Interest Rates and The Credit Crunch: New Formulas and Market Models

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Abstract

We start by describing the major changes that occurred in the quotes of market rates after the 2007 subprime mortgage crisis. We comment on their lost analogies and consistencies, and hint on a possible, simple way to formally reconcile them. We then show how to price interest rate swaps under the new market practice of using different curves for generating future LIBOR rates and for discounting cash flows. Straightforward modifications of the market formulas for caps and swaptions will also be derived.

Finally, we will introduce a new LIBOR market model, which will be based on modeling the joint evolution of FRA rates and forward rates belonging to the discount curve. We will start by analyzing the basic lognormal case and then add stochastic volatility. The dynamics of FRA rates under different measures will be obtained and closed form formulas for caplets and swaptions derived in the lognormal and Heston (1993) cases.

1 Introduction

Before the credit crunch of 2007, the interest rates quoted in the market showed typical consistencies that we learned on books. We knew that a floating rate bond, where rates are set at the beginning of their application period and paid at the end, is always worth par at inception, irrespectively of the length of the underlying rate (as soon as the payment schedule is re-adjusted accordingly). For instance, Hull (2002) recites: "The floating-rate bond underlying the swap pays LIBOR. As a result, the value of this bond equals the swap

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principal." We also knew that a forward rate agreement (FRA) could be replicated by going long a deposit and selling short another with maturities equal to the FRA's maturity and reset time.

These consistencies between rates allowed the construction of a well-defined zero-coupon curve, typically using bootstrapping techniques in conjunction with interpolation methods.[1] Differences between similar rates were present in the market, but generally regarded as negligible. For instance, deposit rates and OIS (EONIA) rates for the same maturity would chase each other, but keeping a safety distance (the basis) of a few basis points. Similarly, swap rates with the same maturity, but based on different lengths for the underlying floating rates, would be quoted at a non-zero (but again negligible) spread.

Then, August 2007 arrived, and our convictions became to weaver. The liquidity crisis widened the basis, so that market rates that were consistent with each other suddenly revealed a degree of incompatibility that worsened as time passed by. For instance, the forward rates implied by two consecutive deposits became different than the quoted FRA rates or the forward rates implied by OIS (EONIA) quotes. Remarkably, this divergence in values does not create arbitrage opportunities when credit or liquidity issues are taken into account.[2] As an example, a swap rate based on semiannual payments of the six-month LIBOR rate can be different (and higher) than the same-maturity swap rate based on quarterly payments of the three-month LIBOR rate.

These stylized facts suggest that the consistent construction of a yield curve is possible only thanks to credit and liquidity theories justifying the simultaneous existence of different values for same-tenor market rates. Morini (2008) is, to our knowledge, the first to design a theoretical framework that motivates the divergence in value of such rates. To this end, he introduces a stochastic default probability and, assuming no liquidity risk and that the risk in the FRA contract exceeds that in the LIBOR rates, obtains patterns similar to the market's.[3] However, while waiting for a combined credit-liquidity theory to be produced and become effective, practitioners seem to agree on an empirical approach, which is based on the construction of as many curves as possible rate lengths (e.g. 1m, 3m, 6m, 1y). Future cash flows are thus generated through the curves associated to the underlying rates and then discounted by another curve, which we term "discount curve".

Assuming different curves for different rate lengths immediately invalidates the classic pricing approaches, which were built on the cornerstone of a unique, and fully consistent, zero-coupon curve, used both in the generation of future cash flows and in the calculation of their present value. This paper shows how to generalize the main (interest rate) market models so as to account for the new market practice of using multiple curves for each single currency.

[1]The bootstrapping aimed at inferring the discount factors (zero-coupon bond prices) for the market maturities (pillars). Interpolation methods were needed to obtain interest rate values between two market pillars or outside the quoted interval.

[2]Also the constraints imposed by regulatory requirements play a fundamental role in this market environment.

[3]We also hint at a possible solution in Section 2.2. Compared to Morini, we consider simplified assumptions on defaults, but allow the interbank counterparty to change over time.

The valuation of interest rate derivatives under different curves for generating future rates and for discounting received little attention in the (non-credit related) financial literature, and mainly concerning the valuation of cross currency swaps, see Fruchard et al. (1995), Boenkost and Schmidt (2005) and Kijima et al. (2009). To our knowledge, Henrard (2007) is the first to apply the methodology to the single-currency case, whereas Bianchetti (2009) is the first to deal with the post subprime-crisis environment. In this article, we will extend the approach proposed by Kijima et al. (2009), and derive "market" formulas for interest rate swaps, caps and swaptions in a two-curve setting. Similar approaches have recently been followed by Chibane and Sheldon (2009) and Henrard (2009). Both put emphasis on the bootstrapping procedure and extend our work by deriving a closed-form formula for Eurodollar futures in a one-factor HJM setting.[4] Henrard (2009), starting from slightly different assumptions than ours, also analyses delta hedging and basis modeling issues. Chibane and Sheldon (2009) address the problem of simultaneous single-currency and cross-currency swap calibration.

In the second part of our paper, we will then show how to extend the (single-currency) LIBOR market model (LMM) consistently with the two-curve assumption. Our extended version of the LMM is based on the joint evolution of FRA rates, namely of the fixed rates that give zero value to the related forward rate agreements.[5] In the single-curve case, an FRA rate can be defined by the expectation of the corresponding LIBOR rate under a given forward measure, see *e.g.* Brigo and Mercurio (2006). In our multi-curve setting, an analogous definition applies, but with the complication that the LIBOR rate and the forward measure belong, in general, to different curves. FRA rates thus become different objects than forward LIBOR rates, and as such can be modeled with their own dynamics. In fact, FRA rates are martingales under the associated forward measure for the discount curve, but modeling their joint evolution is not equivalent to defining their instantaneous covariation structure. In this article, we will start by considering the basic example of lognormal dynamics and then introduce general stochastic volatility processes. The dynamics of FRA rates under non-canonical measures will be shown to be similar to those in the classic LMM. The main difference is given by the drift rates that depend on the relevant forward rates for the discount curve, rather then the other FRA rates in the considered family.

A last remark is in order. Also when we price interest rate derivatives under credit risk we eventually deal with two curves, one for generating cash flows and the other for discounting, see *e.g.* the LMM of Schönbucher (2000). However, in this article we do not want to model the yield curve of a given risky issuer or counterparty. We rather acknowledge that distinct rates in the market account for different credit or liquidity effects, and we start from this stylized fact to build a new LMM consistent with it.

The article is organized as follows. Section 2 briefly describes the changes in the main interest rate quotes occurred after August 2007, proposing a simple formal explanation

[4]More precisely, Chibane and Sheldon (2009) resort to the time-dependent Hull and White model (1990).

[5]These forward rate agreements are actually swaplets, in that, contrary to market FRAs, they pay at the end of the application period.

for their differences. It also describes the market practice of building different curves and motivates the approach we follow in the article. Section 3 introduces the main definitions and notations. Section 4 shows how to value interest rate swaps when future LIBOR rates are generated with a corresponding yield curve but discounted with another. Section 5 extends the market Black formulas for caplets and swaptions to the two-curve case. Section 6 introduces the extended lognormal LIBOR market model and derives the FRA and forward rates dynamics under different measures and the pricing formulas for caplets and swaptions. Section 7 introduces stochastic volatility and derives the dynamics of rates and volatilities under generic forward and swap measures. The derivation of pricing formulas for caps and swaptions is then highlighted in the specific case of the Wu and Zhang (2006) model. Section 8 concludes the article.

2 Credit-crunch interest-rate quotes

An immediate consequence of the 2007 credit crunch was the divergence of rates that until then closely chased each other, either because related to the same time interval or because implied by other market quotes. Rates related to the same time interval are, for instance, deposit and OIS rates with the same maturity. Another example is given by swap rates with the same maturity, but different floating legs (in terms of payment frequency and length of the paid rate). Rates implied by other market quotes are, for instance, FRA rates, which we learnt to be equal to the forward rate implied by two related deposits. All these rates, which were so closely interconnected, suddenly became different objects, each one incorporating its own liquidity or credit premium.[6] Historical values of some relevant rates are shown in Figures 1 and 2.

In Figure 1 we compare the "last" values of one-month EONIA rates and one-month deposit rates, from November 14th, 2005 to November 12, 2008. We can see that the basis was well below ten bp until August 2007, but since then started moving erratically around different levels.

In Figure 2 we compare the "last" values of two two-year swap rates, the first paying quarterly the three-month LIBOR rate, the second paying semiannually the six-month LIBOR rate, from November 14th, 2005 to November 12, 2008. Again, we can notice the change in behavior occurred in August 2007.

In Figure 3 we compare the "last" values of 3x6 EONIA forward rates and 3x6 FRA rates, from November 14th, 2005 to November 12, 2008. Once again, these rates have been rather aligned until August 2007, but diverged heavily thereafter.

[6]Futures rates are less straightforward to compare because of their fixed IMM maturities and their implicit convexity correction. Their values, however, tend to be rather close to the corresponding FRA rates, not displaying the large discrepancies observed with other rates.

