LIBOR Market Models with Stochastic Basis

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Abstract

We extend the LIBOR market model to accommodate the new market practice of using different forward and discount curves in the pricing of interest-rate derivatives. Our extension is based on modeling the joint evolution of forward rates belonging to the discount curve and corresponding spreads with FRA rates. We start by considering general stochastic-volatility dynamics and show how to address both the caplet and swaption pricing problems in general. We then consider specific examples, including a model for the simultaneous evolution of different rate and spread tenors. We conclude the article with an example of calibration to real market data.

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1 Introduction

Until the 2007 credit crunch, market quotes of interest rates consistently followed classic no-arbitrage rules. For instance, a floating rate bond where rates are set in advance and paid in arrears, was worth par at inception, irrespectively of the underlying tenor. Also, a forward rate agreement (FRA) could be replicated by long and short positions in two deposits, with the implied forward rate differing only slightly from the corresponding quantity obtained through OIS rates.

When August 2007 arrived, the market had to face an unprecedented scenario. Interest rates that until then had been almost equivalent, suddenly became unrelated, with the degree of incompatibility that worsened as time passed by. For instance, the forward rate implied by two deposits, the corresponding FRA rate and the forward rate implied by the corresponding OIS quotes became substantially different, and started to be quoted with

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large, non-negligible spreads. This discrepancy of values immediately raised issues in the construction of zero-coupon curves, which clearly could no longer be based on traditional bootstrapping procedures. A new judgment was needed, for example, to decide what market rates made sense to calibrate to and for what purpose.

In fact, differences between (same-currency) rates referring to the same time interval have always been present in the market. For instance, deposit rates and OIS rates for the same maturity would closely track each other, but keeping a distance (spread) of few basis points. Likewise, swap rates with the same maturity, but based on LIBOR rates with different tenors, would be quoted at a non-zero (basis) spread. All these spreads were generally regarded as negligible and, in fact, often assumed to be zero when constructing zero-coupon curves or pricing interest-rate derivatives.

To comply with the new market features, as far as yield curves are concerned, practitioners seem to agree on an empirical approach. For each given contract, they select a specific discount curve, which they use to calculate the net present value (NPV) of the contract's future payments, consistently with the contract's features and the counterparty in question. They then build as many forward LIBOR curves as given market tenors (1m, 3m, 6m, 1y), see *e.g.* Ametrano and Bianchetti (2009). With this approach, future cash flows are generated by the curves associated with the underlying rate tenors and their NPV is calculated through the selected discount curve.[1]

The assumption of distinct discount and forward curves, for a same currency and in absence of default risk, immediately invalidates the classic pricing principles, which were built on the cornerstone of a unique, and fully consistent, zero-coupon curve, containing all relevant information about the (risk-neutral) projection of future rates and the NPV calculation of associated pay-outs. A new model paradigm is thus needed to accommodate the market practice of using multiple interest-rate curves for each given currency.

In this article, we will show how to extend the general (stochastic-volatility) LIBOR market model (LMM) to the multi-curve setting. Our extended version of the LMM is based on modeling the joint evolution of FRA rates, that is the fixed rates that give zero value to the related forward rate agreements, and forward rates belonging to the selected discount curve. This extension was first proposed by Mercurio (2009, 2010), who considered lognormal dynamics for given-tenor FRA rates, and then added stochastic volatility to their evolution. We here follow a different approach, and explicitly model the basis between OIS and FRA rates. This makes our LMM extension closer to the market practice of building (forward) LIBOR curves at a spread over the OIS one. Remarkably, introducing a stochastic basis adds flexibility to the model, without compromising its tractability, as we will show by deriving closed-form formulas for cap and swaption prices and by considering an example of calibration to market caplet data. A similar approach has been recently proposed by Fujii et al. (2009b) who model stochastic basis spreads in a HJM framework both in single- and multi-currency cases, but without providing examples of dynamics or

[1] Different curves for generating future rates and for discounting have been used to value cross currency swaps by Fruchard et al. (1995), Boenkost and Schmidt (2005) and Kijima et al. (2009). To our knowledge, Henrard (2007) is the first to apply the methodology to the single-currency case, whereas Bianchetti (2009) is the first to deal with the post subprime-crisis environment.

explicit formulas for the main calibration instruments. An alternative route is chosen by Henrard (2009) who hints at the modeling of basis swap spreads, but without addressing typical issues of a market model, such as the modeling of joint dynamics or the pricing of plain-vanilla derivatives.

Modeling a stochastic basis creates no issue as far as the calibration of our extended LMM is concerned. In particular, no market data on basis volatility is needed to fit the model parameters of the basis. In fact, a LIBOR rate can be decomposed as the sum of the respective OIS rate and basis, so that the stochastic basis can be viewed as a factor driving the evolution of LIBOR rates (in conjunction with OIS rates). This is similar to what we observe in some short rate models, where the instantaneous short rate is defined as the sum of two (or more) additive factors. Such factors do not need specific options to be calibrated to but their parameters can be fitted to market quotes of standard (LIBOR-based) caps and swaptions.

In this article, we will assume that the discount curve coincides with that stripped from OIS swap rates. Since OIS rates can be regarded as the best available proxy for risk-neutral rates, this amounts to assume zero counterparty risk in the market plain-vanilla instruments (swaps, caps, swaptions). This assumption is reasonable due to the current practice of underwriting collateral agreements to mitigate, possibly eliminate, the counterparty risk affecting a given transaction between banks. When cash, the interest rate earned by the collateral is the overnight rate. Other collateral rates are present in the market, with clear implications as far as derivative pricing is concerned, see *e.g.* Johannes and Sundaresan (2007) or the more recent works by Fujii et al. (2009a, 2009b) and Piterbarg (2010). Here, however, we will assume that collateral rates coincide with overnight rates, which will allow us to work in a risk-neutral environment. This can also be viewed as the necessary initial step for a sensible valuation of deals affected by counterparty risk, which may be in part, but not completely, immunized by the collateral agreement in place. In fact, one may first obtain risk-neutral parameters by calibrating his/her model to the relevant market data and then apply suitable corrections to the risk-neutral prices of contracts that are characterized by collateral rates different than overnight rates.

The article is organized as follows. In Section 2, we describe stylized facts of the market and introduce our assumptions on the discount curve. We then define FRA rates, describe the valuation formula for swaps in a multi-curve context, and hint at the dual-curve bootstrapping of LIBOR projections from market interest-rate data. In Section 3, we illustrate possible ways of extending the LMM, and analyze pros and cons of the different formulations. In Section 4, we introduce the framework assumed in this article, namely a model for the joint evolution of forward OIS rates and related basis spreads for a given tenor. We derive caplet pricing formulas for general stochastic-volatility models and consider a specific example based on SABR dynamics. We then describe a general methodology for pricing swaptions in closed-form, analyzing the particular case of spreads that evolve according to a one-factor model. In Section 5, we propose a specific model for the joint evolution of rates and spreads based on different tenors. Section 6 considers a simple example of calibration to real market data. Section 7 concludes the paper.

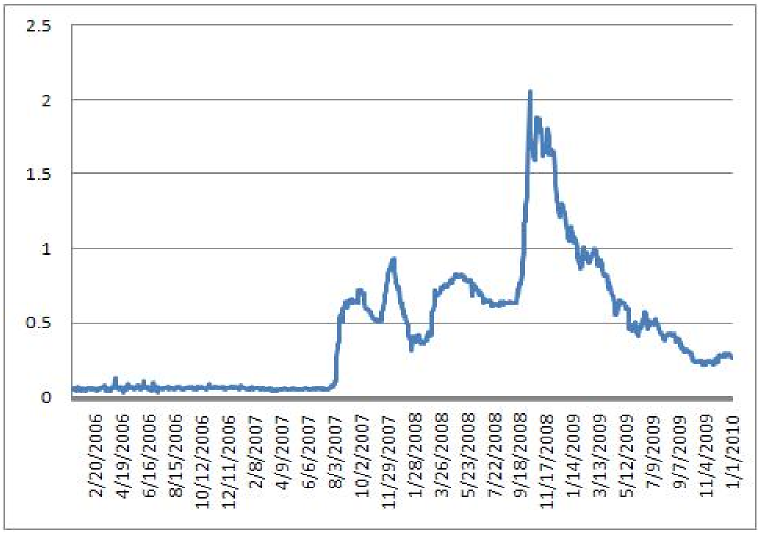

Figure 1: Basis between 3m EONIA rates and 3m deposit rates, from 2 Jan, 2006 to 2 Jan, 2010, EUR market. Source: Bloomberg.

2 The new multiple-curve environment

With the 2007 credit crunch, the basis between market rates referring to the same time interval, started to diverge sensibly. As an example, historical difference of deposit and OIS rates with the same maturity and of swap rates with the same maturity, but different floating legs (in terms of payment frequency and tenor of the paid rate) are plotted in Figures 1 and 2. In Figure 3, we plot the historical difference between 6mx12m forward EONIA rates and 6mx12m FRA rates.

The widening of the basis, and the consequent divergence of previously-equivalent market rates, can be formally explained in terms of credit, liquidity and other effects.[2] However, instead of resorting to fancy and sophisticated hybrid models, financial institutions have adopted two main empirical solutions to comply with this unprecedented interest rate scenario. The first is the separation of rate projection from NPV calculation (discounting): future rates are forecast using a corresponding zero-coupon curve and discounted using another. The rationale behind this is that LIBOR rates incorporate risk premiums that may be different from those embedded in the rates to be used for discounting. The second is the segmentation of market rates, which are grouped into separate classes according to the tenor of the underlying rate, typically 1, 3, 6 months and 1 year. For instance, the three-month bucket can be defined by the market quotes of the three-month deposit, the (3-month-LIBOR) futures (or 3-month FRAs) for the liquid maturities, and the swaps whose floating legs pay quarterly (in arrears) the 3-month-LIBOR rate (set in advance). This bucketing procedure is a direct consequence of the incompatibility arisen between

[2] A possible solution in this direction is provided by Morini (2008, 2009) and Mercurio (2009) who consider simplified settings where only credit risk is modeled.

Figure 2: Basis between 5y swap rates (3m vs 6m), from 2 Jan, 2006 to 2 Jan, 2010, EUR market. Source: Bloomberg.

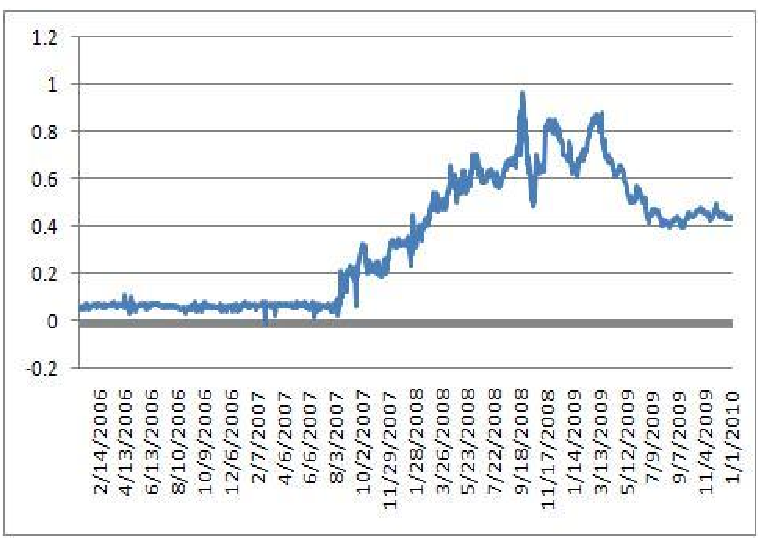

Figure 3: Basis between 6mx12m forward EONIA rates and 6mx12m FRA rates, from 2 Jan, 2006 to 2 Jan, 2010, EUR market. Source: Bloomberg.

market rates whose underlying tenors are different. In fact, compounding, for instance, two consecutive 3-month forward LIBOR rates does not yield any longer the corresponding 6-month forward LIBOR rate.

Segmenting market rates in terms of their underlying tenor naturally leads to the construction of as many different forward curves as considered buckets (tenors). Forward curves can be built in two alternative ways. The first mimics the traditional single-curve construction, but applied only to market rates that are based on the same LIBOR tenor. To this end, standard bootstrapping techniques can be employed and no modification of existing formulas and routines is required. The second is theoretically more sound and addresses the main flaw of the former procedure, that is the dependence of discount factors on the tenor of the forward LIBOR to be bootstrapped. The rationale behind this latter approach is that the NPV of (constant) future cash flows should be uniquely defined (in a default-free setting). Accordingly, the fixed-leg payments in a default-free interest rate swap (IRS) should be discounted with the same curve irrespectively of the frequency of the floating leg.

In this article, we work in the context of this second approach. In fact, given that the swap rates quoted by the market refer to deals with generic interbank counterparties, it makes sense to discount market IRS future payments with the same discount curve.[3] This will result in a modified bootstrapping procedure for each given tenor, based on stripping forward LIBOR rates by using the new IRS formula derived under the assumption of pre-assigned discount factors (calculated consistently with the given discount curve). Such a formula will be reviewed in Section 2.3 below, where hints on the new bootstrapping procedure will also be provided. But before, we need to specify our assumptions on the discount curve and introduce our definition of FRA rate.

