{k,V}$ between interest rates and volatility is zero, the volatility $V(t)$ has the same driftless dynamics under all pricing measures. We start from this assumption, so as to assess the behaviour of the swaption volatility formula of Section 3.3 with no influence of the drift-approximations introduced after for the dynamics of the stochastic volatility.

	Mean Error
SABR Swap model ($\rho_{k,V} = 0$)	23.14 bps
SABR Libor model ($\rho_{k,V} = 0$)	23.24 bps

As we expect for correct market models, modulo the swaption approximation the fitting of the LMM and the fitting of the SMM are practically identical. We also need to assess if the parameters obtained are regular and financially significant, in particular for the LMM and for the parameters which have a more direct link with market quantities, the deterministic volatilities σ_k.

The volatility coefficients $\sigma_{a,b}$ of a SMM tend to reflect the patterns that we observe on ATM swaption volatilities:

Det.Vol	ten=2	ten=5	ten=10
mat=1	0.0289	0.0291	0.0271
mat=5	0.0280	0.0262	0.0247
mat=10	0.0239	0.0222	0.0213

Instead the relationship between ATM swaption volatilities and Libor parameters σ_k is more involved, and the above regularities can be spoiled by the repartition of swap term volatility into Libor instantaneous volatilities. The term structure of Libor volatilities we obtain is the following

Mat	Det.Vol	10	0.115
1	0.028	11	0.040
2	0.030	12	0.018
3	0.034	13	0.021
4	0.035	14	0.025
5	0.018	15	0.014
6	0.022	16	0.017
7	0.025	17	0.020
8	0.028	18	0.024
9	0.008	19	0.029

Considering that $\beta = 0.47$, so that the above volatilities are intermediate between absolute and relative volatility parameters, they appear to be in a reasonable range of values. However they oscillate in a remarkable way, which is an unusual pattern for Libor volatilities and not consistent with market swaption volatilities. This could be due to the fact that our Libor model is overparameterized with respect to the number of data we have to fit: we consider 19 forward rates with only 9 smile curves, rather largely spaced in terms of maturities and tenors. It is expected that volatility distributes unevenly across different forward rates. It would be beneficial to increase the number of swaptions considered. In the following we assess results when including in calibration not only the available swaption smiles, but also all ATM quotations of a standard 10×10 swaption matrix. The term structure of volatilities is now

Mat	Det.Vol	10	0.023
1	0.028	11	0.022
2	0.028	12	0.022
3	0.028	13	0.021
4	0.027	14	0.018
5	0.028	15	0.021
6	0.026	16	0.020
7	0.025	17	0.018
8	0.024	18	0.019
9	0.023	19	0.016

This term structure is smoother, regular, and consistent with market patterns.

In these first tests we have set $\rho_{k,V} = 0$. This constraint makes the model arbitrage-free even without corrections to the volatility, however it has a strong influence both on the dynamical properties of the model and on the fitting capability, as shown in the tables below

Calibration to 9 smiles	Mean Error
SABR Libor Formula (not arbitrage free)	8.64 bps
SABR Libor Formula/model ($\rho_{k,V} = 0$, arbitrage free)	23.24 bps

Calibration to 9 smiles and 10×10 ATM	Mean Error
SABR Libor Formula (not arbitrage free)	9.03 bps
SABR Libor Formula/model ($\rho_{k,V} = 0$, arbitrage free)	17.31 bps

We see that a general correlation $\rho_{k,V} \neq 0$ improves the fitting. In this case, however, without no-arbitrage corrections to the volatility dynamics the formula is not compatible with an arbitrage-free Libor model. If we introduce the corrections to the volatility dynamics that we have developed, the model will be, at least approximately, arbitrage free.

With this regard, there are two main things that we want to assess:

1. if the market fitting remain good and LMM parameters remains regular even with no-arbitrage correction

2. what is the relevance of this correction on the price of financial products out of the calibration sample. If such prices turned out to be unchanged, we could consider the possibility of neglecting such correction drifts even for $\rho_{k,V} \neq 0$.

Now we let $\rho_{k,V}$ to be free parameters[4] and we take into account the corresponding no-arbitrage corrections. We first consider the Equation-Based approximation (12). The fitting amounts to 9.5 bps, namely the consistency to the market has not been altered by the introduction of the no-arbitrage drifts. As for the resulting parameters, the first expected finding is that we obtain higher β, $\beta = 0.817$ for the calibration including 10×10 ATM volatilities. Indeed we here allow $\rho_{k,V}$ to be negative, and this is concurrent with $\beta < 1$ for fitting market skew. Higher β implies that deterministic volatility coefficients will be higher than above, since now they are more similar to relative volatilities. Precisely, they are

SABR LMM (Eq approx) (adding 10×10 ATM)			
Mat	Det.Vol	10	0.069
1	0.089	11	0.068
2	0.088	12	0.066
3	0.088	13	0.070
4	0.087	14	0.069
5	0.085	15	0.070
6	0.078	16	0.064
7	0.076	17	0.056
8	0.074	18	0.056
9	0.090	19	0.056

We see that when the model is correctly parameterized, again volatility parameters are regular and consistent with market patterns, even if we have allowed for general correlation structure and we have modified volatility through approximation for taking no-arbitrage constraints into account. We can also assess the size of the volatility corrections $\bar{V}_T$ that we have obtained.