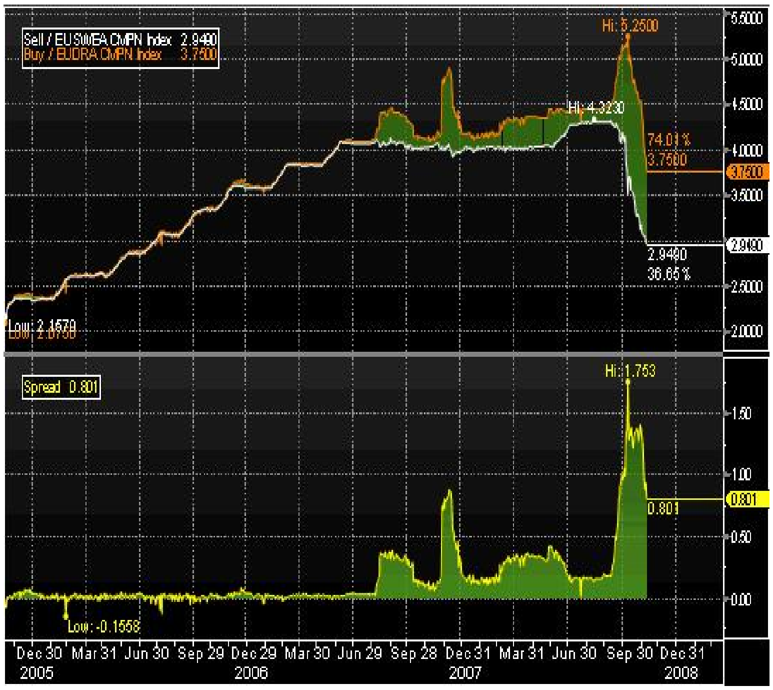

Figure 1: Euro 1m EONIA rates vs 1m deposit rates, from 14 Nov 2005 to 12 Nov 2008. Source: Bloomberg.

2.1 Divergence between FRA rates and forward rates implied by deposits

The closing values of the three-month and six-month EUR deposits on November 12, 2008 were, respectively, 4.286% and 4.345%. Assuming, for simplicity, 30/360 as day-count convention (the actual one for the EUR LIBOR rate is ACT/360), the implied three-month forward rate in three months is 4.357%, whereas the value of the corresponding FRA rate was 1.5% lower, quoted at 2.85%. Surprisingly enough, these values do not necessarily lead to arbitrage opportunities. In fact, let us denote the FRA rate and the forward rate implied by the two deposits with maturity T_1 and T_2 by F_X and F_D, respectively, and assume that $F_D > F_X$. One may then be tempted to implement the following strategy ($\tau_{1,2}$ is the year fraction for $(T_1, T_2]$):

a) Buy $(1 + \tau_{1,2} F_D)$ deposits with maturity T_2, paying

$$(1 + \tau_{1,2} F_D) D(0, T_2) = D(0, T_1)$$

dollars, where $D(t, T)$ denotes the time-t deposit price for maturity T;

b) Sell 1 deposit with maturity T_1, receiving $D(0, T_1)$ dollars;

c) Enter a (payer) FRA, paying out at time T_1

$$\frac{\tau_{1,2}(L(T_1, T_2) - F_X)}{1 + \tau_{1,2} L(T_1, T_2)}$$

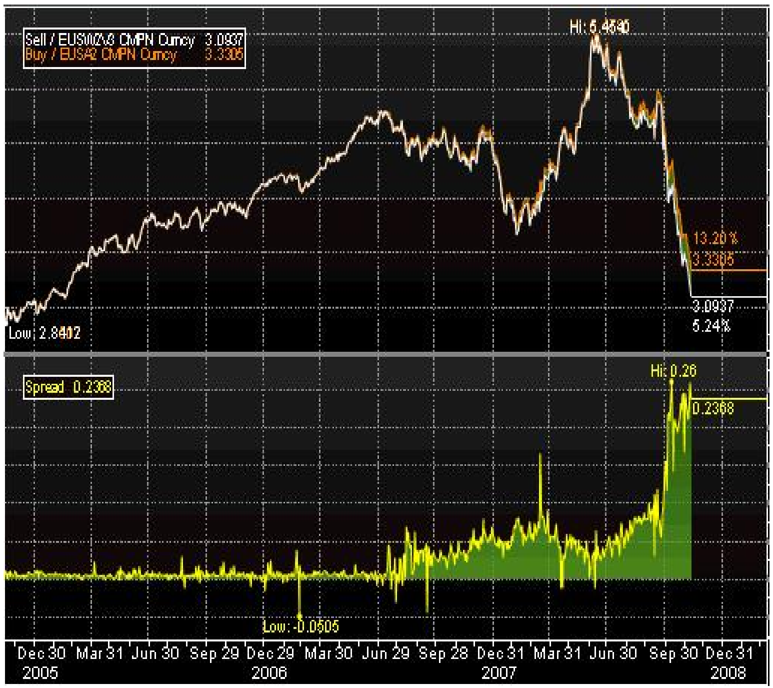

Figure 2: Euro 2y swap rates (3m vs 6m), from 14 Nov 2005 to 12 Nov 2008. Source: Bloomberg.

where $L(T_1, T_2)$ is the LIBOR rate set at T_1 for maturity T_2, namely

$$L(T_1, T_2) = \frac{1}{\tau_{1,2}} \left[\frac{1}{D(T_1, T_2)} - 1 \right]$$

The value of this strategy at the current time is zero. At time T_1, b) plus c) yields

$$\frac{\tau_{1,2}(L(T_1, T_2) - F_X)}{1 + \tau_{1,2} L(T_1, T_2)} - 1 = -\frac{1 + \tau_{1,2} F_X}{1 + \tau_{1,2} L(T_1, T_2)},$$

which is negative if rates are assumed to be positive. To pay this residual debt, we sell the $1 + \tau_{1,2} F_D$ deposits with maturity T_2, remaining with

$$\frac{1 + \tau_{1,2} F_D}{1 + \tau_{1,2} L(T_1, T_2)} - \frac{1 + \tau_{1,2} F_X}{1 + \tau_{1,2} L(T_1, T_2)} = \frac{\tau_{1,2}(F_D - F_X)}{1 + \tau_{1,2} L(T_1, T_2)} > 0$$

in cash at T_1, which is equivalent to $\tau_{1,2}(F_D - F_X)$ received at T_2. This is clearly an arbitrage, since a zero investment today produces a (stochastic but) positive gain at time T_1 or, equivalently, a deterministic positive gain at T_2 (with no intermediate net cash flows). However, there are issues that, in the current market environment, can not be neglected any more (we assume that the FRA is default-free):

i) Possibility of default before T_2 of the counterparty we lent money to;

Figure 3: 3x6 EONIA forward rates vs 3x6 FRA rates, from 14 Nov 2005 to 12 Nov 2008. Source: Bloomberg.

ii) Possibility of liquidity crunch at times 0 or T_1;

iii) Regulatory requirements, etc.

If either events occur, we can end up with a loss at final time T_2 that may outvalue the positive gain $\tau_{1,2}(F_D - F_X)$.[7] Therefore, we can conclude that the strategy above does not necessarily constitute an arbitrage opportunity. The forward rates F_D and F_X are in fact "allowed" to diverge, and their difference can be seen as representative of the market estimate of future credit and liquidity issues.

2.2 Explaining the difference in value of similar rates

The difference in value between formerly equivalent rates can be explained by means of a simple credit model, which is based on assuming that the generic interbank counterparty is subject to default risk.[8] To this end, let us denote by τ_t the default time of the generic interbank counterparty at time t, where the subscript t indicates that the random variable τ_t can be different at different times. Assuming independence between default and interest

[7]Even assuming we can sell back at T_1 the T_2-deposits to the counterparty we initially lent money to, default still plays against us.

[8]Morini (2008) develops a similar approach with stochastic probability of default. In addition to ours, he considers bilateral default risk. His interbank counterparty is, however, kept the same, and his definition of FRA contract is different than that used by the market.

rates and denoting by R the (assumed constant) recovery rate, the value at time t of a deposit starting at that time and with maturity T is

$$D(t,T) = E\Big[e^{-\int_t^T r(u)\,du}\big(R+(1-R)1_{\{\tau_t>T\}}\big)|\mathcal{F}_t\Big] = RP(t,T)+(1-R)P(t,T)E\big[1_{\{\tau_t>T\}}|\mathcal{F}_t\big],$$

where E denotes expectation under the risk-neutral measure, r is the default-free instantaneous interest rate, $P(t,T)$ is the price of a default-free zero-coupon bond at time t for maturity T and $\mathcal{F}_t$ is the information available in the market at time t.[9]

Setting

$$Q(t,T) := E\big[1_{\{\tau_t>T\}}|\mathcal{F}_t\big],$$

the LIBOR rate $L(T_1,T_2)$, which is the simple interest earned by the deposit $D(T_1,T_2)$, is given by

$$L(T_1,T_2) = \frac{1}{\tau_{1,2}}\left[\frac{1}{D(T_1,T_2)}-1\right] = \frac{1}{\tau_{1,2}}\left[\frac{1}{P(T_1,T_2)}\,\frac{1}{R+(1-R)Q(T_1,T_2)}-1\right].$$