2.1 Assumptions on the discount curve

We introduce the following assumption on the (assumed single) discount curve, for a given currency:

A.1 The discount curve is the OIS zero-coupon curve, stripped from market OIS swap rates and defined for every possible maturity T:

$$T \mapsto P_D(0,T) = P^{\mathrm{OIS}}(0,T),$$

where $P_D(t,T)$ denotes the discount factor (zero-coupon bond) at time t for maturity T, which is assumed to coincide with the corresponding OIS-based zero-coupon bond for maturity T. The subscript D stands for "discount curve".

The rationale behind this assumption is that in the interbank derivatives market, a collateral agreement (CSA) is often negotiated between two counterparties. The CSA is set to mitigate the credit risk of both parties, allowing them to establish bilateral mark-to-market

[3]See also the following section for our assumptions on the discount curve.

collateral arrangements. We here assume that the collateral, typically a bond or cash, is revalued daily at a rate equal (or close) to the overnight rate, which can thus justify the use of OIS rates for discounting.

One can also say that the presence of a CSA reduces the counterparty risk of the transaction (almost) to zero. If this is the case, it makes sense, therefore, to discount future payments by using the OIS curve. In fact, OIS rates can be regarded as the best available proxy for risk-neutral rates, since the credit risk embedded in an overnight loan can be deemed to be (almost) negligible.

The OIS curve can be stripped from OIS swap rates using standard (single-curve) bootstrapping methods. For the EUR market, EONIA swaps are quoted up to 30 years, so that the stripping procedure presents no new issues. Different is the case of other currencies, even major ones like USD or JPY, where OIS rates are quoted only up to a relatively short maturity. In such cases, one has to resort to alternative constructions, by modeling, for instance, the spread between OIS (forward) rates and corresponding (forward) LIBOR rates or by adding quotes of cross-currency swaps.

In the following, as in Kijima et al. (2009), the pricing measures we will consider are those associated with the discount curve. This is also consistent with the results of Fujii et al. (2009a) and Piterbarg (2010), since we assume CSA agreements where the collateral rate to be paid equals the (assumed risk-free) overnight rate.

2.2 Definition of FRA rate and its properties

The following definition of FRA rate is a standard one.[4]

Definition 1 *Consider times t, T_1 and T_2, $t \leq T_1 < T_2$. The time-t FRA rate $\mathbf{FRA}(t;T_1,T_2)$ is defined as the fixed rate to be exchanged at time T_2 for the LIBOR rate $L(T_1,T_2)$ so that the swap has zero value at time t.*

Denoting by Q_D^T the T-forward measure with numeraire the zero-coupon bond $P_D(t,T)$, by (risk-adjusted) no-arbitrage pricing, we immediately have

$$\mathbf{FRA}(t;T_1,T_2) = E_D^{T_2}\big[L(T_1,T_2)|\mathcal{F}_t\big], \tag{1}$$

where E_D^T denotes expectation under Q_D^T and $\mathcal{F}_t$ denotes the "information" available in the market at time t.

In the classic single-curve valuation, *i.e.* when the LIBOR curve corresponding to tenor $T_2 - T_1$ coincides with the discount curve, the FRA rate $\mathbf{FRA}(t;T_1,T_2)$ coincides with the forward rate

$$F_D(t;T_1,T_2) = \frac{1}{T_2 - T_1}\left[\frac{P_D(t,T_1)}{P_D(t,T_2)} - 1\right]. \tag{2}$$

[4]This definition of FRA rate slightly differs from that implied by the actual market contract. This slight abuse of terminology is justified because this "theoretical" FRA rate and the market one coincide in a single-curve setting. In our multi-curve case, they are different, but their difference can be shown to be negligible under typical market conditions, see Appendix A.

In fact, the LIBOR rate $L(T_1, T_2)$ can be defined by the classic relation

$$L(T_1, T_2) = \frac{1}{T_2 - T_1}\left[\frac{1}{P_D(T_1, T_2)} - 1\right] = F_D(T_1; T_1, T_2), \tag{3}$$

so that we can write

$$\mathbf{FRA}(t; T_1, T_2) = E_D^{T_2}\left[F_D(T_1; T_1, T_2)|\mathcal{F}_t\right].$$

Since $F_D(t; T_1, T_2)$ is a martingale under $Q_D^{T_2}$, we can then conclude that

$$\mathbf{FRA}(t; T_1, T_2) = F_D(t; T_1, T_2).$$

In our dual-curve setting, however, (3) does not hold any more, since the simply-compounded rates defined by the discount curve are different, in general, from the corresponding LIBOR fixings.

Our FRA rate is the natural generalization of a forward rate to the dual-curve case. In particular, we notice that, at its reset time T_1, the FRA rate $\mathbf{FRA}(T_1; T_1, T_2)$ coincides with the LIBOR rate $L(T_1, T_2)$. Moreover, the FRA rate is a martingale under the corresponding pricing measure. These properties will prove to be very convenient when pricing swaps and options on LIBOR rates.

2.3 The pricing of interest rate swaps

Let us consider a set of times $T_a, \ldots, T_b$ compatible with a given tenor,[5] and an IRS where the floating leg pays at each time T_k the LIBOR rate $L(T_{k-1}, T_k)$ set at the previous time T_{k-1}, $k = a+1, \ldots, b$, and the fixed leg pays the fixed rate K at times $T_{c+1}^S, \ldots, T_d^S$.

Under our assumptions on the discount curve, the swap valuation is straightforward.[6] Applying Definition 1 and setting

$$L_k(t) := \mathbf{FRA}(t; T_{k-1}, T_k) = E_D^{T_k}\left[L(T_{k-1}, T_k)|\mathcal{F}_t\right],$$

the IRS time-t value, to the fixed-rate payer, is given by

$$\mathbf{IRS}(t, K; T_a, \ldots, T_b, T_{c+1}^S, \ldots, T_d^S) = \sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t) - K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S).$$

where τ_k and τ_j^S denote, respectively, the floating-leg year fraction for the interval $(T_{k-1}, T_k]$ and the fixed-leg year fraction for the interval $(T_{j-1}^S, T_j^S]$.

The corresponding forward swap rate, that is the fixed rate K that makes the IRS value equal to zero at time t, is then defined by

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}. \tag{4}$$

[5]For instance, if the tenor is three months, the times T_k must be three-month spaced.

[6]Details of the derivation can be found, for instance, in Chibane and Sheldon (2009), Henrard (2009), Kijima et al. (2009) and Mercurio (2009).

In the particular case of a spot-starting swap, with payment times for the floating and fixed legs given, respectively, by $T_1, \ldots, T_b$ and $T_1^S, \ldots, T_d^S$, with $T_b = T_d^S$, the swap rate becomes:

$$S_{0,b,0,d}(0) = \frac{\sum_{k=1}^{b} \tau_k P_D(0, T_k) L_k(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)}, \tag{5}$$

where $L_1(0)$ is the constant first floating payment (known at time 0).

As already noticed by Kijima et at. (2009), neither leg of a spot-starting swap needs be worth par (when a fictitious exchange of notionals is introduced at maturity). However, this is not a problem, since the only requirement for quoted spot-starting swaps is that their initial NPV must be equal to zero.

2.4 Stripping the LIBOR projections

As traditionally done in any bootstrapping algorithm, equation (5) can be used to infer the expected (risk-free) rates L_k implied by the market quotes of spot-starting swaps, which by definition have zero value. Given that, by assumption, the discount curve has already been bootstrapped from market OIS rates, the discount factors $P_D(0,T)$, $T \in \{T_1, \ldots, T_b, T_1^S, \ldots, T_d^S\}$, entering formula (5), are all known. The FRA rates $L_k(0)$ can thus be iteratively derived by matching the market quotes of rates based on the same LIBOR tenor as the one under consideration. To this end, besides (5), one can use the formulas derived in Appendix A and Appendix B, where market FRA and futures rates are expressed as functions of rates $L_k(0)$ and corresponding forward OIS rates. Details on a similar curve construction methodology can be found, for instance, in Chibane and Sheldon (2009), Henrard (2009) and Fujii et al. (2009a). The analysis in Fujii et al. (2009a) is more thorough since they consider a general collateral rate, dealing also with a multi-currency environment.

The bootstrapped L_k can then be used, in conjunction with any interpolation tool, to price off-the-market swaps based on the same underlying tenor. As already noticed by Boenkost and Schmidt (2005) and by Kijima et al. (2009), these other swaps will have different values, in general, than those obtained by stripping discount factors through a classic (single-curve) bootstrapping method applied to swap rates

$$S_{0,d}(0) = \frac{1 - P_D(0, T_d^S)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)}.$$

Notice, in fact, that

$$\begin{aligned}
\mathbf{IRS}(0, K; T_1, \ldots, T_b, T_1^S, \ldots, T_c^S) &= \sum_{k=1}^{b} \tau_k P_D(0, T_k) L_k(0) - K \sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S) \\
&= \big[S_{0,b,0,d}(0) - K\big] \sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)
\end{aligned}$$

so that the choice of discount factors $P_D(0, T_j^S)$ heavily affects the IRS value for off-the-market fixed rates K.

3 Extending the LMM

As is well known, the classic (single-curve) LMMs are based on modeling the joint evolution of a set of consecutive forward LIBOR rates, as defined by a given time structure.[7] Forward LIBOR rates are "building blocks" of the modeled yield curve, and their dynamics can be conveniently used to generate future LIBOR rates and discount factors defining swap rates. When moving to a multi-curve setting, we immediately face two complications. The first is the existence of several yield curves (one discount curve and as many forward curves as market tenors), which multiplies the number of building blocks (the "old" forward rates) that one needs to jointly model. The second is the impossibility to apply the old definitions, which were based on the equivalence between forward LIBOR rates and the corresponding ones defined by the discount-curve.

The former issue can be trivially addressed by adding extra dimensions to the vector of modeled rates, and by suitably modeling their instantaneous covariance structure. The second, instead, is less straightforward, requiring a new definition of forward rates, which needs to be compatible with the existence of different curves for discounting and for projecting future LIBORs.

A natural extension of the definition of forward rate to a multi-curve setting is given by the FRA rate defined in Section 2.2. In fact, FRA rates reduce to "old" forward rates when the particular case of a single-curve framework is assumed. Moreover, they have the property to coincide with the corresponding LIBOR rates at their reset times and the advantage to be martingales, by definition, under the corresponding forward measures. Finally, by (4), swap rates can be written as a (stochastic) linear combination of FRA rates, with coefficients solely depending on discount-curve zero-coupon bonds.

A consistent extension of a LMM to the multi-curve case can then be obtained by modeling the joint dynamics of FRA rates for different tenors and of forward rates belonging to the discount curve. The reason for modeling OIS rates in addition to FRA rates is twofold. First, by assumption, our pricing measures are related to the discount curve. Since the associated numeraires are portfolios of zero-coupon bonds $P_D(t, T)$, the FRA drift corrections implied by a measure change will depend on the (instantaneous) covariation between FRA rates and corresponding OIS forward rates, see Appendix D. Second, swap rates explicitly depend on zero-coupon bonds $P_D(t, T)$, and, clearly, can only be simulated if the relevant OIS forward rates are simulated too.

[7] The LMM was introduced in the financial literature by Miltersen et al. (1997) and Brace et al. (1997) by assuming lognormal-type dynamics. It was then extended by Jamshidian (1997), who considered a general local-volatility formulation and by a number of authors who assumed stochastic volatility, see *e.g.* Andersen and Andreasen (2002), Piterbarg (2005), Wu and Zhang (2006), Zhu (2007), Henry-Labordère (2007), Rebonato (2007), Hagan and Lesniewski (2008), Mercurio and Morini (2007, 2009) and Rebonato et al. (2009). Other extensions include jumps or Levy-driven processes.

The extended LMM will be based on modeling the joint evolution of FRA rates and corresponding OIS forward rates, either directly or through their spreads. Pros and cons of the possible different formulations are analyzed in the following.

3.1 Alternative formulations

Let us fix, for the moment, a given tenor x and consider a time structure $\mathcal{T}^x = \{0 < T_0^x, \ldots, T_M^x\}$ compatible with x, where typically $x \in \{1m, 3m, 6m, 1y\}$. Let us then define the OIS forward rate

$$F_k^x(t) := F_D(t; T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^x}\left[\frac{P_D(t, T_{k-1}^x)}{P_D(t, T_k^x)} - 1\right] \tag{6}$$

where τ_k^x is the corresponding year fraction for the interval $(T_{k-1}^x, T_k^x]$, and denote by $S_k^x(t)$ the spread, at time t, between the FRA rate $L_k^x(t) = \mathbf{FRA}(t, T_{k-1}^x, T_k^x)$ and the OIS forward rate $F_k^x(t)$, that is

$$S_k^x(t) := L_k^x(t) - F_k^x(t) \tag{7}$$

By definition, both L_k^x and F_k^x are martingales under the forward measure $Q_D^{T_k^x}$, and hence their difference S_k^x is a $Q_D^{T_k^x}$-martingale, too.

Extending the LMM to the multi-curve case can be done essentially in three different ways, that is by:[8]

1. Modeling the joint evolution of rates L_k^x and F_k^x, $k = 1, \ldots, M$.
2. Modeling the joint evolution of rates L_k^x and spreads S_k^x, $k = 1, \ldots, M$.
3. Modeling the joint evolution of rates F_k^x and spreads S_k^x, $k = 1, \ldots, M$.