	1	2	3	4	5	6	7	8	9	10
1	1.00000	1.00006	1.00006	1.00007	1.00013	1.00020	1.00026	1.00029	1.00033	1.00043
2	1.00013	1.00005	1.00006	1.00021	1.00036	1.00049	1.00055	1.00063	1.00085	1.00104
3	0.99997	1.00005	1.00038	1.00065	1.00087	1.00097	1.00109	1.00145	1.00176	1.00207
4	1.00013	1.00075	1.00115	1.00146	1.00155	1.00170	1.00223	1.00267	1.00311	1.00340
5	1.00140	1.00176	1.00208	1.00209	1.00222	1.00295	1.00352	1.00410	1.00446	1.00496
7	1.00287	1.00245	1.00260	1.00410	1.00511	1.00608	1.00659	1.00737	1.00840	1.00958
10	1.01092	1.01138	1.01245	1.01250	1.01350	1.01517	1.01717	1.01923	1.02090	1.02226

The corrections appear quite stable across maturity and tenors, related to the level of rate-volatility correlation (that in turn are related to the market skew) and rather small. Such limited size may rise suspicion that they are irrelevant, and we could have neglected such correction even for $\rho_{k,V} \neq 0$.

In order to make a more precise assessment of this point, we considered the prices of European swaptions not included in our calibration set. We report results regarding swaptions with the first maturity not included in our smile calibration set: the maturity 2 years, intermediate between maturities 1 and 5 years included in the calibration set. For the 2 year tenor we have:

[4] It is clear that introducing this additional correlation parameter arises an issue of correct modeling of the global correlation structure in the model. In particular, the correlation matrix including cross-rate correlations and rate-volatility correlations must be positive semidefinite. We tackle this problem by using the correlation structure described in Mercurio and Morini (2007).

Strikes	Volatilities **with Eq Corr**	Volatilities **without** Corr	**Difference** (bps)
-1.00%	17.34%	17.70%	35.57
-0.50%	16.43%	16.57%	13.87
-0.25%	16.18%	16.21%	2.18
0.00%	16.05%	15.96%	9.46
0.25%	16.03%	15.82%	20.54
0.50%	16.08%	15.78%	30.69
1.00%	16.38%	15.91%	47.51
2.00%	17.34%	16.65%	69.02

We see that, even for a simple European swaption with low maturity, implied volatilities show some differences compared to the values computed neglecting no-arbitrage correction, and for some strikes these differences are relevant.

We now consider the second approximation we have developed, the formula (13) that we expect to be more accurate since based on the actual solution of the volatility SDE, and we see if these results are confirmed.

First we compute the fitting error, which is 9 bps, a minimal difference from what we found before. We then check if the parameters obtained are again financially significant. This approximation appears less stable in calibration, in particular when β is very small (very small β obviously increases the magnitude of all no arbitrage corrections, because of (4)). Thus we keep a constant β to the level 0.817 that we found before. With this provision calibration is fast and stable, and we obtain analogous regularity patterns

SABR LMM (Sol Corr) (adding 10×10 ATM)			
k	σ_k	10	0.069
1	0.089	11	0.068
2	0.088	12	0.067
3	0.088	13	0.072
4	0.085	14	0.067
5	0.086	15	0.066
6	0.080	16	0.062
7	0.082	17	0.056
8	0.079	18	0.056
9	0.081	19	0.054

The corrections $\bar{V}$ are

	1	2	3	4	5	6	7	8	9	10
1	1.00000	1.00006	1.00006	1.00009	1.00011	1.00018	1.00022	1.00031	1.00044	1.00051
2	1.00013	1.00005	1.00011	1.00016	1.00031	1.00039	1.00060	1.00088	1.00103	1.00118
3	0.99996	1.00016	1.00028	1.00056	1.00069	1.00105	1.00153	1.00178	1.00200	1.00225
4	1.00038	1.00047	1.00090	1.00107	1.00161	1.00231	1.00265	1.00296	1.00329	1.00348
5	1.00056	1.00134	1.00151	1.00230	1.00331	1.00373	1.00411	1.00453	1.00475	1.00516
7	1.00184	1.00390	1.00589	1.00635	1.00679	1.00734	1.00754	1.00812	1.00898	1.00997
10	1.00779	1.00842	1.00944	1.00941	1.01046	1.01212	1.01398	1.01583	1.01735	1.01865

So, in spite of the theoretically possible explosive behaviour, corrections are bounded and rather low. When we use this more accurate approximation, the implied volatilities for the out of sample swaptions are:

Strikes	Volatilities **with Sol Corr**	Volatilities **without** Corr	**Difference** (bps)
-1.00%	17.44%	17.70%	25.71
-0.50%	16.46%	16.57%	11.54
-0.25%	16.17%	16.21%	3.94
0.00%	15.99%	15.96%	3.63
0.25%	15.93%	15.82%	10.84
0.50%	15.95%	15.78%	17.44
1.00%	16.19%	15.91%	28.38
2.00%	17.08%	16.65%	42.22

Although it is clear that no-arbitrage corrections have some relevance, one may wonder if these correction are in the "right direction" or are only an additional noise. It is possible to give an answer to this question, because the quotations for this 2×2 swaption are actually provided by the market, although not included in our calibration set. We can check out how market quotations compare with the results implied by the calibrated models in the three cases of interest: no-correction, equation-based correction, solution-based correction.