Assuming that the above FRA has no counterparty risk, its time-0 value can be written as

$$\begin{aligned}
0 &= E\left[e^{-\int_0^{T_1} r(u)\,du}\frac{\tau_{1,2}(L(T_1,T_2)-F_X)}{1+\tau_{1,2}L(T_1,T_2)}\right]\\
&= E\left[e^{-\int_0^{T_1} r(u)\,du}\left(1-\frac{1+\tau_{1,2}F_X}{1+\tau_{1,2}L(T_1,T_2)}\right)\right]\\
&= E\Big[e^{-\int_0^{T_1} r(u)\,du}\big(1-(1+\tau_{1,2}F_X)P(T_1,T_2)(R+(1-R)Q(T_1,T_2))\big)\Big]\\
&= P(0,T_1)-(1+\tau_{1,2}F_X)P(0,T_2)\big(R+(1-R)E\big[Q(T_1,T_2)\big]\big)
\end{aligned}$$

which yields the value of the FRA rate F_X:

$$F_X = \frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)}\,\frac{1}{R+(1-R)E\big[Q(T_1,T_2)\big]}-1\right].$$

Since

$$0 \le R \le 1, \quad 0 < Q(T_1,T_2) < 1,$$

then

$$0 < R+(1-R)E\big[Q(T_1,T_2)\big] < 1$$

so that

$$F_X > \frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)}-1\right]. \tag{1}$$

Therefore, the FRA rate F_X is larger than the forward rate implied by the default-free bonds $P(0,T_1)$ and $P(0,T_2)$.

If the OIS (EONIA) swap curve is elected to be the risk-free curve, which is reasonable since the credit risk in an overnight rate is deemed to be negligible even in this new market

[9] We also refer to the next section for all definitions and notations.

Rate	Classic formulas	Model formulas
$F_{FRA}(0; T_1, T_2)$	$\frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)} - 1\right]$	$\frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)}\,\frac{1}{E[Q(T_1,T_2)]} - 1\right]$
$F_{OIS}(0; T_1, T_2)$	$\frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)} - 1\right]$	$\frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)} - 1\right]$
$F_{Depo}(0; T_1, T_2)$	$\frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)} - 1\right]$	$\frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)}\,\frac{Q(0,T_1)}{Q(0,T_2)} - 1\right]$

Table 1: Comparison between classic and model formulas for forward rates (under zero recovery).

situation, then (1) explains that the FRA rate F_X can be (arbitrarily) higher than the corresponding forward OIS rate if the default risk implicit in the LIBOR rate is taken into account. Similarly, the forward rate implied by the two deposits $D(0, T_1)$ and $D(0, T_2)$, *i.e.*

$$F_D = \frac{1}{\tau_{1,2}}\left[\frac{D(0,T_1)}{D(0,T_2)} - 1\right] = \frac{1}{\tau_{1,2}}\left[\frac{R + (1-R)Q(0,T_1)}{R + (1-R)Q(0,T_2)}\,\frac{P(0,T_1)}{P(0,T_2)} - 1\right]$$

will be larger than the FRA rate F_X if

$$\frac{R + (1-R)Q(0,T_1)}{R + (1-R)Q(0,T_2)} > \frac{1}{R + (1-R)E\big[Q(T_1,T_2)\big]}.$$

This happens, for instance, when $R < 1$ and the market expectation for the future credit premium from T_1 to T_2 (inversely proportional to $Q(T_1, T_2)$) is low compared to the value implied by the spot quantities $Q(0, T_1)$ and $Q(0, T_2)$.[10]

A summary of the different forward-rate formulas we have obtained is provided in Table 1 under the simplification of a zero recovery. The “classic” common value is also reported for comparison.

Further degrees of freedom to be calibrated to market quotes can be added by also modeling liquidity risk.[11] A thorough and sensible treatment of liquidity effects, is however beyond the scope of this work.

[10] Even though the quantities $Q(T_1, T_2)$ and $Q(0, T_2)$ refer to different default times τ_0 and τ_{T_1}, they can not be regarded as completely unrelated to each other, since they both depend on the credit worthiness of the generic interbank counterparty from T_1 to T_2.

[11] Liquidity effects are modeled, among others, by Cetin et al. (2006) and Acerbi and Scandolo (2007).

2.3 Using multiple curves

The analysis just performed is meant to provide a simple theoretical justification for the current divergence of market rates that refer to the same time interval. Such rates, in fact, become compatible with each other as soon as credit and liquidity risks are taken into account. However, instead of explicitly modeling credit and liquidity effects, practitioners seem to deal with the above discrepancies by segmenting market rates, labeling them differently according to their application period. This results in the construction of different zero-coupon curves, one for each possible rate length considered. One of these curves, or any version obtained by mixing "inhomogeneous rates", is then elected to act as the discount curve.

As far as derivatives pricing is concerned, however, it is still not clear how to account for these new market features and practice. When pricing interest rate derivatives with a given model, the usual first step is the model calibration to the term structure of market rates. This task, before August 2007, was straightforward to accomplish thanks to the existence of a unique, well defined yield curve. When dealing with multiple curves, however, not only the calibration to market rates but also the modeling of their evolution becomes a non-trivial task. To this end, one may identify two possible solutions:

i) Modeling default-free rates in conjunction with default times τ_t and/or liquidity or other effects.

ii) Modeling the joint, but distinct, evolution of rates that refer to the same time interval.

The former choice is consistent with the above procedure to justify the simultaneous existence of formerly equivalent rates. However, devising a sensible model for the evolution of default times may not be so obvious. Notice, in fact, that the standard theories on credit risk do not immediately apply here, since the default time does not refer to a single credit entity, but it is representative of a generic sector, the interbank one. The random variable τ_t, therefore, does not change over time because the credit worthiness of the reference entity evolves stochastically, but because the counterparty is generic and a new default time τ_t is generated at each time t to assess the credit premium in the LIBOR rate at that time.

In this article, we prefer to follow the latter approach and apply a logic similar to that used in the yield curves construction. In fact, given that practitioners build different curves for different rate tenors, it is quite reasonable to introduce an interest rate model where such curves are modeled jointly but distinctly. To this end, we will model forward rates with a given tenor in conjunction with those implied by the discount curve. This will be achieved in the spirit of Kijima et al (2009).

The forward (or "growth") curve associated to a given rate tenor can be constructed with standard bootstrapping techniques. The main difference with the methodology followed in the pre-credit-crunch situation is that now only the market quotes corresponding to the given tenor are employed in the stripping procedure. For instance, the three-month curve can be constructed by bootstrapping zero-coupon rates from the market quotes of the three-month deposit, the futures (or 3m FRAs) for the main maturities and the liquid swaps (vs 3m).

The discount curve, instead, can be selected in several different ways, depending on the contract to price. For instance, in absence of counterparty risk or in case of collateralized derivatives, it can be deemed to be the classic risk-neutral curve, whose best proxy is the OIS swap curve, obtained by suitably interpolating and extrapolating OIS swap quotes.[12] For a contract signed with a generic interbank counterparty without collateral, the discount curve should reflect the fact that future cash flows are at risk and, as such, must be discounted at LIBOR, which is the rate reflecting the credit risk of the interbank sector. In such a case, therefore, the discount curve may be bootstrapped (and extrapolated) from the quoted deposit rates. In general, the discount curve can be selected as the yield curve associated to the counterparty in question.[13]

In the following, we will assume that future cash flows are all discounted with the same discount curve. The extension to a more general case involves a heavier notation and here neglected for simplicity. For similar reasons, we will only consider one forward curve at a time (modeled jointly with the discount one), leaving aside the issue of modeling the spread between two different forward curves.

3 Basic definitions and notation

Let us assume that, in a single-currency economy, we have selected N different interest-rate lengths $\delta_1, \ldots, \delta_N$ and constructed the corresponding yield curves with standard (single-curve) bootstrapping techniques. The curve associated to length δ_i will be shortly referred to as curve i.[14] We denote by $P_i(t,T)$ the associated discount factor (equivalently, zero-coupon bond price) at time t for maturity T. We also assume we are given a curve D for discounting future cash flows. We denote by $P_D(t,T)$ the curve-D discount factor at time t for maturity T.

We will consider the time structures $\{T_0^i, T_1^i, \ldots\}$, where the superscript i denotes the curve it belongs to, and $\{T_0^S, T_1^S, \ldots\}$, which includes the payment times of a swap's fixed leg.

Forward rates can be defined for each given curve. Precisely, for each curve $x \in \{1, 2, \ldots, N, D\}$, the (simply-compounded) forward rate prevailing at time t and applied to the future time interval $[T,S]$ is defined by

$$F_x(t;T,S) := \frac{1}{\tau_x(T,S)}\left[\frac{P_x(t,T)}{P_x(t,S)} - 1\right], \tag{2}$$

where $\tau_x(T,S)$ is the year fraction for the interval $[T,S]$ under the convention of curve x.[15]

[12]Notice that OIS rates carry the credit risk of an overnight rate, which may be regarded as negligible in most situations.

[13]A detailed description of a possible methodology for constructing forward and discount curves is outlined in Ametrano and Bianchetti (2009). In general, bootstrapping multiple curves, for the same currency, involves plenty of technicalities and subjective choices.

[14]In the next section, we will hint at a possible bootstrap methodology.