Let us assume that the modeled variables follow stochastic-volatility processes. These three choices present different advantages and drawbacks, which we summarize in the following.

The first choice, which has been proposed by Mercurio (2009, 2010), is the most convenient in terms of model tractability and calibration to market data. In fact, modeling the relevant FRA rates directly, allows for a straightforward modification of the cap and swaption pricing formulas in the corresponding single-curve LMM, where forward LIBORs follow the same (stochastic-volatility) dynamics as FRA rates in the extended setting. The problem with this choice is that there is no guarantee that the implied basis spreads will have a realistic behavior in the future, preserving in particular the positive sign that is typically observed in the market.

[8] Clearly, modeling the dynamics of two out of the three processes L_k^x, F_k^x and S_k^x yields, by (7), the dynamics of the third process, either as a difference or as a sum. These three possibilities are obviously equivalent in that the dynamics of two processes uniquely identify the dynamics of their difference or sum. What we mean here, by presenting these different cases, is the possibility to explicitly model, in each case, the selected variables with processes known in the financial literature.

The second choice has the same advantages of the first, as far as the derivation of closed-form formulas for caps is concerned, but some additional complication may arise in the derivation of swaption prices. Moreover, the implied forward rates F_k^x may go negative, even when FRA rates and spreads are modeled with processes whose support is the positive half-line. Another drawback of this formulation is that the volatility dynamics under different forward and swap measures is likely to be more involved than in the first case, especially when stochastic volatilities are instantaneously correlated with the corresponding rates.

The third approach has the advantage to be more realistic, being inspired by the market practice of building LIBOR curves at a spread over the OIS one. Moreover, since historical spreads have (almost) always been positive, and there are sound financial reasons why their sign is likely to be preserved in the future, it is more reasonable to directly model spreads S_k^x with positive-valued stochastic processes, rather than modeling (L_k^x, F_k^x) hoping for their difference to remain positive in the future, too. An apparent drawback of this approach is that the derivation of closed-form formulas for caps and swaption is more involved than in the previous cases. However, as we will show in the following sections, a smart choice of model dynamics can actually add flexibility without compromising tractability.

In this article we will follow the third approach and model forward OIS rates jointly with basis spreads. This is also inspired by the historical pattern of the (forward) basis, as showed in Figure 3. We will assume general stochastic-volatility dynamics, but also consider specific examples. We start by focusing on the single-tenor case and then propose a model for the joint evolution of rates and spreads with different tenors.

4 The extended LMM with stochastic basis

Under the assumptions of the previous section, we start by assuming general stochastic-volatility dynamics for each F_k^x and S_k^x under the associated forward measure $Q_D^{T_k^x}$, $k = 1, \ldots, M$:

$$\begin{aligned} \mathrm{d}F_k^x(t) &= \phi_k^F(t, F_k^x(t))\psi_k^F(t, V_k^F(t))\,\mathrm{d}Z_k^F(t) \\ \mathrm{d}V_k^F(t) &= a_k^F(t, V_k^F(t))\,\mathrm{d}t + b_k^F(t, V_k^F(t))\,\mathrm{d}W_k^F(t) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mathrm{d}S_k^x(t) &= \phi_k^S(t, S_k^x(t))\psi_k^S(t, V_k^S(t))\,\mathrm{d}Z_k^S(t) \\ \mathrm{d}V_k^S(t) &= a_k^S(t, V_k^S(t))\,\mathrm{d}t + b_k^S(t, V_k^S(t))\,\mathrm{d}W_k^S(t) \end{aligned} \tag{9}$$

where ϕ_k^F, ψ_k^F, a_k^F, b_k^F, ϕ_k^S, ψ_k^S, a_k^S and b_k^S are deterministic functions of the respective arguments, for each k, and Z_k^F, W_k^F, Z_k^S and W_k^S are $Q_D^{T_k^x}$-Brownian motions.

For computational purposes, we then assume that both dynamics (8) and (9) have known marginal density (equivalently, known caplet prices),[9] and that the Brownian motions Z_k^S and W_k^S are independent of Z_h^F and W_h^F, for each $h, k = 1, \ldots, M$.

[9] In principle, by Breeden and Litzenberger (1978), knowing the marginal density is equivalent to know-

The FRA rates F_k^x are allowed to be (instantaneously) correlated with their own volatility and with one another. A similar assumption holds for spreads S_k^x, too. Clearly, the constraint to be fulfilled, when modeling these correlations, is that the overall correlation matrix, including all cross correlations, must be positive semidefinite. This is assumed to hold true.

Remark 2 *It is a standard practice to define a (single-curve) LMM by modeling the joint evolution of forward rates under a given reference measure, mostly the so-called spot-LIBOR measure, see Jamshidian (1997). Assume one-factor stochastic-volatility dynamics, correlated with forward rates. Moving to a given forward measure leads to a volatility drift correction that also depends on the relevant forward rates. In our multi-curve case, the stochastic volatility V_k^F has a similar behavior, with related drift corrections that will depend on rates F_k^x. Therefore, a more thorough specification of volatility dynamics is obtained by assuming that a_k^F is a general adapted process. In this case, the dynamics of V_k^F in (8) can be viewed as an approximation of the true ones.*

4.1 Caplet pricing

Let us denote by $L^x(T_{k-1}^x, T_k^x)$ the x-tenor LIBOR rate set at time T_{k-1}^x with maturity T_k^x, and consider the associated strike-K caplet, which pays out at time T_k^x

$$\tau_k^x[L^x(T_{k-1}^x, T_k^x) - K]^+ = \tau_k^x[L_k^x(T_{k-1}^x) - K]^+. \tag{10}$$

Our assumptions on the discount curve imply that the caplet price at time t is given by

$$\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[L_k^x(T_{k-1}^x) - K]^+ | \mathcal{F}_t\right\} \tag{11}$$

Since $L_k^x(T_{k-1}^x) = F_k^x(T_{k-1}^x) + S_k^x(T_{k-1}^x)$, by the independence of $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$, the density $f_{L_k^x(T_{k-1}^x)}$ is equal to the convolution of densities $f_{F_k^x(T_{k-1}^x)}$ and $f_{S_k^x(T_{k-1}^x)}$, where we denote by f_X the density function of the random variable X under $Q_D^{T_k^x}$, conditional on $\mathcal{F}_t$. We can then write:

$$\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \int_{-\infty}^{+\infty} (l - K)^+ f_{L_k^x(T_{k-1}^x)}(l)\, \mathrm{d}l \tag{12}$$

In general, however, deriving the convolution $f_{L_k^x(T_{k-1}^x)}$ and integrating numerically (12) may not be the most efficient way to calculate the caplet price. In fact, an alternative derivation is based on applying the tower property of conditional expectations:

$$\begin{aligned}
&\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) \\
&= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) + S_k^x(T_{k-1}^x) - K]^+ | \mathcal{F}_t\right\} \\
&= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - S_k^x(T_{k-1}^x))]^+ | \mathcal{F}_t\right\} \\
&= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - S_k^x(T_{k-1}^x))]^+ | \mathcal{F}_t \vee S_k^x(T_{k-1}^x)\right\} | \mathcal{F}_t\right\}
\end{aligned} \tag{13}$$

ing the corresponding caplet prices for all possible strikes, and vice versa. However, from a numerical point of view, the equivalence may easily break down, especially when densities are approximated or need numerical integration, as is the case of both the Heston (1993) and the Hagan et al. (2002) models.

where $\mathcal{F}_t \vee S_k^x(T_{k-1}^x)$ denotes the sigma-algebra generated by $\mathcal{F}_t$ and $S_k^x(T_{k-1}^x)$.

The inner expectation in the RHS of (13) can easily be calculated thanks to the independence of the random variables $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$. We have:

$$\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \int_{-\infty}^{+\infty} E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - z)]^+ | \mathcal{F}_t\right\} f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \tag{14}$$

In particular, if the support of $f_{F_k^x(T_{k-1}^x)}$ is the positive half-line, then

$$\begin{aligned}
\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) &= \tau_k^x P_D(t, T_k^x) \bigg[\int_{-\infty}^{K} E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - z)]^+ | \mathcal{F}_t\right\} f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \\
&\qquad + \int_{K}^{+\infty} [F_k^x(t) - (K - z)] f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \bigg] \\
&= \int_{-\infty}^{K} \mathbf{Cplt}^F(t, K - z; T_{k-1}^x, T_k^x) f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \\
&\qquad + \tau_k^x P_D(t, T_k^x)(F_k^x(t) - K) Q_{S_k^x(T_{k-1}^x)}(t, K) \\
&\qquad + \tau_k^x P_D(t, T_k^x) \int_{K}^{+\infty} z f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \\
&= \int_{-\infty}^{K} \mathbf{Cplt}^F(t, K - z; T_{k-1}^x, T_k^x) f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \\
&\qquad - F_k^x(t) \frac{\partial}{\partial K} \mathbf{Cplt}^S(t, K; T_{k-1}^x, T_k^x) + \mathbf{Cplt}^S(t, K; T_{k-1}^x, T_k^x)
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\mathbf{Cplt}^F(t, \kappa; T_{k-1}^x, T_k^x) &= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - \kappa]^+ | \mathcal{F}_t\right\} \\
\mathbf{Cplt}^S(t, \kappa; T_{k-1}^x, T_k^x) &= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[S_k^x(T_{k-1}^x) - \kappa]^+ | \mathcal{F}_t\right\}
\end{aligned}$$

and

$$\begin{aligned}
Q_{S_k^x(T_{k-1}^x)}(t, K) &= E_D^{T_k^x}\left\{S_k^x(T_{k-1}^x) \geq K | \mathcal{F}_t\right\} = \int_{K}^{+\infty} f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z \\
&= -\frac{1}{\tau_k^x P_D(t, T_k^x)} \frac{\partial}{\partial K} \mathbf{Cplt}^S(t, K; T_{k-1}^x, T_k^x)
\end{aligned}$$

If $F_k^x(T_{k-1}^x)$ can assume negative values, as in the case of Gaussian or (negatively) shifted-lognormal distributions, the calculation of the integral

$$\int_{K}^{+\infty} E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - z)]^+ | \mathcal{F}_t\right\} f_{S_k^x(T_{k-1}^x)}(z)\, \mathrm{d}z$$

is slightly more involved, depending on the support of $f_{F_k^x(T_{k-1}^x)}$. However, it can still be written explicitly in terms of the caplet prices and densities related to $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$.

A different, but equivalent, characterization of the caplet price can be obtained by exploiting the symmetry of roles between OIS forward rate and spread in the pricing formula (13).[10] In fact, one can switch $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$, thus writing

$$\begin{aligned}
&\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) \\
&= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x} \Big\{ E_D^{T_k^x} \left\{ [S_k^x(T_{k-1}^x) - (K - F_k^x(T_{k-1}^x))]^+ | \mathcal{F}_t \vee F_k^x(T_{k-1}^x) \right\} | \mathcal{F}_t \Big\} \\
&= \tau_k^x P_D(t, T_k^x) \int_{-\infty}^{+\infty} E_D^{T_k^x} \left\{ [S_k^x(T_{k-1}^x) - (K - z)]^+ | \mathcal{F}_t \right\} f_{F_k^x(T_{k-1}^x)}(z) \, \mathrm{d}z
\end{aligned} \tag{16}$$

If we now assume that the support of $f_{S_k^x(T_{k-1}^x)}$ is the positive half-line, and apply the same steps leading to (15), we get:

$$\begin{aligned}
\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = &\int_{-\infty}^{K} \mathbf{Cplt}^S(t, K - z; T_{k-1}^x, T_k^x) f_{F_k^x(T_{k-1}^x)}(z) \, \mathrm{d}z \\
&+ \tau_k^x P_D(t, T_k^x) S_k^x(t) Q_{F_k^x(T_{k-1}^x)}(t, K) + \mathbf{Cplt}^F(t, K; T_{k-1}^x, T_k^x)
\end{aligned} \tag{17}$$

where

$$Q_{F_k^x(T_{k-1}^x)}(t, K) = E_D^{T_k^x} \left\{ F_k^x(T_{k-1}^x) \geq K | \mathcal{F}_t \right\} = -\frac{1}{\tau_k^x P_D(t, T_k^x)} \frac{\partial}{\partial K} \mathbf{Cplt}^F(t, K; T_{k-1}^x, T_k^x)$$

The caplet pricing formulas (15) and (17) coincide when both $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$ are positive valued. In general, we will use either (15) or (17), depending on whether $F_k^x(T_{k-1}^x)$ or $S_k^x(T_{k-1}^x)$ is positive valued. In case both $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$ can take negative values, one can then calculate (14) or (16) consistently with the assumed density supports. These calculations are here omitted for brevity.