Strikes	**Market Vols**	**Error with NoCorr**	**Error with EqCorr**	**Error with SolCorr**
-1.00%	17.5%	19.9534	15.6207	5.7544
-0.50%	16.4%	17.3287	3.4610	5.7901
-0.25%	16.2%	0.5012	1.6772	3.4369
0.00%	16.0%	4.2299	5.2286	0.6025
0.25%	16.0%	18.0093	2.5341	7.1708
0.50%	16.0%	22.3187	8.3730	4.8761
1.00%	16.3%	39.1237	8.3909	10.7483
2.00%	17.3%	64.5629	4.4560	22.3452

We see that even in this first simple case, the consistency with market quotations out of the calibration sample is always improved by the introduction of the no-arbitrage corrections.

Now we test if these results are confirmed considering a swaption of higher maturity, in this case 5 years, after calibrating only to maturities 1,2, and 10 years. First we check the relevance of the correction introduced by the solution-based dynamics.

Strikes	Volatilities **with Sol Corr**	Volatilities **without** Corr	**Difference** (bps)
-1.00%	17.16%	17.77%	61.08
-0.50%	15.93%	16.28%	34.78
-0.25%	15.48%	15.68%	19.37
0.00%	15.14%	15.16%	2.75
0.25%	14.89%	14.75%	14.56
0.50%	14.74%	14.42%	31.90
1.00%	14.69%	14.04%	64.06
2.00%	15.17%	14.06%	110.09

The relevance of the correction increases with maturity, with a difference to the non-corrected (and not arbitrage-free) case which can exceed 100bps. In the table below we show also for this higher maturity how real market quotations compare with the results implied by the model, in the no-correction case and with the more accurate solution-based correction.

Strikes	**Market Vols**	Error: **Zero Drift**	Error: **Dyn-based**	Error: **Sol-based**
$S_{a,b}(0)$-1.00%	16.86%	91.5	40.7	30.4
$S_{a,b}(0)$-0.50%	15.81%	47.2	17.5	12.4
$S_{a,b}(0)$-0.25%	15.45%	22.5	5.3	3.1
$S_{a,b}(0)$+0.00%	15.20%	3.7	7.5	6.5
$S_{a,b}(0)$+0.25%	15.04%	29.3	19.0	14.8
$S_{a,b}(0)$+0.50%	14.96%	53.6	28.9	21.7
$S_{a,b}(0)$+1.00%	15.01%	96.7	45.4	32.6
$S_{a,b}(0)$+2.00%	15.57%	150.8	61.0	40.7
Average error (bps)		61.9	28.2	20.3

Previous considerations appear confirmed. Further analysis of this issue should assess on a larger range of products, including exotic derivatives, the actual relevance of no-arbitrage corrections on pricing and hedging; the analysis should be repeated for different trading days, so as to assess the stability in time of the regularities appeared in the above tests. This is under analysis.

6 Conclusions

Some of the open issues regarding the SABR model, whose closed-form formula is now a market standard for dealing with swaption smile, are well outlined in the introduction of Zhu (2007). Zhu (2007) recalls that "the volatility in SABR model displays no mean-reversion on the contrary to the well documented empirical features of a volatility process", "it is not clear which processes should Libors and stochastic volatilities follow under a unique forward measure" and "there is no link between cap pricing and swaption pricing in SABR model".

In this work we address the issues related to the arbitrage-free implementation of a term structure model consistent with SABR dynamics, designed to remain simple and synthetic but providing at the same time good fit to market data. Using change of numeraire theory, we compute the joint dynamics followed by Libor rates and stochastic volatility of SABR kind under the general pricing measures used for interest rate derivatives. The stochastic volatility SDE under a forward or swap measure turns out to be non-standard compared to results in the related literature. Based on the analysis of the equation found, we develop and justify theoretically a few approximations aimed at making these no-arbitrage adjustments compatible with the use of the SABR closed-form formula, which is the standard for smile-consistent pricing in the swaption market.

Then the formulas developed above are confronted both with alternative numerical implementations and with market data. We verify that the formulas for no-arbitrage corrections are acceptably precise, maintain the good fitting allowed by standard SABR swaption implementation, and produce regular Libor parameters. We also quantify their effect on the price of derivatives out of the calibration sample.

The latter tests show that our no-arbitrage corrections have a non-negligible impact also on vanilla option, and that the corrections make model out-of-sample prices closer to market quotations, compared to prices implied by a model neglecting such correction.

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