[15]In practice, for curves $i = 1, \ldots, N$, we will consider only intervals where $S = T + \delta_i$, whereas for curve D the interval $[S,T]$ may be totally arbitrary.

Given the times $t \leq T^i_{k-1} < T^i_k$ and the curve $x \in \{1, \dots, N, D\}$, we will make use of the following short-hand notation:

$$F^x_k(t) := F_x(t; T^i_{k-1}, T^i_k) = \frac{1}{\tau^x_k}\left[\frac{P_x(t, T^i_{k-1})}{P_x(t, T^i_k)} - 1\right] \tag{3}$$

where τ^x_k is the year fraction for the interval $[T^i_{k-1}, T^i_k]$ for curve x, namely $\tau^x_k := \tau_x(T^i_{k-1}, T^i_k)$.

As in Kijima et al. (2009), the pricing measures we will consider are those associated to the discount curve D. To denote these measures we will adopt the notation Q^z_x, where the subscript x (mainly D) identifies the underlying yield curve, and the superscript z defines the measure in question. More precisely, we denote by:

- Q^T_D the T-forward measure, whose numeraire is the zero-coupon bond $P_D(\cdot, T)$.
- $Q^{\mathcal{T}}_D$ the spot LIBOR measure associated to times $\mathcal{T} = \{T^i_0, \dots, T^i_M\}$, whose numeraire is the discretely-rebalanced bank account $B^{\mathcal{T}}_D$:

$$B^{\mathcal{T}}_D(t) = \frac{P_D(t, T^i_m)}{\prod_{j=0}^m P_D(T^i_{j-1}, T^i_j)}, \quad T^i_{m-1} < t \leq T^i_m, \; m = 1, \dots, M.$$

- $Q^{c,d}_D$ the forward swap measure defined by the time structure $\{T^S_c, T^S_{c+1}, \dots, T^S_d\}$, whose numeraire is the annuity

$$C^{c,d}_D(t) = \sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j),$$

  where $\tau^S_j := \tau_D(T^S_{j-1}, T^S_j)$.

The expectation under the generic measure Q^z_x will be denoted by E^z_x, where again the indices x and z identify, respectively, the underlying yield curve and the measure in question. The information available in the market at each time t will be described by the filtration $\mathcal{F}_t$.

4 The valuation of interest rate swaps

In this section, we show how to value linear interest rate derivatives under our assumption of distinct forward and discount curves. To this end, let us consider a set of times $T^i_a, \dots, T^i_b$ compatible with curve i,[16] and an interest rate swap where the floating leg pays at each time T^i_k the LIBOR rate of curve i set at the previous time T^i_{k-1}, $k = a+1, \dots, b$. In formulas, the time-T^i_k payoff of the floating leg is

$$\mathbf{FL}(T^i_k; T^i_{k-1}, T^i_k) = \tau^i_k F^i_k(T^i_{k-1}) = \frac{1}{P_i(T^i_{k-1}, T^i_k)} - 1. \tag{4}$$

[16]For instance, if i denotes the three-month curve, then the times T^i_k must be three-month spaced.

The time-t value, $\mathbf{FL}(t; T^i_{k-1}, T^i_k)$, of such a payoff can be obtained by taking the discounted expectation under the forward measure $Q_D^{T^i_k}$:[17]

$$\mathbf{FL}(t; T^i_{k-1}, T^i_k) = \tau^i_k P_D(t, T^i_k) E_D^{T^i_k}\big[F^i_k(T^i_{k-1})|\mathcal{F}_t\big].$$

Defining the time-t FRA rate as the fixed rate to be exchanged at time T^i_k for the floating payment (4) so that the swap has zero value at time t,[18] *i.e*

$$L^i_k(t) := \mathbf{FRA}(t; T^i_{k-1}, T^i_k) = E_D^{T^i_k}\big[F^i_k(T^i_{k-1})|\mathcal{F}_t\big],$$

we can write

$$\mathbf{FL}(t; T^i_{k-1}, T^i_k) = \tau^i_k P_D(t, T^i_k) L^i_k(t). \tag{5}$$

In the classic single-curve valuation ($i \equiv D$), the forward rate F^i_k is a martingale under the associated T^i_k-forward measure (coinciding with $Q_D^{T^i_k}$), so that the expected value $L^i_k(t)$ coincides with the current forward rate:

$$L^i_k(t) = F^i_k(t).$$

Accordingly, as is well known, the present value of each payment in the swap's floating leg can be simplified as follows:

$$\mathbf{FL}(t; T^i_{k-1}, T^i_k) = \tau^i_k P_D(t, T^i_k) L^i_k(t) = \tau^i_k P_i(t, T^i_k) F^i_k(t) = P_i(t, T^i_{k-1}) - P_i(t, T^i_k),$$

which leads to the classic result that the LIBOR rate set at time T^i_{k-1} and paid at time T^i_k can be replicated by a long position in a zero-coupon bond expiring at time T^i_{k-1} and a short position in another bond with maturity T^i_k.

In the situation we are dealing with, however, curves i and D are different, in general. The forward rate F^i_k is not a martingale under the forward measure $Q_D^{T^i_k}$, and the FRA rate $L^i_k(t)$ is different from $F^i_k(t)$. Therefore, the present value of a future LIBOR rate is no longer obtained by discounting the corresponding forward rate, but by discounting the corresponding FRA rate.

The net present value of the swap's floating leg is simply given by summing the values (5) of single payments:

$$\mathbf{FL}(t; T^i_a, \ldots, T^i_b) = \sum_{k=a+1}^{b} \mathbf{FL}(t; T^i_{k-1}, T^i_k) = \sum_{k=a+1}^{b} \tau^i_k P_D(t, T^i_k) L^i_k(t), \tag{6}$$

which, for the reasons just explained, will be different in general than $P_D(t, T^i_a) - P_D(t, T^i_b)$ or $P_i(t, T^i_a) - P_i(t, T^i_b)$.

[17] For most swaps, thanks to the presence of collaterals or netting clauses, curve D can be assumed to be the risk-free one (as obtained from OIS swap rates).

[18] This FRA rate is slightly different than that defined by the market, see Section 2.2. This slight abuse of terminology is justified by the definition that applies when payments occur at the end of the application period (like in this case).

Let us then consider the swap's fixed leg and denote by K the fixed rate paid on the fixed leg's dates $T_c^S, \dots, T_d^S$. The present value of these payments is immediately obtained by discounting them with the discount curve D:

$$\sum_{j=c+1}^{d} \tau_j^S K P_D(t, T_j^S) = K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S),$$

where we remember that $\tau_j^S = \tau_D(T_{j-1}^S, T_j^S)$.

Therefore, the interest rate swap value, to the fixed-rate payer, is given by

$$\mathbf{IRS}(t, K; T_a^i, \dots, T_b^i, T_c^S, \dots, T_c^S) = \sum_{k=a+1}^{b} \tau_k^i P_D(t, T_k^i) L_k^i(t) - K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S).$$

We can then calculate the corresponding forward swap rate as the fixed rate K that makes the IRS value equal to zero at time t. We get:

$$S_{a,b,c,d}^i(t) = \frac{\sum_{k=a+1}^{b} \tau_k^i P_D(t, T_k^i) L_k^i(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}. \tag{7}$$

This is the forward swap rate of an interest rate swap where cash flows are generated through curve i and discounted with curve D.

In the particular case of a spot-starting swap, with payment times for the floating and fixed legs given, respectively, by $T_1^i, \dots, T_b^i$ and $T_1^S, \dots, T_d^S$, with $T_b^i = T_d^S$, the swap rate becomes:

$$S_{0,b,0,d}^i(0) = \frac{\sum_{k=1}^{b} \tau_k^i P_D(0, T_k^i) L_k^i(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)}, \tag{8}$$

where $L_1(0)$ is the constant first floating payment (known at time 0). As already noticed by Kijima et at. (2009), neither leg of a spot-starting swap needs be worth par (when a fictitious exchange of notionals is introduced at maturity). However, this is not a problem, since the only requirement for quoted spot-starting swaps is that their net present value must be equal to zero.

Remark 1 *As traditionally done in any bootstrapping algorithm, equation (8) can be used to infer the expected rates L_k^i implied by the market quotes of spot-starting swaps, which by definition have zero value. For such swaps, curve D can be typically assumed to be the risk-free one (as obtained from OIS swap rates). The bootstrapped L_k^i can then be used, in conjunction with any interpolation tool, to price other swaps based on curve i. As already noticed by Boenkost and Schmidt (2005) and by Kijima et al. (2009), these other swaps will have different values, in general, than those obtained through classic bootstrapping methods applied to swap rates*

$$S_{0,d}(0) = \frac{1 - P_D(0, T_d^S)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)}.$$

Swap rate	Formulas
OLD	$\frac{\sum_{k=a+1}^{b} \tau_k^i P_i(t,T_k^i) F_k^i(t)}{\sum_{j=c+1}^{d} \tau_j^S P_i(t,T_j^S)} = \frac{P_i(t,T_a^i)-P_i(t,T_b^i)}{\sum_{j=1}^{d} \tau_j^S P_i(t,T_j^S)}$
NEW	$\frac{\sum_{k=a+1}^{b} \tau_k^i P_D(t,T_k^i) L_k^i(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)}$

Table 2: Comparison between "old" and "new" formulas for forward swap rates.