4.2 A specific example

As a specific example, assume that OIS forward rates satisfy the following SABR dynamics:

$$\begin{aligned}
\mathrm{d}F_k^x(t) &= \left(F_k^x(t)\right)^{\beta_k} V_k^F(t) \, \mathrm{d}Z_k^F(t) \\
\mathrm{d}V_k^F(t) &= \epsilon_k V_k^F(t) \, \mathrm{d}W_k^F(t), \quad V_k^F(0) = \alpha_k
\end{aligned} \tag{18}$$

with $\mathrm{d}Z_k^F(t) \, \mathrm{d}W_k^F(t) = \rho_k \, dt$, and that spreads are given by (driftless) geometric Brownian motions

$$\mathrm{d}S_k^x(t) = \sigma_k S_k^x(t) \, \mathrm{d}Z_k^S(t) \tag{19}$$

[10] This is equivalent to apply the commutative property of convolutions.

where $\alpha_k > 0$, $\beta_k \in (0,1]$, $\epsilon_k > 0$, $\rho_k \in [-1,1]$ and $\sigma_k > 0$ are constants. Formula (15), in this case, becomes:

$$
\begin{aligned}
&\mathbf{Cplt}(t,K;T^x_{k-1},T^x_k)\\
&= \int_0^K \frac{\mathbf{Cplt}^{\mathrm{SABR}}(t,K-z;T^x_{k-1},T^x_k)}{z\sigma_k\sqrt{T^x_{k-1}-t}\sqrt{2\pi}}\exp\Bigg\{-\frac{1}{2}\frac{\big(\ln\frac{z}{S^x_k(t)}+\frac{1}{2}\sigma_k^2(T^x_{k-1}-t)\big)^2}{\sigma_k^2(T^x_{k-1}-t)}\Bigg\}\,\mathrm{d}z\\
&\quad + \tau^x_k P_D(t,T^x_k)(F^x_k(t)-K)\Phi\Bigg(\frac{\ln\frac{S^x_k(t)}{K}-\frac{1}{2}\sigma_k^2(T^x_{k-1}-t)}{\sigma_k\sqrt{T^x_{k-1}-t}}\Bigg)\\
&\quad + \tau^x_k P_D(t,T^x_k)S^x_k(t)\Phi\Bigg(\frac{\ln\frac{S^x_k(t)}{K}+\frac{1}{2}\sigma_k^2(T^x_{k-1}-t)}{\sigma_k\sqrt{T^x_{k-1}-t}}\Bigg)
\end{aligned}
\tag{20}
$$

where Φ denotes the standard normal distribution function and

$$
\mathbf{Cplt}^{\mathrm{SABR}}(t,K;T^x_{k-1},T^x_k) = \tau^x_k P_D(t,T^x_k)\big[F^x_k(t)\Phi(d_1)-K\Phi(d_2)\big] \tag{21}
$$

with

$$
\begin{aligned}
d_{1,2} &:= \frac{\ln(F^x_k(t)/K)\pm\frac{1}{2}\sigma^{\mathrm{SABR}}(K,F^x_k(t))^2(T^x_{k-1}-t)}{\sigma^{\mathrm{SABR}}(K,F^x_k(t))\sqrt{T^x_{k-1}-t}}\\
\sigma^{\mathrm{SABR}}(K,F) &:= \frac{\alpha_k}{(FK)^{\frac{1-\beta_k}{2}}\Big[1+\frac{(1-\beta_k)^2}{24}\ln^2\big(\frac{F}{K}\big)+\frac{(1-\beta_k)^4}{1920}\ln^4\big(\frac{F}{K}\big)+\cdots\Big]}\frac{\zeta}{x(\zeta)}\\
&\quad\cdot\Bigg\{1+\Bigg[\frac{(1-\beta_k)^2\alpha_k^2}{24(FK)^{1-\beta_k}}+\frac{\rho_k\beta_k\epsilon_k\alpha_k}{4(FK)^{\frac{1-\beta_k}{2}}}+\epsilon_k^2\frac{2-3\rho_k^2}{24}\Bigg]T^x_{k-1}+\cdots\Bigg\}\\
\zeta &:= \frac{\epsilon_k}{\alpha_k}(FK)^{\frac{1-\beta_k}{2}}\ln\left(\frac{F}{K}\right)\\
x(\zeta) &:= \ln\left\{\frac{\sqrt{1-2\rho_k\zeta+\zeta^2}+\zeta-\rho_k}{1-\rho_k}\right\}
\end{aligned}
$$

4.3 Swaption pricing

Let us consider a (payer) swaption, which gives the right to enter at time $T^x_a = T^S_c$ an IRS with payment times for the floating and fixed legs given, respectively, by $T^x_{a+1},\ldots,T^x_b$ and $T^S_{c+1},\ldots,T^S_d$, with $T^x_b = T^S_d$ and where the fixed rate is K. We assume that each T^S_j belongs to $\{T^x_a,\ldots,T^x_b\}$.[11] Then, for each j, there exists an index i_j such that $T^S_j = T^x_{i_j}$.

The swaption payoff at time $T^x_a = T^S_c$ is given by

$$
[S_{a,b,c,d}(T^x_a)-K]^+\sum_{j=c+1}^{d}\tau^S_j P_D(T^S_c,T^S_j), \tag{22}
$$

[11] This assumption is motivated by the measure change, from forward to swap measure, which is needed in the approximation of the swaption price. See Mercurio (2009, 2010) or Appendix D.

where, see (4),

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}.$$

Setting

$$C_D^{c,d}(t) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_{i_j}^x),$$

the swaption payoff (22) is conveniently priced under the swap measure $Q_D^{c,d}$, whose associated numeraire is the annuity $C_D^{c,d}(t)$. In fact, denoting by $E_D^{c,d}$ expectation under $Q_D^{c,d}$, we have:

$$\begin{aligned}
&\mathbf{PS}(t, K; T_a^x, \ldots, T_b^x, T_{c+1}^S, \ldots, T_d^S) \\
&= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\Big\{ \frac{[S_{a,b,c,d}(T_a^x) - K]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S)}{C_D^{c,d}(T_c^S)} |\mathcal{F}_t \Big\} \\
&= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\big\{ [S_{a,b,c,d}(T_a^x) - K]^+ \,|\mathcal{F}_t \big\}
\end{aligned} \tag{23}$$

so that, also in a multi-curve environment, pricing a swaption is equivalent to pricing an option on the underlying swap rate.

To calculate the last expectation, we proceed as follows. We set

$$\omega_k(t) := \frac{\tau_k^x P_D(t, T_k^x)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} \tag{24}$$

and write:[12]

$$S_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \omega_k(t) L_k^x(t) = \sum_{k=a+1}^{b} \omega_k(t) F_k^x(t) + \sum_{k=a+1}^{b} \omega_k(t) S_k^x(t) \tag{25}$$

The swap rate $S_{a,b,c,d}$ is, by definition, a martingale under the swap measure $Q_D^{c,d}$, and so is the process $\bar{F}(t) := \sum_{k=a+1}^{b} \omega_k(t) F_k^x(t)$, which represents the corresponding swap rate associated with the discount (OIS) curve. As a consequence, also the process $\bar{S}(t) := \sum_{k=a+1}^{b} \omega_k(t) S_k^x(t)$ is a martingale under $Q_D^{c,d}$.

Process $\bar{F}$ is equal to the classic single-curve forward swap rate that is defined by OIS discount factors, and whose reset and payment times are given by $T_c^S, \ldots, T_d^S$. If dynamics (8), which define a standard (single-curve) LMM based on OIS rates, are sufficiently tractable, we can approximate $\bar{F}(t)$ by a driftless stochastic-volatility process, $\tilde{F}(t)$, of the

[12]See also Fujii et al. (2009) for a similar decomposition.

same type as (8). This property holds for the majority of LMMs in the financial literature,[13] so that we can safely assume it also applies to our dynamics (8).

The case of process $\bar{S}$ is slightly more involved. In fact, contrary to $\bar{F}$, $\bar{S}$ explicitly depends both on OIS discount factors, defining the weights ω_k, and on basis spreads. However, this issue can easily be addressed by resorting to a standard approximation as far as swaption pricing in a LMM is concerned, that is by freezing the ω_k at their time-0 value, thus removing the dependence of $\bar{S}$ on OIS discount factors. This produces an approximating process $\sum_{k=a+1}^{b}\omega_k(0)S_k^x(t)$, which is a martingale under $Q_D^{c,d}$ thanks to the independence of the Brownian motions in (9) from OIS forward rates, see also Appendix D. We can then assume we can further approximate $\bar{S}$ with a dynamics $\tilde{S}$ similar to (9), for instance by matching instantaneous variations.

After the approximations just described, the swaption price becomes

$$\begin{aligned}&\mathbf{PS}(t,K;T_a^x,\dots,T_b^x,T_{c+1}^S,\dots,T_d^S)\\&=\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)\,E_D^{c,d}\big\{\big[\tilde{F}(T_a^x)+\tilde{S}(T_a^x)-K\big]^+|\mathcal{F}_t\big\}\end{aligned}\tag{26}$$

and can then be calculated exactly in the same way as the caplet price (14). Notice, in fact, that the two random variables $\tilde{F}(T_a^x)$ and $\tilde{S}(T_a^x)$ are independent, under $Q_D^{c,d}$, as a consequence of the weight-freezing approximation on $\bar{S}$.

4.4 A one-factor model for the spread dynamics

The swaption pricing problem above can be simplified by conveniently assuming that the evolution of spreads is modeled by the same stochastic-volatility factor, independent of OIS rates:

$$S_k^x(t)=S_k^x(0)M^S(t),\quad k=1,\dots,M\tag{27}$$

where M^S is a (continuous) martingale under each forward measure $Q_D^{T_k^x}$, independent of rates F_k^x. Clearly, $M^S(0)=1$.

From (27), we immediately have:

$$\bar{S}(t)=\sum_{k=a+1}^{b}\omega_k(t)S_k^x(t)=\sum_{k=a+1}^{b}\omega_k(t)S_k^x(0)M^S(t)\approx M^S(t)\sum_{k=a+1}^{b}\omega_k(0)S_k^x(0)=\bar{S}(0)M^S(t)$$

Given the independence between M^S and OIS rates, the dynamics of M^S under the swap

[13] This is the case, for instance, of the LMMs of Andersen and Andreasen (2002), Piterbarg (2005) and Wu and Zhang (2006). The LMMs of Henry-Labordère (2007), Mercurio and Morini (2009) and Rebonato et al. (2009) are slightly more involved to deal with, because of the assumed non-zero correlation between rates and associated stochastic volatility (Rebonato et al. (2009) also have a multi-factor volatility process). However, also in these latter cases, one can resort to efficient approximations.

measure $Q_D^{c,d}$ does not change. The swaption price (26) can then be expressed as follows:

$$\begin{aligned}
&\mathbf{PS}(t, K; T_a^x, \ldots, T_b^x, T_{c+1}^S, \ldots, T_d^S) \\
&= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\big\{\big[\tilde{F}(T_a^x) + \bar{S}(0) M^S(T_a^x) - K\big]^+ | \mathcal{F}_t\big\} \\
&= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\Big\{E_D^{c,d}\big\{\big[\tilde{F}(T_a^x) + \bar{S}(0) M^S(T_a^x) - K\big]^+ | \mathcal{F}_t \vee M^S(T_a^x)\big\} | \mathcal{F}_t\Big\}
\end{aligned} \tag{28}$$

which, again, can be calculated in the same way as the caplet price (13).

As an example, we can assume that M^S follows the SABR process

$$\begin{aligned}
\mathrm{d}M^S(t) &= \big(M^S(t)\big)^{\beta} V(t)\, \mathrm{d}Z_k^S(t) \\
\mathrm{d}V(t) &= \epsilon V(t)\, \mathrm{d}W_k^S(t), \quad V(0) = \alpha
\end{aligned} \tag{29}$$

under every forward measure $Q_D^{T_k^x}$. Since

$$\mathrm{d}S_k^x(t) = S_k^x(0)\mathrm{d}M^S(t) = (S_k^x(0))^{1-\beta}\big(S_k^x(t)\big)^{\beta} V(t)\, \mathrm{d}Z_k^S(t)$$

the resulting spread dynamics, in this case, are given by

$$\begin{aligned}
\mathrm{d}S_k^x(t) &= \big(S_k^x(t)\big)^{\beta} V^S(t)\, \mathrm{d}Z_k^S(t) \\
\mathrm{d}V^S(t) &= \epsilon V^S(t)\, \mathrm{d}W_k^S(t), \quad V^S(0) = \alpha (S_k^x(0))^{1-\beta}
\end{aligned}$$

5 Modeling different tenors simultaneously

The single-tenor case considered in the previous sections has the advantage that one can model forward OIS rates of a given length without worrying about the implications on other tenors. When modeling multiple tenors simultaneously, instead, one has to properly account for possible no-arbitrage relations that hold across different time intervals. In particular, we can not trivially extend dynamics (8) and (9) to other tenors as if different-tenor rates were totally unrelated to one another. As an example, assume that T_0, T_1, T_2 are three-month spaced, *i.e.* $T_0 = 3m$, $T_1 = 6m$, $T_2 = 9m$, and consider the three-month forward OIS rates $F_1^{3m}(t) = F_D(t, T_0, T_1)$, $F_2^{3m}(t) = F_D(t, T_1, T_2)$, and the six-month forward OIS rate $F^{6m}(t) = F_D(t, T_0, T_2)$. Clearly, these three rates are not free to vary independently from one another since, by classic no-arbitrage relations applied to the OIS curve, we must have:

$$[1 + \tau_1 F_1^{3m}(t)][1 + \tau_2 F_2^{3m}(t)] = 1 + (\tau_1 + \tau_2) F^{6m}(t) \tag{30}$$

Therefore, if the dynamics of F_1^{3m} and F_2^{3m} are given, the dynamics of F^{6m} is fully specified by (30). This implies that the stochastic process governing the evolution of the six-month rate will not belong, in general, to the same family as that of the processes of the three-month rates. For instance, if both F_1^{3m} and F_2^{3m} have SABR dynamics under the respective forward measures, F^{6m} will not have SABR dynamics even for particular values of the model parameters.