However, this is perfectly reasonable since we are here using an alternative, and more general, approach.

A recipe for a two-curve bootstrapping procedure, in the single-currency case, can be found in Henrard (2009). When also calibrating cross-currency swaps, a related bootstrapping technique is detailed in Chibane and Sheldon (2009).

A comparison between the two swap-rate formulas in the single- and two-curve setting is provided in Table 2.

5 The pricing of caplets and swaptions

Similarly to what we just did for interest rate swaps, the purpose of this section is to derive pricing formulas for options on the main interest rates, which will result in modifications of the corresponding Black-like formulas governed by our two-curve paradigm.

As is well known, the formal justifications for the use of Black-like formulas for caps and swaptions come, respectively, from the lognormal LMM of Brace et al. (1997) and Miltersen et al. (1997) and the lognormal swap model of Jamshidian (1997).[19] To be able to adapt such formulas to our two-curve case, we will have to reformulate accordingly the corresponding market models.

Again, the choice of the discount curve D depends on the credit worthiness of the counterparty and on the possible presence of a collateral mitigating the credit risk exposure.

5.1 Market formula for caplets

We first consider the case of a caplet paying out at time T_k^i

$$\tau_k^i [F_k^i(T_{k-1}^i) - K]^+. \tag{9}$$

[19] It is worth mentioning that the first proof that Black-like formulas for caps and swaptions are arbitrage free is due to Jamshidian (1996).

To price such payoff in the basic single-curve case, one notices that the forward rate F_k^i is a martingale under the T_k^i-forward measure $Q_i^{T_k^i}$ for curve i, and then calculates the time-t caplet price

$$\mathbf{Cplt}(t, K; T_{k-1}^i, T_k^i) = \tau_k^i P_i(t, T_k^i) E_i^{T_k^i}\big\{[F_k^i(T_{k-1}^i) - K]^+ | \mathcal{F}_t\big\}$$

according to the chosen dynamics. For instance, the classic choice of a driftless geometric Brownian motion[20]

$$\mathrm{d}F_k^i(t) = \sigma_k F_k^i(t)\,\mathrm{d}Z_k(t), \quad t \leq T_{k-1}^i,$$

where σ_k is a constant and Z_k is a $Q_i^{T_k^i}$-Brownian motion, leads to Black's pricing formula:

$$\mathbf{Cplt}(t, K; T_{k-1}^i, T_k^i) = \tau_k^i P_i(t, T_k^i)\,\mathrm{Bl}\big(K, F_k^i(t), \sigma_k\sqrt{T_{k-1}^i - t}\big) \tag{10}$$

where

$$\mathrm{Bl}(K, F, v) = F\Phi\bigg(\frac{\ln(F/K) + v^2/2}{v}\bigg) - K\Phi\bigg(\frac{\ln(F/K) - v^2/2}{v}\bigg),$$

and Φ denotes the standard normal distribution function.

In our two-curve setting, the caplet valuation requires more attention. In fact, since the pricing measure is now the forward measure $Q_D^{T_k^i}$ for curve D, the caplet price at time t becomes

$$\mathbf{Cplt}(t, K; T_{k-1}^i, T_k^i) = \tau_k^i P_D(t, T_k^i) E_D^{T_k^i}\big\{[F_k^i(T_{k-1}^i) - K]^+ | \mathcal{F}_t\big\}.$$

As already explained in the IRS case, the problem with this new expectation is that the forward rate F_k^i is not, in general, a martingale under $Q_D^{T_k^i}$. A possible way to value it is to model the dynamics of F_k^i under its own measure $Q_i^{T_k^i}$ and then to model the Radon-Nikodym derivative $\mathrm{d}Q_i^{T_k^i}/\mathrm{d}Q_D^{T_k^i}$ that defines the measure change from $Q_i^{T_k^i}$ to $Q_D^{T_k^i}$. This is the approach proposed by Bianchetti (2009), who uses a foreign-currency analogy and derives a quanto-like correction for the drift of F_k^i under $Q_D^{T_k^i}$. Here, instead, we take a different route.

Our idea is to follow a conceptually similar approach as in the classic LMM. There, the trick was to replace the LIBOR rate entering the caplet payoff with the equivalent forward rate, since the latter has "better" dynamics (a martingale) under the reference pricing measure. Here, we make a step forward, and replace the forward rate with its conditional expected value (the FRA rate). The purpose is the same as before, namely to introduce an underlying asset whose dynamics is easier to model.

Since

$$L_k^i(t) = E_D^{T_k^i}\big[F_k^i(T_{k-1}^i) | \mathcal{F}_t\big],$$

at the reset time T_{k-1}^i the two rates F_k^i and L_k^i coincides:

$$L_k^i(T_{k-1}^i) = F_k^i(T_{k-1}^i).$$

[20]We will use the symbol "d" to denote differentials as opposed to d, which instead denotes the index of the final date in the swap's fixed leg.

We can, therefore, replace the payoff (9) with

$$\tau_k^i [L_k^i(T_{k-1}^i) - K]^+ \tag{11}$$

and view the caplet as a call option no more on $F_k^i(T_{k-1}^i)$ but on $L_k^i(T_{k-1}^i)$. This leads to:

$$\mathbf{Cplt}(t, K; T_{k-1}^i, T_k^i) = \tau_k^i P_D(t, T_k^i) E_D^{T_k^i}\big\{[L_k^i(T_{k-1}^i) - K]^+ | \mathcal{F}_t\big\}. \tag{12}$$

The FRA rate $L_k^i(t)$ is, by definition, a martingale under the measure $Q_D^{T_k^i}$. If we smartly choose the dynamics of such a rate, we can value the last expectation analytically and obtain a closed-form formula for the caplet price. For instance, the obvious choice of a driftless geometric Brownian motion

$$\mathrm{d}L_k^i(t) = v_k L_k^i(t)\,\mathrm{d}Z_k(t), \quad t \leq T_{k-1}^i$$

where v_k is a constant and Z_k is now a $Q_D^{T_k^i}$-Brownian motion, leads to the following pricing formula:

$$\mathbf{Cplt}(t, K; T_{k-1}^i, T_k^i) = \tau_k^i P_D(t, T_k^i)\,\mathrm{Bl}\big(K, L_k^i(t), v_k\sqrt{T_{k-1}^i - t}\big).$$

Therefore, under lognormal dynamics for the rate L_k^i, the caplet price is again given by Black's formula with an implied volatility v_k. The differences with respect to the classic formula (10) are given by the underlying rate, which here is the FRA rate L_k^i, and by the discount factor, which here belongs to curve D. Moreover, the volatility v_k can in principle be different than σ_k. Notice in fact, that the moneyness of the strike K changes when moving from the single to the two-curve setting.

5.2 Market formula for swaptions

The other plain-vanilla option in the interest rate market is the European swaption. A payer swaption gives the right to enter at time $T_a^i = T_c^S$ an IRS with payment times for the floating and fixed legs given, respectively, by $T_{a+1}^i, \ldots, T_b^i$ and $T_{c+1}^S, \ldots, T_d^S$, with $T_b^i = T_d^S$ and where the fixed rate is K. Its payoff at time $T_a^i = T_c^S$ is therefore

$$\big[S_{a,b,c,d}^i(T_a^i) - K\big]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S), \tag{13}$$

where, see (7),

$$S_{a,b,c,d}^i(t) = \frac{\sum_{k=a+1}^{b} \tau_k^i P_D(t, T_k^i) L_k^i(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}.$$

Setting

$$C_D^{c,d}(t) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)$$

the payoff (13) is conveniently priced under the swap measure $Q_D^{c,d}$, whose associated numeraire is the annuity $C_D^{c,d}(t)$. In fact, we get:

$$\begin{aligned}
&\mathbf{PS}(t, K; T_{a+1}^i, \dots, T_b^i, T_{c+1}^S, \dots, T_d^S) \\
&= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E^{Q_D^{c,d}} \left\{ \frac{\left[S_{a,b,c,d}^i(T_a^i) - K\right]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S)}{C_D^{c,d}(T_c^S)} | \mathcal{F}_t \right\} \\
&= \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E^{Q_D^{c,d}} \left\{ \left[S_{a,b,c,d}^i(T_a^i) - K\right]^+ | \mathcal{F}_t \right\}
\end{aligned} \tag{14}$$

so that, also in our multi-curve paradigm, pricing a swaption is equivalent to pricing an option on the underlying swap rate.