While the evolution of forward OIS rates with different tenors is constrained by no-arbitrage relations like (30), the associated spreads are relatively free to move independently of each other (though not being necessarily stochastically independent). In fact, FRA rates with different tenors belong to different curves, which can in principle be highly correlated with one another, but on which no functional dependence must be imposed a priori to fulfill no-arbitrage requirements.[14]

Our objective, when modeling multiple tenors, is to preserve the tractability of the single-tenor case, that is the possibility to price in closed form both caps and swaptions. In theory, the pricing formulas for a tenor may be different than those of another tenor, for instance because the corresponding spreads are modeled by different stochastic-volatility processes. However, in this article, we follow a simpler and more consistent approach, and choose dynamics of forward OIS rates and related spreads that are similar for all considered tenors, with the general form of condition (30) being satisfied by construction.

5.1 A tractable model for the multi-tenor case

Let us consider a time structure $\mathcal{T} = \{0 < T_0, \dots, T_M\}$ and different tenors $x_1 < x_2 < \dots < x_n$ with associated time structures $\mathcal{T}^{x_i} = \{0 < T_0^{x_i}, \dots, T_{M_i}^{x_i}\}$. We assume that each x_i is a multiple of the preceding tenor x_{i-1}, and that $\mathcal{T}^{x_n} \subset \mathcal{T}^{x_{n-1}} \subset \dots \subset \mathcal{T}^{x_1} = \mathcal{T}$. For instance, for typical market tenors, we can have

$$\begin{aligned}
\mathcal{T}^{1m} &= \{1/12,\ 2/12,\ 3/12, \dots\} \\
\mathcal{T}^{3m} &= \{1/12,\ 4/12,\ 7/12, \dots\} \\
\mathcal{T}^{6m} &= \{1/12,\ 7/12,\ 13/12, \dots\} \\
\mathcal{T}^{1y} &= \{1/12,\ 13/12,\ 25/12, \dots\}
\end{aligned}$$

For each tenor x_i, forward OIS rates are defined by (6), *i.e.*

$$F_k^{x_i}(t) := F_D(t; T_{k-1}^{x_i}, T_k^{x_i}) = \frac{1}{\tau_k^{x_i}} \left[\frac{P_D(t, T_{k-1}^{x_i})}{P_D(t, T_k^{x_i})} - 1 \right] \tag{31}$$

where $\tau_k^{x_i}$ is the year fraction for the interval $(T_{k-1}^{x_i}, T_k^{x_i}]$, and basis spreads are defined by (7), *i.e.*

$$S_k^{x_i}(t) = \mathbf{FRA}(t, T_{k-1}^{x_i}, T_k^{x_i}) - F_k^{x_i}(t) = L_k^{x_i}(t) - F_k^{x_i}(t) \tag{32}$$

[14] The extent at which two-tenor curves deviate from each other can be measured by the market quotes of corresponding basis swaps, where payments based on the former tenor are exchanged for payments based on the latter, see also Section 5.2 below.

We then assume that, for each tenor x_i, the corresponding OIS forward rates $F_k^{x_i}$, $k = 1, \ldots, M_i$, follow shifted-lognormal stochastic-volatility processes

$$\mathrm{d}F_k^{x_i}(t) = \sigma_k^{x_i}(t) V^F(t) \Big[\frac{1}{\tau_k^{x_i}} + F_k^{x_i}(t)\Big] \,\mathrm{d}Z_k^{F,x_i}(t) \tag{33}$$

where, for each k and x_i, $\sigma_k^{x_i}$ is a deterministic function, Z_k^{F,x_i} is a standard Brownian motion under the forward measure $Q_D^{T_k^{x_i}}$, and the stochastic volatility V^F is a one-factor process (common to all OIS forward rates, for all considered tenors), instantaneously uncorrelated with Z_k^{F,x_i} and with $V^F(0) = 1$.

For simplicity, we assume that, for each tenor x_i, forward rates $F_k^{x_i}$ are (instantaneously) perfectly correlated. This assumption is here introduced for notational convenience, and can in fact be easily relaxed.

Functions $\sigma_k^{x_i}$ are tenor-dependent. In order to meet no-arbitrage constraints like (30), they must satisfy the relation (60), proved in Appendix C. That is, if rate $F_k^{x_i}$ (with tenor x_i) can be obtained by compounding consecutive rates $F_j^{x_1}$ (with smallest tenor x_1), then the volatility coefficient $\sigma_k^{x_i}$ of $F_k^{x_i}$ must be equal to the sum of the volatility coefficients $\sigma_j^{x_1}$ of the rates $F_j^{x_1}$.

We notice that (33) are the simplest stochastic-volatility dynamics that are consistent across different tenors. This means, for example, that if three-month rates follow shifted lognormal processes with common stochastic volatility, the same type of dynamics is also followed by six-month rates.

As far as spread dynamics are concerned, a convenient choice is to assume, for each tenor x_i, one-factor models like (27), that is

$$S_k^{x_i}(t) = S_k^{x_i}(0) M^{x_i}(t), \quad k = 1, \ldots, M_i \tag{34}$$

where, for each x_i, M^{x_i} is a (continuous) martingale under each forward measure $Q_D^{T_k^{x_i}}$, $k = 1, \ldots, M_i$, independent of rates $F_k^{x_i}$. Clearly, $M^{x_i}(0) = 1$. The martingales $M^{x_1}, \ldots, M^{x_n}$ can be (instantaneously) correlated, to capture relative movements between curves based on different tenors.

Mimicking example (29), we can assume, for instance, that each M^{x_i} follows a SABR process

$$\begin{aligned} \mathrm{d}M^{x_i}(t) &= \big(M^{x_i}(t)\big)^{\beta^{x_i}} V^{x_i}(t) \,\mathrm{d}Z^{x_i}(t) \\ \mathrm{d}V^{x_i}(t) &= \epsilon^{x_i} V^{x_i}(t) \,\mathrm{d}W^{x_i}(t), \quad V^{x_i}(0) = \alpha^{x_i} \end{aligned}$$

where $\mathrm{d}Z^{x_i}(t)\,\mathrm{d}W^{x_i}(t) = \rho^{x_i}\,\mathrm{d}t$, and the parameters are tenor dependent.

To price caps and swaptions under (33) and (34), we just have to apply the formulas previously derived in the single-tenor case. In fact, given that rates and spreads with different x_i's follow the same type of dynamics, caps and swaptions based on different tenors will have similar pricing formulas. This is particularly convenient when simultaneously pricing options with different tenors, either for calibration purposes or because one wants to price options based on non-standard tenors given the market quotes of standard-tenor ones.

5.2 The pricing of basis swaps

A popular market contract based on different LIBOR tenors, in the same currency, is a basis swap, which is composed of two floating legs where payments set on a given LIBOR tenor are exchanged for payments set on another tenor. For instance, one can receive quarterly the 3-month LIBOR rate and pay semiannually the 6-month LIBOR rate, both set in advance and paid in arrears. The market actively quotes basis swaps, at least for the main tenors (3m vs 6m). These quotes are typically positive, meaning that a positive spread has to be added to the smaller-tenor leg to match the NPV of the larger-tenor leg.

Let us be given two tenors x_1 and x_2 with $x_1 < x_2$ and the associated time structures $\mathcal{T}^{x_1} = \{0 < T_0^{x_1}, \ldots, T_{M_1}^{x_1}\}$ and $\mathcal{T}^{x_2} = \{0 < T_0^{x_2}, \ldots, T_{M_2}^{x_2}\}$. We assume that $\mathcal{T}^{x_2} \subset \mathcal{T}^{x_1}$ and that $T_{M_1}^{x_1} = T_{M_2}^{x_2}$.

Let us then consider the two floating legs in the basis swap where x_1-rates are exchanged for x_2-rates. The x_1-leg pays at each time $T_i^{x_1}$, $i = 0, \ldots, M_1$, the x_1-LIBOR rate $L^{x_1}(T_{i-1}^{x_1}, T_i^{x_1})$. Likewise, the x_2-leg pays at each time $T_j^{x_2}$, $j = 0, \ldots, M_2$, the x_2-LIBOR rate $L^{x_2}(T_{j-1}^{x_2}, T_j^{x_2})$, where we set $T_{-1}^{x_1} = T_{-1}^{x_2} := 0$. The NPVs of the two legs at time 0 are:

$$\sum_{i=0}^{M_k} \tau_i^{x_k} P_D(0, T_i^{x_k}) L_i^{x_k}(0), \quad k = \{1, 2\}.$$

As mentioned above, typical market quotes imply that:

$$\sum_{j=0}^{M_2} \tau_j^{x_2} P_D(0, T_j^{x_2}) L_j^{x_2}(0) > \sum_{i=0}^{M_1} \tau_i^{x_1} P_D(0, T_i^{x_1}) L_i^{x_1}(0)$$

or, equivalently, that

$$\sum_{j=0}^{M_2} \tau_j^{x_2} P_D(0, T_j^{x_2}) S_j^{x_2}(0) > \sum_{i=0}^{M_1} \tau_i^{x_1} P_D(0, T_i^{x_1}) S_i^{x_1}(0)$$

These time-0 conditions are satisfied by our multi-tenor model (34) by construction. However, there is no guarantee that the corresponding conditions at a future time t will also hold true. In fact, the spread dynamics (34) may in principle generate unrealistic future scenarios.[15] If we want to preserve the positivity of basis spreads, we then have to constrain the joint evolution of processes M^{x_1} and M^{x_2}, for instance by assuming a very high correlation between them.

6 An example of calibration to real market data

We now consider a simple example of calibration to market caplet data of the LMM described by dynamics (33) and (34). In particular, we fix a tenor x and an index k and

[15] We define "unrealistic" a feature that has never (or very rarely) observed in the market until the present moment. As we have learned from the recent credit crisis, this does not necessarily mean that unrealistic features will never occur in the future.

assume that the corresponding OIS forward rate follow the shifted-lognomal process:[16]

$$\mathrm{d}F_k^x(t) = \sigma_k^x \Big[\frac{1}{\tau_k^x} + F_k^x(t)\Big]\,\mathrm{d}Z_k^{F,x}(t) \tag{35}$$

where σ_k^x is a positive constant and $Z_k^{F,x}$ is a standard $Q_D^{T_k^x}$-Brownian motion. This corresponds to assuming that $V^F \equiv 1$ in (33).

The related spread (equivalently, process M^x) is assumed to follow SABR dynamics:

$$\begin{aligned}
&\mathrm{d}S_k^x(t) = \big(S_k^x(t)\big)^{\beta_k} V_k^S(t)\,\mathrm{d}Z_k^S(t)\\
&\mathrm{d}V_k^S(t) = \epsilon_k V_k^S(t)\,\mathrm{d}W_k^S(t), \quad V_k^S(0) = \alpha_k, \quad \mathrm{d}Z_k^S(t)\,\mathrm{d}W_k^S(t) = \rho_k\,\mathrm{d}t
\end{aligned} \tag{36}$$

where $\alpha_k > 0$, $\beta_k \in (0,1]$, $\epsilon_k > 0$, $\rho_k \in [-1,1]$ are constants, and Z_k^S and $W_k^S(t)$ are standard $Q_D^{T_k^x}$-Brownian motions.

The price of the caplet $\tau_k^x[F_k^x(T_{k-1}^x)+S_k^x(T_{k-1}^x)-K]^+$ is then given by (17) where $\mathbf{Cplt}^S$ is the SABR option price associated to (36), $f_{F_k^x(T_{k-1}^x)}$ is the shifted-lognormal density coming from (35) and $\mathbf{Cplt}^F$ is the related caplet price, *i.e.*

$$\begin{aligned}
&\mathbf{Cplt}(t,K;T_{k-1}^x,T_k^x)\\
&= \int_0^{K+\frac{1}{\tau_k^x}} \frac{\mathbf{Cplt}^{\mathrm{SABR}}(t, K+\frac{1}{\tau_k^x}-z; T_{k-1}^x, T_k^x)}{z\sigma_k\sqrt{T_{k-1}^x - t}\sqrt{2\pi}} \exp\Bigg\{-\frac{1}{2}\frac{\big(\ln\frac{z}{F_k^x(t)+1/\tau_k^x} + \frac{1}{2}\sigma_k^2(T_{k-1}^x - t)\big)^2}{\sigma_k^2(T_{k-1}^x - t)}\Bigg\}\,\mathrm{d}z\\
&\quad + \tau_k^x P_D(t,T_k^x)\big(S_k^x(t) - K - 1/\tau_k^x\big)\Phi\left(\frac{\ln\frac{F_k^x(t)+1/\tau_k^x}{K+1/\tau_k^x} - \frac{1}{2}\sigma_k^2(T_{k-1}^x - t)}{\sigma_k\sqrt{T_{k-1}^x - t}}\right)\\
&\quad + \tau_k^x P_D(t,T_k^x)\big(F_k^x(t) + \frac{1}{\tau_k^x}\big)\Phi\left(\frac{\ln\frac{F_k^x(t)+1/\tau_k^x}{K+1/\tau_k^x} + \frac{1}{2}\sigma_k^2(T_{k-1}^x - t)}{\sigma_k\sqrt{T_{k-1}^x - t}}\right)
\end{aligned} \tag{37}$$

We want to test the flexibility of the caplet pricing function (37) by calibrating EUR caplet data as of February 8th, 2010. In particular, we calibrate the market caplets with reset date at $T_{k-1}^x = 3$ (years), for which $L_k^x(T_{k-1}^x) = 3.07\%$ and $F_k^x(T_{k-1}^x) = 2.50\%$, so that $S_k^x(T_{k-1}^x) = 0.57\%$. The quoted strikes and corresponding Black volatilities are shown in Table 1.