As in the single-curve case, the forward swap rate $S_{a,b,c,d}^i(t)$ is a martingale under the swap measure $Q_D^{c,d}$. In fact, by (6), $S_{a,b,c,d}^i(t)$ is equal to a tradable asset (the floating leg of the swap) divided by the numeraire $C_D^{c,d}(t)$:

$$S_{a,b,c,d}^i(t) = \frac{\sum_{k=a+1}^{b} \tau_k^i P_D(t, T_k^i) L_k^i(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} = \frac{\mathbf{FL}(t; T_a^i, \dots, T_b^i)}{C_D^{c,d}(t)}. \tag{15}$$

Assuming that the swap rate $S_{a,b,c,d}^i$ evolves, under $Q_D^{c,d}$, according to a driftless geometric Brownian motion:

$$\mathrm{d}S_{a,b,c,d}^i(t) = \nu_{a,b,c,d} S_{a,b,c,d}^i(t)\, \mathrm{d}Z_{a,b,c,d}(t), \quad t \le T_a^i$$

where $\nu_{a,b,c,d}$ is a constant and $Z_{a,b,c,d}$ is a $Q_D^{c,d}$-Brownian motion, the expectation in (14) can be explicitly calculated as in the caplet case, leading to the generalized Black formula:

$$\mathbf{PS}(t, K; T_{a+1}^i, \dots, T_b^i, T_{c+1}^S, \dots, T_d^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, \mathrm{Bl}\big(K, S_{a,b,c,d}^i(t), \nu_{a,b,c,d}\sqrt{T_a^i - t}\big).$$

Therefore, the two-curve swaption price is still given by a Black-like formula, with the only differences with respect to the basic case that discounting is done through curve D and that the swap rate $S_{a,b,c,d}^i(t)$ has a more general definition. Similarly to the caplet case, the volatility $\nu_{a,b,c,d}$ can be different than that in the single-curve paradigm.

A recap of the market formulas for caplets and swaptions in the single- and two-curve setting is provided in Table 3, where the "old" (single-curve) swap rate is defined as in Table 2, *i.e.*

$$S_{c,d}^i(t) = \frac{P_i(t, T_c^S) - P_i(t, T_d^S)}{\sum_{j=c+1}^{d} \tau_j^S P_i(t, T_j^S)},$$

and $\sigma_{c,d}$ denotes the corresponding implied volatility.

Type	Formulas
OLD **Cplt**	$\tau_k^i P_i(t,T_k^i)\,\mathrm{Bl}\big(K, F_k^i(t), \sigma_k\sqrt{T_{k-1}^i - t}\big)$
NEW **Cplt**	$\tau_k^i P_D(t,T_k^i)\,\mathrm{Bl}\big(K, L_k^i(t), v_k\sqrt{T_{k-1}^i - t}\big)$
OLD **PS**	$\sum_{j=c+1}^d \tau_j^S P_i(t,T_j^S)\,\mathrm{Bl}\big(K, S_{c,d}^i(t), \sigma_{c,d}\sqrt{T_a^i - t}\big)$
NEW **PS**	$\sum_{j=c+1}^d \tau_j^S P_D(t,T_j^S)\,\mathrm{Bl}\big(K, S_{a,b,c,d}^i(t), \nu_{a,b,c,d}\sqrt{T_a^i - t}\big)$

Table 3: Comparison between "old" and "new" formulas for caplets and swaptions.

After having derived market formulas for caps and swaptions under distinct discount and forward curves, we are now ready to extend the basic LMMs. We start by considering the fundamental case of lognormal dynamics, and then introduce stochastic volatility in a rather general fashion.

6 The two-curve lognormal LMM

In the classic (single-curve) LMM, one models the joint evolution of a set of consecutive forward LIBOR rates under a common pricing measure, typically some "terminal" forward measure or the spot LIBOR measure corresponding to the set of times defining the family of forward rates. Denoting by $\mathcal{T} = \{T_0^i, \ldots, T_M^i\}$ the times in question, one then jointly models rates F_k^i, $k = 1, \ldots, M$, under the forward measure $Q_i^{T_M^i}$ or under the spot LIBOR measure $Q_i^{\mathcal{T}}$. Using measure change techniques, one finally derives pricing formulas for the main calibration instruments (caps and swaptions) either in closed form or through efficient approximations.

To extend the LMM to the multi-curve case, we first need to identify the rates we need to model. The previous section suggests that the FRA rates L_k^i are convenient rates to model as soon as we have to price a payoff, like that of a caplet, which depends on LIBOR rates belonging to the same curve i. Moreover, in case of a swap-rate dependent payoff, we notice we can write

$$S_{a,b,c,d}^i(t) = \frac{\sum_{k=a+1}^b \tau_k^i P_D(t,T_k^i) L_k^i(t)}{\sum_{j=c+1}^d \tau_j^S P_D(t,T_j^S)} = \sum_{k=a+1}^b \omega_k(t) L_k^i(t), \tag{16}$$

where the weights ω_k are defined by

$$\omega_k(t) := \frac{\tau_k^i P_D(t, T_k^i)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}. \tag{17}$$

Characterizing the forward swap rate $S_{a,b,c,d}^i(t)$ as a linear combination of FRA rates $L_k^i(t)$ gives another argument supporting the modeling of FRA rates as fundamental bricks to generate sensible future payoffs in the pricing of interest rate derivatives. Notice, also the consistency with the standard single-curve case, where the forward LIBOR rates $F_k^i(t)$ and the FRA rates $L_k^i(t)$ coincide by definition.

However, there is a major difference with respect to the single-curve case, namely that forward rates belonging to the discount curve need to be modeled too. In fact, as is evident from equation (16), future swap rates also depend on future discount factors which, unless we unrealistically assume a deterministic discount curve, will evolve stochastically over time. Moreover, we will show below that the dynamics of FRA rates under typical pricing measures depend on forward rates of curve D, so that also path-dependent payoffs on LIBOR rates will depend on the dynamics of the discount curve.

6.1 The model dynamics

The LMM was introduced in the financial literature by Brace et al. (1997) and Miltersen et al. (1997) by assuming that forward LIBOR rates have a lognormal-type diffusion coefficient.[21] Here, we extend their approach to the case where the curve used for discounting is different than that used to generate the relevant future rates. For simplicity, we stick to the case where these rates belong to the same curve i.

Let us consider a set of times $\mathcal{T} = \{0 < T_0^i, \ldots, T_M^i\}$, which we assume to be compatible with curve i. We assume that each rate $L_k^i(t)$ evolves, under the forward measure $Q_D^{T_k^i}$, as a driftless geometric Brownian motion:

$$\mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t)\, \mathrm{d}Z_k(t), \quad t \le T_{k-1}^i \tag{18}$$

where the instantaneous volatility $\sigma_k(t)$ is deterministic and Z_k is the k-th component of an M-dimensional $Q_D^{T_k^i}$-Brownian motion Z with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\ldots,M}$, namely $\mathrm{d}Z_k(t)\, \mathrm{d}Z_j(t) = \rho_{k,j}\, \mathrm{d}t$.

In a two-curve setting, we also need to model the evolution of rates

$$\begin{aligned} F_k^D(t) &= F_D(t; T_{k-1}^i, T_k^i) = \frac{1}{\tau_k^D} \left[\frac{P_D(t, T_{k-1}^i)}{P_D(t, T_k^i)} - 1 \right] \\ \tau_k^D &= \tau_D(T_{k-1}^i, T_k^i) \end{aligned}$$

[21] This implies that each forward LIBOR rate evolves according to a geometric Brownian motion under its associated forward measure.

To this end, we assume that the dynamics of each rate F_h^D under the associated forward measure $Q_D^{T_h^i}$ is given by:

$$\mathrm{d}F_h^D(t) = \sigma_h^D(t) F_h^D(t)\,\mathrm{d}Z_h^D(t), \quad t \le T_{h-1}^i \tag{19}$$

where the instantaneous volatility $\sigma_h^D(t)$ is deterministic and Z_h^D is the h-th component of an M-dimensional $Q_D^{T_h^i}$-Brownian motion Z^D whose correlations are

$$\begin{aligned} \mathrm{d}Z_k^D(t)\,\mathrm{d}Z_h^D(t) &= \rho_{k,h}^{D,D}\,\mathrm{d}t \\ \mathrm{d}Z_k(t)\,\mathrm{d}Z_h^D(t) &= \rho_{k,h}^{i,D}\,\mathrm{d}t \end{aligned}$$

Clearly, correlations $\rho = (\rho_{k,j})_{k,j=1,\dots,M}$, $\rho^{D,D} = (\rho_{k,h}^{D,D})_{k,h=1,\dots,M}$ and $\rho^{i,D} = (\rho_{k,h}^{i,D})_{k,h=1,\dots,M}$ must be chosen so as to ensure that the global matrix

$$R := \left[\begin{array}{c|c} \rho & \rho^{i,D} \\ \hline \left(\rho^{i,D}\right)' & \rho^{D,D} \end{array}\right]$$

is positive (semi)definite ($'$ denotes transposition).