The calibration is performed by minimizing the sum of squared differences between model and market prices. To convert the market Black volatilities v_k into prices we use the market caplet formula that holds in a multi-curve setting, and under OIS discounting:[17]

$$\mathbf{Cplt}^{\mathrm{mkt}}(0,K;T_{k-1}^x,T_k^x) = \tau_k^x P_D(0,T_k^x)\,\mathrm{Bl}\big(K, L_k^x(0), v_k\sqrt{T_{k-1}^x}\big)$$

[16] By the consistency result of Appendix C, the other forward OIS rates will follow similar deterministic-volatility dynamics.

[17] The derivation of this formula can be found in Mercurio (2009). Bianchetti (2009) derives an equivalent formula, which is based on a different underlying rate to which a quanto-like correction must be applied to obtain our FRA rate.

K	Black vol
1	47.48
2	37.03
3	30.58
4	27.71
5	26.54
6	26.15
7	26.13
8	26.2
9	26.38

Table 1: EUR market caplets as of February 8th, 2010. Strikes and volatilities are in percentage points.

where

$$\mathrm{Bl}(K, L, v) = L\Phi\left(\frac{\ln(L/K) + v^2/2}{v}\right) - K\Phi\left(\frac{\ln(L/K) - v^2/2}{v}\right)$$

Our model specification fits the considered market data almost perfectly. In fact, as is typical of the SABR functional form, we have equivalently good fits for different choices of the parameter β_k. In Fig. 4, we show our calibration result corresponding to the choice of $\beta_k = 0.5$. For comparison purposes, we also plot the calibrated volatilities implied by directly assuming SABR dynamics for the FRA rate L_k^x, where the corresponding β parameter is set to 0.5 and the correlation between rate and volatility to zero:

$$\begin{aligned}
&\mathrm{d}L_k^x(t) = \big(L_k(t)\big)^{\beta_k^L} V_k^L(t)\,\mathrm{d}Z_k^L(t) \\
&\mathrm{d}V_k^L(t) = \epsilon_k^L V_k^L(t)\,\mathrm{d}W_k^L(t), \quad V_k^L(0) = \alpha_k^L, \quad \mathrm{d}Z_k^L(t)\,\mathrm{d}W_k^L(t) = 0
\end{aligned} \tag{38}$$

where $\beta_k^L = 1/2$, $\alpha_k^L > 0$ and $\epsilon_k^L > 0$ are constants, and Z_k^L and $W_k^L(t)$ are independent standard $Q_D^{T_k^x}$-Brownian motions. The implied volatilities associated to (38) are given by:

$$\begin{aligned}
\sigma^{\mathrm{SABR}}(K, L_k^x(0)) &:= \frac{\alpha_k^L}{(L_k^x(0)K)^{\frac{1}{4}}\left[1 + \frac{1}{96}\ln^2\left(\frac{L_k^x(0)}{K}\right) + \frac{1}{30720}\ln^4\left(\frac{L_k^x(0)}{K}\right) + \cdots\right]} \frac{\zeta}{x(\zeta)} \\
&\cdot \left\{1 + \left[\frac{(\alpha_k^L)^2}{96(L_k^x(0)K)^{\frac{1}{2}}} + (\epsilon_k^L)^2 \frac{1}{12}\right] T_{k-1}^x + \cdots\right\} \\
\zeta &:= \frac{\epsilon_k^L}{\alpha_k^L}(L_k^x(0)K)^{\frac{1}{4}} \ln\left(\frac{L_k^x(0)}{K}\right) \\
x(\zeta) &:= \ln\left\{\sqrt{1+\zeta^2} + \zeta\right\}
\end{aligned}$$

Setting to zero the correlation between Z_k^L and W_k^L is a common choice in many stochastic-volatility LMMs, for the purpose of keeping the volatility dynamics unchanged

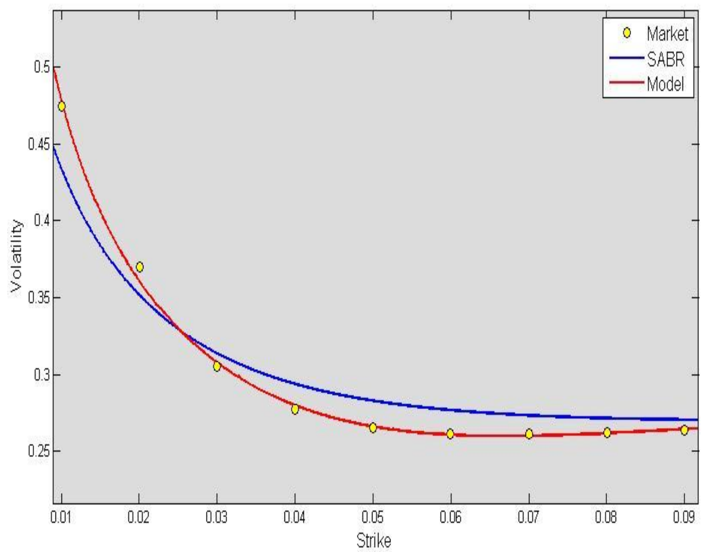

Figure 4: Comparison between market caplet volatilities with the calibrated volatilities implied by (35) and (36) (Model) and by (38) (SABR). EUR market data as of February 8th, 2010.

after a measure change. Here, however, we have the freedom to use a non-zero ρ_k and, at the same time, set to zero the correlation between stochastic volatility and forward OIS rates so as to keep the same volatility dynamics under different forward and swap measures.[18] Our model, therefore, has one extra degree of freedom with respect to (38). In general, the zero-correlation SABR model applied to L_k^x must use the parameter β_k^L to calibrate the negative slope of implied volatilities at the at-the-money level. This task in our LMM can be performed by ρ_k, whereas β_k can either be fixed a priori, as in our calibration example, or used to calibrate other market data, like for instance CMS swap spreads.

In Fig. 5, we compare the values of our calibrated volatilities coming from the SABR approximation with those obtained with Monte Carlo (MC) generation of the model dynamics. The MC window we plot has been obtained by applying a simple Euler scheme with a time step of $3/100,000$ and by simulating 1,000,000 paths.

7 Conclusions

In this article, we have shown how to extend the LMM to price interest rate derivatives under distinct yield curves, used for generating future LIBOR rates and for discounting.

[18]Clearly, some attention is still required since we need to ensure that the overall correlation matrix is positive semi-definite.

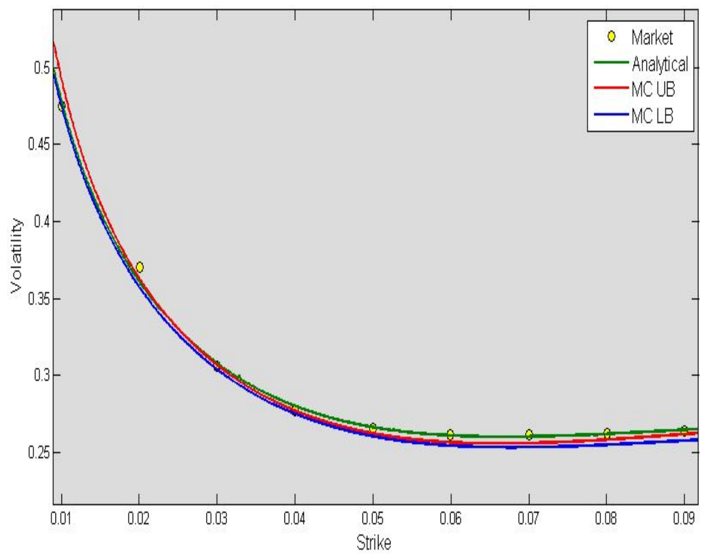

Figure 5: Comparison between prices obtained through formula (17) and 99% Monte-Carlo (MC) window. UB stands for upper bound and LB for lower bound.

To this end, we have chosen to model the joint evolution of OIS forward rates and corresponding basis spreads, under the assumption that the discount curve coincides with the OIS-based one.

We have first modeled the joint evolution of rates and spreads with a given tenor, and then proposed a model for the multi-tenor case. The dynamics we have considered imply the possibility to price in closed-form both caps and swaptions, with procedures that are only slightly more involved than the corresponding ones in the single-curve case. The framework we have introduced is rather general and allows for further extensions based on alternative dynamics.

We have finally considered a simple example of calibration to a market caplet smile. This is to be intended as a preliminary result, since the model robustness and flexibility should be tested on a much broader data set, including swaption smiles and CMS swap spreads.

Another issue that needs further investigation is the modeling of correlations with parametric forms granting the positive definiteness of the overall correlation matrix. To this end, one may try to extend to the multi-curve case the parametrization proposed by Mercurio and Morini (2007) in a single-curve setting.

A Appendix: The pricing of market FRAs

Given times T_1 and T_2, the $T_1 \times T_2$-FRA traded in the market is a contract paying out at time T_1 (to the fixed-rate payer)

$$\frac{\tau_{1,2}(L(T_1,T_2)-K)}{1+\tau_{1,2}L(T_1,T_2)} \tag{39}$$

where K is the fixed (FRA) rate and $\tau_{1,2}$ is the year fraction for the interval $(T_1,T_2]$.

The corresponding time-t FRA rate $\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2)$ is defined as the fixed rate K such that the value of payoff (39) is zero at time t.

Market FRAs are typically collateralized contracts. Assuming that the collateral rate equals the overnight rate, as we do in this article, we can thus resort to risk-neutral pricing theory, and get:

$$\begin{aligned} 0 &= E_D^{T_1}\left[\frac{\tau_{1,2}\big(L(T_1,T_2)-\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2)\big)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right] \\ &= E_D^{T_1}\left[\left(1-\frac{1+\tau_{1,2}\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}\right)|\mathcal{F}_t\right] \end{aligned} \tag{40}$$

so that

$$1 = \big(1+\tau_{1,2}\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2)\big)E_D^{T_1}\left[\frac{1}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right]$$

Therefore, the market FRA rate $\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2)$ is given by

$$\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2) = \frac{1}{\tau_{1,2}E_D^{T_1}\left[\frac{1}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}}. \tag{41}$$

The $Q_D^{T_1}$-expectation in (41) can be converted into a $Q_D^{T_2}$-expectation by a classic chance of measure (equivalently, change of numeraire) technique:

$$\begin{aligned} E_D^{T_1}\left[\frac{1}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right] &= \frac{P_D(t,T_2)}{P_D(t,T_1)}E_D^{T_2}\left[\frac{1}{P_D(T_1,T_2)}\,\frac{1}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right] \\ &= \frac{P_D(t,T_2)}{P_D(t,T_1)}E_D^{T_2}\left[\frac{1+\tau_{1,2}^D L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right], \end{aligned}$$

where we set

$$L_D(T_1,T_2) = \frac{1}{\tau_{1,2}^D}\left[\frac{1}{P_D(T_1,T_2)}-1\right],$$

with $\tau_{1,2}^D$ denoting the year fraction for the interval $(T_1,T_2]$ for the discount curve.

Thus, we can write:

$$\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2) = \frac{1}{\tau_{1,2}\frac{P_D(t,T_2)}{P_D(t,T_1)}E_D^{T_2}\left[\frac{1+\tau_{1,2}^D L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}} \tag{42}$$

Remembering (2), *i.e.*

$$F_D(t;T_1,T_2) = \frac{1}{\tau^D_{1,2}}\Big[\frac{P_D(t,T_1)}{P_D(t,T_2)} - 1\Big],$$

we finally obtain

$$\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2) = \frac{1+\tau^D_{1,2}F_D(t;T_1,T_2)}{\tau_{1,2}E^{T_2}_D\Big[\frac{1+\tau^D_{1,2}L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\Big]} - \frac{1}{\tau_{1,2}} \tag{43}$$

Remark 3 *Under a single-curve setting, the LIBOR rate $L(T_1,T_2)$ coincides with $L_D(T_1,T_2)$, and obviously $\tau^D_{1,2} = \tau_{1,2}$, so that*

$$\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2) = \frac{1+\tau^D_{1,2}F_D(t;T_1,T_2)}{\tau_{1,2}} - \frac{1}{\tau_{1,2}} = F_D(t;T_1,T_2)$$

In this case, we recover the classic well-known result that $\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2)$ coincides with the corresponding (uniquely-defined) forward rate.

Remark 4 *Formula (43) can also be derived by replacing (40) with its equivalent formulation under the forward measure $Q^{T_2}_D$:*

$$\begin{aligned} 0 &= E^{T_2}_D\Big[\frac{L(T_1,T_2)-\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}\,\frac{1}{P_D(T_1,T_2)}|\mathcal{F}_t\Big] \\ &= E^{T_2}_D\Big[\big(L(T_1,T_2)-\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2)\big)\frac{1+\tau^D_{1,2}L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\Big]. \end{aligned}$$

This leads to

$$\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2) = \frac{E^{T_2}_D\Big[L(T_1,T_2)\frac{1+\tau^D_{1,2}L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\Big]}{E^{T_2}_D\Big[\frac{1+\tau^D_{1,2}L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\Big]},$$

which can easily be shown to coincide with (43) since

$$E^{T_2}_D[L_D(T_1,T_2)|\mathcal{F}_t] = F_D(t;T_1,T_2).$$

As is evident from equation (43), the valuation of the market FRA rate $\mathbf{FRA}^{\text{mkt}}(t;T_1,T_2)$ is model dependent and based on the joint distribution of rates $L_D(T_1,T_2)$ and $L(T_1,T_2)$ under the forward measure $Q^{T_2}_D$. Given the nature of the term inside expectation, a very convenient choice is to model the dynamics of the corresponding rates $F_D(t;T_1,T_2)$ and $\mathbf{FRA}(t;T_1,T_2)$ as shifted-lognormal processes:

$$\begin{aligned} \mathrm{d}F_D(t;T_1,T_2) &= \sigma^D_{1,2}\Big[\frac{1}{\tau^D_{1,2}} + F_D(t;T_1,T_2)\Big]\,\mathrm{d}Z^D_2(t) \\ \mathrm{d}\mathbf{FRA}(t;T_1,T_2) &= \sigma_{1,2}\Big[\frac{1}{\tau_{1,2}} + \mathbf{FRA}(t;T_1,T_2)\Big]\,\mathrm{d}Z_2(t) \end{aligned} \tag{44}$$

where $\sigma_{1,2}^D$ and $\sigma_{1,2}$ are constants, and $\mathrm{d}Z_2^D$ and $\mathrm{d}Z_2$ are $Q_D^{T_2}$-Brownian motions with instantaneous correlation $\rho_{1,2}^D$. In fact, integrating dynamics (44) and taking expectation, we obtain

$$
\begin{aligned}
&E_D^{T_2}\left[\frac{1+\tau_{1,2}^D L_D(T_1,T_2)}{1+\tau_{1,2}L(T_1,T_2)}|\mathcal{F}_t\right] = E_D^{T_2}\left[\frac{1+\tau_{1,2}^D F_D(T_1;T_1,T_2)}{1+\tau_{1,2}\mathbf{FRA}(T_1;T_1,T_2)}|\mathcal{F}_t\right] \\
&= E_D^{T_2}\left[\frac{\big(1+\tau_{1,2}^D F_D(t;T_1,T_2)\big)\exp\big\{-\frac{1}{2}(\sigma_{1,2}^D)^2(T_1-t)+\sigma_{1,2}^D(Z_2^D(T_1)-Z_2^D(t))\big\}}{\big(1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)\big)\exp\big\{-\frac{1}{2}(\sigma_{1,2})^2(T_1-t)+\sigma_{1,2}(Z_2(T_1)-Z_2(t))\big\}}|\mathcal{F}_t\right] \\
&= \frac{1+\tau_{1,2}^D F_D(t;T_1,T_2)}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)} \\
&\quad\cdot E_D^{T_2}\left[e^{-\frac{1}{2}[(\sigma_{1,2}^D)^2-(\sigma_{1,2})^2](T_1-t)+\sigma_{1,2}^D(Z_2^D(T_1)-Z_2^D(t))-\sigma_{1,2}(Z_2(T_1)-Z_2(t))}|\mathcal{F}_t\right] \\
&= \frac{1+\tau_{1,2}^D F_D(t;T_1,T_2)}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)}\, e^{\left[(\sigma_{1,2})^2-\rho_{1,2}^D\sigma_{1,2}^D\sigma_{1,2}\right](T_1-t)}
\end{aligned}
\tag{45}
$$

This immediately leads to

$$
\begin{aligned}
\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2) &= \frac{1+\tau_{1,2}^D F_D(t;T_1,T_2)}{\tau_{1,2}\frac{1+\tau_{1,2}^D F_D(t;T_1,T_2)}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)}\, e^{\left[(\sigma_{1,2})^2-\rho_{1,2}^D\sigma_{1,2}^D\sigma_{1,2}\right](T_1-t)}} - \frac{1}{\tau_{1,2}} \\
&= \frac{1}{\tau_{1,2}}\big(1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)\big)\, e^{\left[-(\sigma_{1,2})^2+\rho_{1,2}^D\sigma_{1,2}^D\sigma_{1,2}\right](T_1-t)} - \frac{1}{\tau_{1,2}}
\end{aligned}
\tag{46}
$$

or, equivalently,

$$
\begin{aligned}
&\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2) - \mathbf{FRA}(t;T_1,T_2) \\
&= \frac{1}{\tau_{1,2}}\big(1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)\big)\left[e^{\left[-(\sigma_{1,2})^2+\rho_{1,2}^D\sigma_{1,2}^D\sigma_{1,2}\right](T_1-t)} - 1\right].
\end{aligned}
\tag{47}
$$

The *convexity correction* $\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2) - \mathbf{FRA}(t;T_1,T_2)$ has a sign that depends on the volatility and correlation coefficients. Precisely, given that by no-arbitrage $1 + \tau_{1,2}\mathbf{FRA}(t;T_1,T_2) > 0$, it is strictly positive if $\sigma_{1,2} < \rho_{1,2}^D\sigma_{1,2}^D$ and negative otherwise.

An example of the convexity corrections that can be obtained from formula (47) is shown in Figure 6, where we assume $\sigma_{1,2} = \sigma_{1,2}^D$ and $\rho_{1,2}^D = 0.8$, and calibrate the remaining parameters to EUR market data as of November 11th, 2009. Such data is reported in Table 2. The value of $\sigma_{1,2}$ for each T_1 is found by exact calibration of the shifted-lognormal caplet price implied by (44) to the corresponding at-the-money (ATM) volatility. As we can see, the convexity correction lies well below 1bp especially for short expiries. However, assuming volatilities $\sigma_{1,2}^D$ different than $\sigma_{1,2}$ or other levels of correlations $\rho_{1,2}^D$, can produce much higher corrections, even in the short end of the yield curves.

T_1	$\mathbf{FRA}(0; T_1, T_2)$	ATM vols
0.5	1.02233	59.46
1	1.86174	53.11
1.5	2.50463	49.55
2	2.62471	33.07
2.5	3.14865	30.88
3	3.27457	29.29
3.5	3.34763	29.1
4	3.51263	24.33
4.5	3.66589	23.91
5	3.77789	20.13
5.5	3.86381	19.8
6	3.92827	18.55
6.5	3.99064	18.36
7	4.0522	17.89
7.5	4.1111	17.78
8	4.16227	16.55
8.5	4.21015	16.5
9	4.25924	15.74
9.5	4.33377	15.62

Table 2: EUR market data as of November 11th, 2009 with $T_2 = T_1 + 0.5$. Times are in years. Rates and volatilities are in percentage points.

We may also want to assess the size of convexity corrections in a typical pre-credit-crunch situation, where the basis tended to be constant (and small) over time. This amounts to assume that, for each $t \leq T_1$,

$$\mathbf{FRA}(t; T_1, T_2) - F_D(t; T_1, T_2) = S_{1,2},$$

where $S_{1,2}$ is a (positive) constant.[19] Formula (43) then becomes

$$\begin{aligned}\mathbf{FRA}^{\mathrm{mkt}}(t; T_1, T_2) &= \frac{1 + \tau_{1,2}^D \big[\mathbf{FRA}(t; T_1, T_2) - S_{1,2}\big]}{\tau_{1,2} E_D^{T_2}\left[\frac{1+\tau_{1,2}^D (L(T_1,T_2) - S_{1,2})}{1+\tau_{1,2} L(T_1,T_2)} | \mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}} \\ &= \frac{1 + \tau_{1,2}^D \big[\mathbf{FRA}(t; T_1, T_2) - S_{1,2}\big]}{\tau_{1,2}^D - (\tau_{1,2}^D - \tau_{1,2} + \tau_{1,2}\tau_{1,2}^D S_{1,2}) E_D^{T_2}\left[\frac{1}{1+\tau_{1,2} L(T_1,T_2)} | \mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}}\end{aligned}$$

[19] Alternatively, one may assume that

$$\tau_{1,2}\mathbf{FRA}(t; T_1, T_2) - \tau_{1,2}^D F_D(t; T_1, T_2) = \tau_{1,2} S_{1,2}.$$

This leads to a slightly different formula than (49) below.

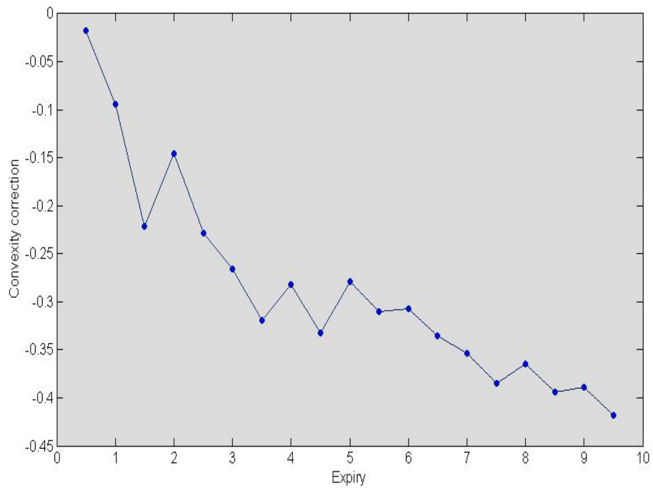

Figure 6: Difference in bp between $\mathbf{FRA}^{\mathrm{mkt}}(0; T_1, T_2)$ and $\mathbf{FRA}(0; T_1, T_2)$ as from formula (47) with $\tau_{1,2} = 0.5$ and $T_2 = T_1 + 0.5$. EUR market data as of November 11th, 2009.

Under the shifted-lognormal dynamics (44) for $\mathbf{FRA}(t; T_1, T_2)$, we finally get:

$$\mathbf{FRA}^{\mathrm{mkt}}(t; T_1, T_2) = \frac{1 + \tau_{1,2}^D \big[\mathbf{FRA}(t; T_1, T_2) - S_{1,2}\big]}{\tau_{1,2}^D - (\tau_{1,2}^D - \tau_{1,2} + \tau_{1,2}\tau_{1,2}^D S_{1,2}) \frac{1}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)} e^{\sigma_{1,2}^2 T_1}} - \frac{1}{\tau_{1,2}} \tag{48}$$

Assuming, for simplicity, that $\tau_{1,2}^D = \tau_{1,2}$, this formula for the market FRA rate simplifies to

$$\begin{aligned}
\mathbf{FRA}^{\mathrm{mkt}}(t; T_1, T_2) &= \frac{1 + \tau_{1,2}\big[\mathbf{FRA}(t; T_1, T_2) - S_{1,2}\big]}{\tau_{1,2} - \tau_{1,2}^2 S_{1,2} \frac{1}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)} e^{\sigma_{1,2}^2 T_1}} - \frac{1}{\tau_{1,2}} \\
&= \frac{1 + \tau_{1,2}\big[\mathbf{FRA}(t; T_1, T_2) - S_{1,2}\big] - 1 + \tau_{1,2} S_{1,2} \frac{1}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)} e^{\sigma_{1,2}^2 T_1}}{\tau_{1,2} - \tau_{1,2}^2 S_{1,2} \frac{1}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)} e^{\sigma_{1,2}^2 T_1}} \\
&= \frac{\big[\mathbf{FRA}(t; T_1, T_2) - S_{1,2}\big]\big[1 + \tau_{1,2}\mathbf{FRA}(t; T_1, T_2)\big] + S_{1,2} e^{\sigma_{1,2}^2 T_1}}{1 + \tau_{1,2}\mathbf{FRA}(t; T_1, T_2) - \tau_{1,2} S_{1,2} e^{\sigma_{1,2}^2 T_1}}
\end{aligned} \tag{49}$$

Therefore, the corresponding *convexity correction* is

$$
\begin{aligned}
&\mathbf{FRA}^{\mathrm{mkt}}(t;T_1,T_2) - \mathbf{FRA}(t;T_1,T_2) \\
&= \frac{\big[\mathbf{FRA}(t;T_1,T_2) - S_{1,2}\big]\big[1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)\big] + S_{1,2}e^{\sigma_{1,2}^2 T_1}}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2) - \tau_{1,2}S_{1,2}e^{\sigma_{1,2}^2 T_1}} \\
&\quad - \frac{\mathbf{FRA}(t;T_1,T_2)\big[1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2) - \tau_{1,2}S_{1,2}e^{\sigma_{1,2}^2 T_1}\big]}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2) - \tau_{1,2}S_{1,2}e^{\sigma_{1,2}^2 T_1}} \\
&= \frac{S_{1,2}\big[1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)\big]\big(e^{\sigma_{1,2}^2 T_1}-1\big)}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2) - \tau_{1,2}S_{1,2}e^{\sigma_{1,2}^2 T_1}} \\
&= S_{1,2}\big(e^{\sigma_{1,2}^2 T_1}-1\big) + \frac{\tau_{1,2}S_{1,2}^2 e^{\sigma_{1,2}^2 T_1}\big(e^{\sigma_{1,2}^2 T_1}-1\big)}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2) - \tau_{1,2}S_{1,2}e^{\sigma_{1,2}^2 T_1}} \\
&\approx S_{1,2}\big(e^{\sigma_{1,2}^2 T_1}-1\big)
\end{aligned}
\tag{50}
$$

where, in the approximation, we only keep the first-order term in $S_{1,2}$, neglecting higher order ones.