Remark 2 *In some situations, it may be more realistic to resort to an alternative approach and model either curve i or D jointly with the spread between them, see* e.g. *Kijima (2009) or Schönbucher (2000). This happens, for instance, when one curve is above the other and there are sound financial reasons why the spread should be preserved positive in the future. In such a case, one can assume that each spread $X_k(t) := |L_k^i(t) - F_k^D(t)|$ evolves under the corresponding forward measure $Q_D^{T_k^i}$, according to some*

$$\mathrm{d}X_k(t) = \sigma_k^X(t, X_k(t))\,\mathrm{d}Z_k^X(t),$$

whose solution is positive valued, and where the drift is zero since X_k is either the sum or the difference of two martingales. Sticking to (18), the dynamics (19) of forward rates F_k^D must then be replaced with

$$\mathrm{d}F_k^D(t) = \mathrm{d}L_k^i(t) \pm \mathrm{d}X_k(t),$$

where the sign $\pm$ depends on the relative position of curves i and D. The analysis that follows can be equivalently applied to the new dynamics of rates F_k^D.[22]

6.2 Dynamics under a general forward measure

To derive the dynamics of the FRA rate $L_k^i(t)$ under the forward measure $Q_D^{T_j^i}$ we start from (18) and perform a change of measure from $Q_D^{T_k^i}$ to $Q_D^{T_j^i}$, whose associated numeraires are the curve-D zero-coupon bonds with maturities T_k^i and T_j^i, respectively. To this end, we apply the change-of-numeraire formula relating the drifts of a given process under two

[22] The calculations are essentially the same. Their length depends on the chosen covariation structure.

measures with known numeraires, see for instance Brigo and Mercurio (2006). The drift of $L_k^i(t)$ under $Q_D^{T_j^i}$ is then equal to

$$\mathbf{Drift}(L_k^i; Q_D^{T_j^i}) = -\frac{\mathrm{d}\langle L_k^i, \ln(P_D(\cdot, T_k^i)/P_D(\cdot, T_j^i))\rangle_t}{\mathrm{d}t},$$

where $\langle X, Y\rangle_t$ denotes the instantaneous covariation between processes X and Y at time t.

Let us first consider the case $j < k$. The log of the ratio of the two numeraires can be written as

$$\begin{aligned}\ln(P_D(t, T_k^i)/P_D(t, T_j^i)) &= \ln\left(1/\left[\prod_{h=j+1}^{k}(1+\tau_h^D F_h^D(t))\right]\right)\\ &= -\sum_{h=j+1}^{k}\ln\left(1+\tau_h^D F_h^D(t)\right)\end{aligned}$$

from which we get:

$$\begin{aligned}\mathbf{Drift}(L_k^i; Q_D^{T_j^i}) &= -\frac{\mathrm{d}\langle L_k^i, \ln(P_D(\cdot, T_k^i)/P_D(\cdot, T_j^i))\rangle_t}{\mathrm{d}t} = \sum_{h=j+1}^{k}\frac{\mathrm{d}\langle L_k^i, \ln\left(1+\tau_h^D F_h^D\right)\rangle_t}{\mathrm{d}t}\\ &= \sum_{h=j+1}^{k}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\frac{\mathrm{d}\langle L_k^i, F_h^D\rangle_t}{\mathrm{d}t}.\end{aligned}$$

In the standard LMM, the drift term of L_k^i under $Q_D^{T_j^i}$ depends on the instantaneous covariations between forward rates F_k^i and F_h^i, $h = j+1, \ldots, k$. The initial assumptions on the joint dynamics of forward rates are therefore sufficient to determine such a drift term. Here, however, the situation is different since rates L_k^i and F_h^D belong, in general, to different curves, and to calculate the instantaneous covariations in the drift term, we also need the dynamics of rates F_h^D.

Under (19), we thus obtain

$$\mathbf{Drift}(L_k^i; Q_D^{T_j^i}) = \sigma_k(t)L_k^i(t)\sum_{h=j+1}^{k}\frac{\rho_{k,h}^{i,D}\tau_h^D\sigma_h^D(t)F_h^D(t)}{1+\tau_h^D F_h^D(t)}.$$

The derivation of the drift rate in the case $j > k$ is perfectly analogous.

As to forward rates F_k^D, their $Q_D^{T_j^i}$-dynamics are equivalent to those we obtain in the classic single-curve case, see Brigo and Mercurio (2006), since these probability measures and rates are associated to the same curve D.

The joint evolution of all FRA rates $L_1^i, \ldots, L_M^i$ and forward rates $F_1^D, \ldots, F_M^D$ under a common forward measure is then summarized in the following.

Proposition 3 *The dynamics of L_k^i and F_k^D under the forward measure $Q_D^{T_j^i}$ in the three cases $j < k$, $j = k$ and $j > k$ are, respectively,*

$$
\begin{aligned}
&j < k,\ t \le T_j^i : &&\begin{cases} \mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t) \left[\displaystyle\sum_{h=j+1}^{k} \frac{\rho_{k,h}^{i,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \mathrm{d}Z_k^j(t) \right] \\ \mathrm{d}F_k^D(t) = \sigma_k^D(t) F_k^D(t) \left[\displaystyle\sum_{h=j+1}^{k} \frac{\rho_{k,h}^{D,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \mathrm{d}Z_k^{j,D}(t) \right] \end{cases} \\
&j = k,\ t \le T_{k-1}^i : &&\begin{cases} \mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t)\, \mathrm{d}Z_k^j(t) \\ \mathrm{d}F_k^D(t) = \sigma_k^D(t) F_k^D(t)\, \mathrm{d}Z_k^{j,D}(t) \end{cases} \\
&j > k,\ t \le T_{k-1}^i : &&\begin{cases} \mathrm{d}L_k^i(t) = \sigma_k(t) L_k^i(t) \left[- \displaystyle\sum_{h=k+1}^{j} \frac{\rho_{k,h}^{i,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \mathrm{d}Z_k^j(t) \right] \\ \mathrm{d}F_k^D(t) = \sigma_k^D(t) F_k^D(t) \left[- \displaystyle\sum_{h=k+1}^{j} \frac{\rho_{k,h}^{D,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \mathrm{d}Z_k^{j,D}(t) \right] \end{cases}
\end{aligned}
\tag{20}
$$

where Z_k^j and $Z_k^{j,D}$ are the k-th components of M-dimensional $Q_D^{T_j^i}$-Brownian motions Z^j and $Z^{j,D}$ with correlation matrix R.

Remark 4 *Following the same arguments used in the standard single-curve case, we can easily prove that the SDEs (20) for the FRA rates all admit a unique strong solution if the coefficients σ_h^D are bounded.*

When curves i and D coincide, we have already noticed that the FRA rates L_k^i coincide with the corresponding F_k^i. As a further sanity check, we can also see that the FRA rate dynamics reduce to those of the corresponding forward rates since

$$
i \equiv D \Rightarrow \begin{cases} \rho_{k,h}^{i,D} \to \rho_{k,h} \\ \tau_h^D \to \tau_h^i \\ \sigma_h^D(t) \to \sigma_h(t) \\ F_h^D(t) \to F_h^i(t) \end{cases}
$$

for each h, k.

The extended dynamics (20) may raise some concern on numerical issues. In fact, having doubled the number of rates to simulate, the computational burden of the lognormal LMM (20) is doubled with respect to that of the single-curve case, since the SDEs for the homologues L_k^i and F_k^D share the same structure. However, some smart selection of the correlations between rates can reduce the simulation time. For instance, assuming that $\rho_{k,h}^{i,D} = \rho_{k,h}^{D,D}$ for each h, k, leads to the same drift rates for L_k^i and the corresponding F_k^D, thus halvening the number of drifts to be calculated at each simulation time. This gives a valuable advantage since it is well known that the drift calculations in a LMM are extremely time consuming.

6.3 Dynamics under the spot LIBOR measure

Another measure commonly used for modeling the joint evolution of the given rates and for pricing derivatives is the spot LIBOR measure $Q_D^{\mathcal{T}}$ associated to times $\mathcal{T} = \{T_0^i, \ldots, T_M^i\}$, whose numeraire is the discretely-rebalanced bank account $B_D^{\mathcal{T}}$

$$B_D^{\mathcal{T}}(t) = \frac{P_D(t, T^i_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P_D(T^i_{j-1}, T^i_j)},$$

where $\beta(t) = m$ if $T^i_{m-2} < t \leq T^i_{m-1}$, $m \geq 1$, so that $t \in (T^i_{\beta(t)-2}, T^i_{\beta(t)-1}]$.

Application of the change-of numeraire technique, immediately leads to the following.

Proposition 5 *The dynamics of FRA and forward rates under the spot LIBOR measure $Q_D^{\mathcal{T}}$ are given by:*

$$\begin{aligned}
\mathrm{d}L_k^i(t) &= \sigma_k(t) L_k^i(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{i,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \sigma_k(t) L_k^i(t) \,\mathrm{d}Z_k^d(t) \\
\mathrm{d}F_k^D(t) &= \sigma_k^D(t) F_k^D(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{D,D} \tau_h^D \sigma_h^D(t) F_h^D(t)}{1 + \tau_h^D F_h^D(t)} \,\mathrm{d}t + \sigma_k^D(t) F_k^D(t) \,\mathrm{d}Z_k^{d,D}(t)
\end{aligned} \tag{21}$$

where $Z^d = \{Z_1^d, \ldots, Z_M^d\}$ and $Z^{d,D} = \{Z_1^{d,D}, \ldots, Z_M^{d,D}\}$ are M-dimensional $Q_D^{\mathcal{T}}$-Brownian motions with correlation matrix R.

6.4 Pricing caplets in the lognormal LMM

The pricing of caplets in the LMM is straightforward and follows from the same arguments of Section 5. We get:

$$\mathbf{Cplt}(t, K; T^i_{k-1}, T^i_k) = \tau_k^i P_D(t, T_k^i) \,\mathrm{Bl}(K, L_k^i(t), v_k(t))$$

where

$$v_k(t) := \sqrt{\int_t^{T^i_{k-1}} \sigma_k(u)^2 \, du}$$

As expected, thanks to the lognormality assumption, this formula for caplets (and hence caps) is analogous to that obtained in the basic lognormal LMM. We just have to replace forward rates with FRA rates and use the discount factors coming from curve D.