The convexity correction is, with a very good degree of approximation, an exponential function of the expiry time T_1. This may lead us to suspect that the correction, being an increasing and convex function of maturity, is non-negligible especially in the long end of the yield curve. However, the volatility coefficient $\sigma_{1,2}$ refers to a shifted-lognormal model, and as such is usually much smaller than the corresponding volatility in lognormal terms, which we denote by $\sigma_{1,2}^{LN}$. In fact, one approximately has

$$\sigma_{1,2} \approx \frac{\tau_{1,2}\sigma_{1,2}^{LN}\mathbf{FRA}(t;T_1,T_2)}{1+\tau_{1,2}\mathbf{FRA}(t;T_1,T_2)},$$

so that $\sigma_{1,2}$ is typically (at least) one degree of magnitude smaller than the corresponding $\sigma_{1,2}^{LN}$.

For typical pre-credit crunch values in the major currencies, formula (50) indeed gives negligible corrections even for very long maturities. For instance, setting $S_{1,2} = 0.001$ and $\sigma_{1,2} = 0.01$, both of which are rather conservative values, we get a convexity correction for $T_1 = 50$ that roughly amounts to a twentieth of a basis point. Therefore, if the difference between forward LIBOR rates and corresponding OIS rates remains (roughly) constant over time, we can conclude that market and theoretical FRA rates have basically the same value in non-pathological market conditions. In general, however, the magnitude of the correction can become meaningful in regimes of high volatility or when the constant spread $S_{1,2}$ is much larger than a handful of basis points.

B Appendix: The pricing of futures

A Eurodollar-futures contract gives its owner the payoff

$$1 - L(T_{k-1}, T_k) \tag{51}$$

at the future time T_{k-1}, where we assume a unit notional amount.

The fair price of this contract at time t is

$$V_t = E_t[1 - L(T_{k-1}, T_k)] = 1 - E_t[L(T_{k-1}, T_k)] \tag{52}$$

where continuous rebalancing is assumed and E_t denotes the time t-conditional expectation under the risk-neutral measure.

The purpose of this section is to derive an analytical approximation for the price (52) under the extended market model of Mercurio (2009, 2010). To this end, we assume that the instantaneous volatility of rates is deterministic and constant, and we approximate the risk-neutral expectation in (52) with that under the spot LIBOR measure $Q_D^{\mathcal{T}}$, associated with times $\mathcal{T} = \{0 < T_0, \ldots, T_M\}$, whose numeraire is the discretely-rebalanced bank account $B_D^{\mathcal{T}}$:

$$B_D^{\mathcal{T}}(t) = \frac{P_D(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P_D(T_{j-1}, T_j)},$$

where $\beta(t) = m$ if $T_{m-2} < t \leq T_{m-1}$, $m \geq 1$, so that $t \in (T_{\beta(t)-2}, T_{\beta(t)-1}]$, and $\beta(0) := 0$.

Application of the change-of numeraire technique, immediately leads to the following dynamics of FRA rates under the spot LIBOR measure $Q_D^{\mathcal{T}}$, see Mercurio (2010):

$$\mathrm{d}L_k(t) = \sigma_k L_k(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F} \tau_h^D \sigma_h^D F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \sigma_k L_k(t) \,\mathrm{d}Z_k^d(t) \tag{53}$$

where

- $Z^d = \{Z_1^d, \ldots, Z_M^d\}$ is an M-dimensional $Q_D^{\mathcal{T}}$-Brownian motion;
- $F_h^D(t)$ is the forward rate for the discount curve as defined by

$$F_k^D(t) = F_D(t; T_{k-1}, T_k) = \frac{1}{\tau_k^D} \left[\frac{P_D(t, T_{k-1})}{P_D(t, T_k)} - 1 \right]$$

  with τ_k^D the associated year fraction for the interval $(T_{k-1}, T_k]$;
- σ_k and σ_h^D are, respectively, the (deterministic) volatilities of L_k and F_h^D;
- $\rho_{k,h}^{L,F}$ is the instantaneous correlation between L_k and F_h^D.

For computational purposes, we freeze the drift in (53) to its time-0 value and set

$$\mu_k := \sigma_k \sum_{h=0}^{k} \frac{\rho_{k,h}^{L,F} \tau_h^D \sigma_h^D F_h^D(0)}{1 + \tau_h^D F_h^D(0)}, \tag{54}$$

where $F_0^D(0)$ is the (curve D) spot rate at time 0 for maturity T_0.

The dynamics of FRA rates under the spot LIBOR measure $Q_D^{\mathcal{T}}$ can then be approximated as

$$\mathrm{d}L_k(t) = \mu_k L_k(t) \,\mathrm{d}t + \sigma_k L_k(t) \,\mathrm{d}Z_k^d(t). \tag{55}$$

The valuation of (51) is now straightforward, since it reduces to the calculation of the mean of a lognormal random variable. Denoting by $E_D^{\mathcal{T}}$ expectation under $Q_D^{\mathcal{T}}$, we get:

$$\begin{aligned} V_t &\approx 1 - E_D^{\mathcal{T}}[L(T_{k-1}, T_k)|\mathcal{F}_t] \\ &\approx 1 - E_D^{\mathcal{T}}[L_k(T_{k-1})|\mathcal{F}_t] \\ &\approx 1 - L_k(t) e^{\mu_k (T_{k-1} - t)} \end{aligned} \tag{56}$$

In particular, at time $t = 0$,

$$V_0 = 1 - L_k(0) e^{\mu_k T_{k-1}}.$$

We can thus infer the unknown value of $L_k(0)$ from the corresponding market quote V_0, given volatilities and correlations, by solving this last equation:[20]

$$L_k(0) = (1 - V_0) e^{-\mu_k T_{k-1}}.$$

A better approximation can be obtained by freezing only the forward rates at their time-0 value and not the function $\beta(t)$. Setting

$$\mu_k(t) := \sigma_k \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F} \tau_h^D \sigma_h^D F_h^D(0)}{1 + \tau_h^D F_h^D(0)} \tag{57}$$

leads to the following approximated lognormal dynamics

$$\mathrm{d}L_k(t) = \mu_k(t) L_k(t)\,\mathrm{d}t + \sigma_k L_k(t)\,\mathrm{d}Z_k^d(t). \tag{58}$$

We now have:

$$E_D^{\mathcal{T}}[L_k(T_{k-1})|\mathcal{F}_t] = L_k(t) e^{\int_t^{T_{k-1}} \mu_k(u)\,\mathrm{d}u}.$$

In particular, at time $t = 0$:

$$\begin{aligned} E_D^{\mathcal{T}}[L_k(T_{k-1})] &= L_k(0) e^{\int_0^{T_{k-1}} \mu_k(u)\,\mathrm{d}u} \\ &= L_k(0) \exp\left[\sigma_k \sum_{h=0}^{k-1} \int_{T_{h-1}}^{T_h} \sum_{j=h+1}^{k} \frac{\tau_j^D \rho_{j,k}^{L,F} \sigma_j^D F_j^D(0)}{1 + \tau_j^D F_j^D(0)}\,\mathrm{d}u\right] \\ &= L_k(0) \exp\left[\sigma_k \sum_{h=0}^{k-1} \sum_{j=h+1}^{k} \frac{\tau_j^D \rho_{j,k}^{L,F} \sigma_j^D F_j^D(0)}{1 + \tau_j^D F_j^D(0)} (T_h - T_{h-1})\right] \\ &= L_k(0) \exp\left[\sigma_k \sum_{j=1}^{k} \frac{\tau_j^D \rho_{j,k}^{L,F} \sigma_j^D F_j^D(0)}{1 + \tau_j^D F_j^D(0)} T_{j-1}\right]. \end{aligned} \tag{59}$$

[20]Notice that in the single-curve case the situation is slightly more complicated since μ_k depends on $F_k(0) = L_k(0)$. However, solving the resulting non-linear equation presents no problem because it always admits a unique solution.

As before, the value of $L_k(0)$ can be obtained from the corresponding market quote V_0 by solving the equation

$$V_0 = 1 - E_D^{\mathcal{T}}[L_k(T_{k-1})].$$

We get:

$$L_k(0) = (1 - V_0) \exp\Big[- \sigma_k \sum_{j=1}^{k} \frac{\tau_j^D \rho_{j,k}^{L,F} \sigma_j^D F_j^D(0)}{1 + \tau_j^D F_j^D(0)} T_{j-1} \Big].$$

C Appendix: No-arbitrage conditions for dynamics (33)

Let us fix a tenor x_i. Since $\mathcal{T}^{x_i} \subset \mathcal{T} = \mathcal{T}^{x_1}$, then, for each $j = 0, \dots, M_i$, there exists an index i_j such that $T_j^{x_i} = T_{i_j}$. The generalization of the no-arbitrage constraint (30) to the case of tenors x_1 and x_i reads as

$$\prod_{h=i_{k-1}+1}^{i_k} [1 + \tau_h^{x_1} F_h^{x_1}(t)] = 1 + \tau_k^{x_i} F_k^{x_i}(t)$$

This equality is, by definition, satisfied at time 0. To derive conditions under which the equality holds true for every t, we notice that both of its sides are martingales under the forward measure $Q_D^{T_k^{x_i}} = Q_D^{T_{i_k}}$. Then, we just have to match diffusion coefficients:

$$\sum_{l=i_{k-1}+1}^{i_k} \prod_{\substack{h=i_{k-1}+1 \\ h \neq l}}^{i_k} [1 + \tau_h^{x_1} F_h^{x_1}(t)] \tau_l^{x_1} \sigma_l^{x_1}(t) V^F(t) \Big[\frac{1}{\tau_l^{x_1}} + F_l^{x_1}(t) \Big] = \tau_k^{x_i} \sigma_k^{x_i}(t) V^F(t) \Big[\frac{1}{\tau_k^{x_i}} + F_k^{x_i}(t) \Big]$$

which, after straightforward algebra, becomes

$$\big[1 + \tau_k^{x_i} F_k^{x_i}(t)\big] V^F(t) \sum_{l=i_{k-1}+1}^{i_k} \sigma_l^{x_1}(t) = \sigma_k^{x_i}(t) V^F(t) \big[1 + \tau_k^{x_i} F_k^{x_i}(t)\big]$$

that is

$$\sum_{l=i_{k-1}+1}^{i_k} \sigma_l^{x_1}(t) = \sigma_k^{x_i}(t) \tag{60}$$

D Appendix: Dynamics under different measures

Let us denote by $T_{c+1}^S, \dots, T_d^S$, the fixed-leg payment times of a given forward swap rate, with corresponding year fractions $\tau_{c+1}^S, \dots, \tau_d^S$, and assume that each T_j^S belongs to $\mathcal{T} = \{T_0^x, \dots, T_M^x\}$. Then, for each j, there exists an index i_j such that $T_j^S = T_{i_j}^x$.

Consider the annuity term

$$C_D^{c,d}(t) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_{i_j}^x),$$

which is the numeraire associated to the swap measure $Q_D^{c,d}$.

Let us consider dynamics (8) and (9) for OIS forward rates and related spreads under the forward measure $Q_D^{T_k^x}$. When moving from measure $Q_D^{T_k^x}$ to measure $Q_D^{c,d}$, the drift of a process X changes according to

$$\begin{aligned}
\mathbf{Drift}(X; Q_D^{c,d}) &= \mathbf{Drift}(X; Q_D^{T_k^x}) - \frac{d\langle X, \ln(P_D(\cdot, T_k^x)/C_D^{c,d}(\cdot))\rangle_t}{dt} \\
&= \mathbf{Drift}(X; Q_D^{T_k^x}) + \frac{d\langle X, \ln\left(\sum_{j=c+1}^{d} \tau_j^S \frac{P_D(\cdot, T_{i_j}^x)}{P_D(\cdot, T_k^x)}\right)\rangle_t}{dt}
\end{aligned}$$

where $\langle X, Y\rangle_t$ denotes instantaneous covariation between processes X and Y at time t.

If X is independent of the forward OIS rates F_k^x, the covariation term is zero, and the dynamics of X does not change. This is the case of both S_k^x and V_k^S. In general, the dynamics of X under $Q_D^{c,d}$ can be derived by using a standard change-of-numeraire technique. We get, see Mercurio (2009, 2010):

$$\begin{aligned}
&\mathbf{Drift}(X; Q_D^{c,d}) \\
&= \sum_{j=c+1}^{d} \frac{\tau_j^S P_D(t, T_j^S)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} (1_{\{k>i_j\}} - 1_{\{i_j>k\}}) \sum_{h=\min(i_j,k)+1}^{\max(i_j,k)} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} d\langle X, F_h^D\rangle_t
\end{aligned}$$

This formula gives also the drift correction when moving to a forward measure $Q_D^{T_h^x}$, with $h \neq k$. To this end, we just have to set $c = d-1$, $T_d^S = T_h^x$ and $T_c^S = T_{d-1}^S = T_{h-1}^x$. In fact, in this case, the annuity term reduces to (a multiple of) the zero-coupon bond $P_D(t, T_h^x)$, since

$$C_D^{d-1,d}(t) = \tau_d^S P_D(t, T_d^S) = \tau_d^S P_D(t, T_h^x).$$

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