6.5 Pricing swaptions in the lognormal LMM

An analytical approximation for the implied volatility of swaptions can be derived also in our multi-curve setting. To this end, remember (16) and (17):

$$S^i_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \omega_k(t) L^i_k(t),$$
$$\omega_k(t) = \frac{\tau^i_k P_D(t, T^i_k)}{\sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j)}.$$

The forward swap rate $S^i_{a,b,c,d}(t)$ can be written as a linear combination of FRA rates $L^i_k(t)$. Contrary to the single-curve case, the weights are not a function of the FRA rates only, since they also depend on discount factors calculated on curve D. Therefore we can not write that, under the swap measure $Q^{c,d}_D$, the swap rate $S^i_{a,b,c,d}(t)$ satisfies the S.D.E.

$$\mathrm{d}S^i_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \frac{\partial S^i_{a,b,c,d}(t)}{\partial L^i_k(t)} \sigma_k(t) L^i_k(t)\, \mathrm{d}Z^{c,d}_k(t).$$

However, we can resort to a standard approximation technique and freeze the weights ω_k at their time-zero value. This leads to the approximation

$$S^i_{a,b,c,d}(t) \approx \sum_{k=a+1}^{b} \omega_k(0) L^i_k(t),$$

which enables us to write

$$\mathrm{d}S^i_{a,b,c,d}(t) \approx \sum_{k=a+1}^{b} \omega_k(0) \sigma_k(t) L^i_k(t)\, \mathrm{d}Z^{c,d}_k(t). \tag{22}$$

Notice, in fact, that by freezing the weights, we are also freezing the dependence of $S^i_{a,b,c,d}(t)$ on forward rates F^D_h.

To obtain a closed equation of type

$$\mathrm{d}S^i_{a,b,c,d}(t) = S^i_{a,b,c,d}(t) v_{a,b,c,d}(t)\, \mathrm{d}Z_{a,b,c,d}(t), \tag{23}$$

we equate the instantaneous quadratic variations of (22) and (23):

$$\left[v_{a,b,c,d}(t) S^i_{a,b,c,d}(t)\right]^2 \mathrm{d}t = \sum_{h,k=a+1}^{b} \omega_h(0)\omega_k(0)\sigma_h(t)\sigma_k(t) L^i_h(t) L^i_k(t) \rho_{h,k}\, \mathrm{d}t. \tag{24}$$

Freezing FRA and swap rates at their time-zero value, we obtain this (approximated) $Q^{c,d}_D$-dynamics for the swap rate $S^i_{a,b,c,d}(t)$:

$$\mathrm{d}S^i_{a,b,c,d}(t) = S^i_{a,b,c,d}(t) \sqrt{\frac{\sum_{h,k=a+1}^{b} \omega_h(0)\omega_k(0)\sigma_h(t)\sigma_k(t) L^i_h(0) L^i_k(0) \rho_{h,k}}{(S^i_{a,b,c,d}(0))^2}}\, \mathrm{d}Z_{a,b,c,d}(t).$$

This immediately leads to the following (payer) swaption price at time 0:

$$\mathbf{PS}(0,K;T_{a+1}^i,\ldots,T_b^i,T_{c+1}^S,\ldots,T_d^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(0,T_j^S)\,\mathrm{Bl}\big(K, S_{a,b,c,d}^i(0), V_{a,b,c,d}\big),$$

where the swaption implied volatility (multiplied by $\sqrt{T_a^i}$) is given by

$$V_{a,b,c,d} = \sqrt{\sum_{h,k=a+1}^{b} \frac{\omega_h(0)\omega_k(0)L_h^i(0)L_k^i(0)\rho_{h,k}}{(S_{a,b,c,d}^i(0))^2} \int_0^{T_a^i} \sigma_h(t)\sigma_k(t)\,\mathrm{d}t}. \tag{25}$$

Again, this formula is analogous in structure to that obtained in the classic lognormal LMM, see Brigo and Mercurio (2006). Here, besides the annuity term (based on the discount curve) and the forward swap rate (more general definition), the difference is that the swaption volatility depends both on curves i and D, since weights ω's belong to curve D and the FRA and swap rates are calculated with both curves.

A better approximation for lognormal LMM swaption volatilities can be derived by assuming that each T_j^S belongs to $\mathcal{T} = \{T_0^i,\ldots,T_M^i\}$, so that, for each j, there exists an index i_j such that $T_j^S = T_{i_j}^i$. In this case, we can write:

$$\begin{aligned}\omega_k(t) &= \frac{\tau_k^i P_D(t,T_k^i)}{\sum_{j=c+1}^d \tau_j^S P_D(t,T_j^S)} = \frac{\tau_k^i \dfrac{P_D(t,T_k^i)}{P_D(t,T_a^i)}}{\sum_{j=c+1}^d \tau_j^S \dfrac{P_D(t,T_{i_j}^i)}{P_D(t,T_a^i)}} \\ &= \frac{\tau_k^i \displaystyle\prod_{h=a+1}^{k} \frac{1}{1+\tau_h^D F_h^D(t)}}{\sum_{j=c+1}^d \tau_j^S \displaystyle\prod_{a=c+1}^{i_j} \frac{1}{1+\tau_h^D F_h^D(t)}} =: f(F_{a+1}^D(t),\ldots,F_b^D(t))\end{aligned}$$

where the last equality defines the function f and where the subscripts of rates $F_h^D(t)$ range from $a+1$ to b since $T_d^S = T_b^i$ (namely $i_d = b$).

Under the swap measure $Q_D^{c,d}$, the swap rate $S_{a,b,c,d}^i(t)$ then satisfies the S.D.E.

$$\mathrm{d}S_{a,b,c,d}^i(t) = \sum_{k=a+1}^{b} \frac{\partial S_{a,b,c,d}^i(t)}{\partial L_k^i(t)} \sigma_k(t) L_k^i(t)\,\mathrm{d}Z_k^{c,d}(t) + \sum_{k=a+1}^{b} \frac{\partial S_{a,b,c,d}^i(t)}{\partial F_k^D(t)} \sigma_k^D(t) F_k^D(t)\,\mathrm{d}Z_k^{c,d,D}(t),$$

where $\{Z_1^{c,d},\ldots,Z_M^{c,d}\}$ and $\{Z_1^{c,d,D},\ldots,Z_M^{c,d,D}\}$ are M-dimensional $Q_D^{c,d}$-Brownian motions with correlation matrix R.

Matching instantaneous quadratic variations as in (24) and freezing stochastic quantities at their time-zero value, we can finally obtain another, more accurate, approximation for the implied swaption volatility in the lognormal LMM, which we here omit for brevity.

6.6 The terminal correlation between FRA rates

Assume we are interested to calculate, at time 0, the terminal correlation between the FRA rates L_k^i and L_h^i at time T_j^i, with $j \leq k-1 < h$, under the forward measure $Q_D^{T_r^i}$, with $r \geq j$:

$$\begin{aligned} &\mathrm{Corr}_D^{T_r^i}\left(L_k^i(T_j^i), L_h^i(T_j^i)\right) \\ &= \frac{E_D^{T_r^i}\left[L_k^i(T_j^i) L_h^i(T_j^i)\right] - E_D^{T_r^i}\left[L_k^i(T_j^i)\right] E_D^{T_r^i}\left[L_h^i(T_j^i)\right]}{\sqrt{E_D^{T_r^i}\left[(L_k^i(T_j^i))^2\right] - \left[E_D^{T_r^i}\left(L_k^i(T_j^i)\right)\right]^2}\sqrt{E_D^{T_r^i}\left[(L_h^i(T_j^i))^2\right] - \left[E_D^{T_r^i}\left(L_h^i(T_j^i)\right)\right]^2}} \end{aligned} \tag{26}$$

Mimicking the derivation of the approximation formula in the single-curve lognormal LMM, see Brigo and Mercurio (2006), we first recall the dynamics of L_x^i, $x = k, h$, under $Q_D^{T_r^i}$:

$$\mathrm{d}L_x^i(t) = \mu_x(t) L_x^i(t)\,\mathrm{d}t + \sigma_x(t) L_x^i(t)\,\mathrm{d}Z_x(t)$$

where

$$\mu_x(t) := \begin{cases} \sigma_x(t) \displaystyle\sum_{l=j+1}^{x} \frac{\rho_{x,l}^{i,D} \tau_l^D \sigma_l^D(t) F_l^D(t)}{1 + \tau_l^D F_l^D(t)} & \text{if } r < x \\ 0 & \text{if } r = x \\ -\sigma_x(t) \displaystyle\sum_{l=x+1}^{j} \frac{\rho_{x,l}^{i,D} \tau_l^D \sigma_l^D(t) F_l^D(t)}{1 + \tau_l^D F_l^D(t)} & \text{if } r > x \end{cases}$$

Then, in the drift rate $\mu_x(t)$, we freeze the forward rates $F_l^D(t)$ at their time-0 value to obtain:

$$\mathrm{d}L_x^i(t) = \nu_x(t) L_x^i(t)\,\mathrm{d}t + \sigma_x(t) L_x^i(t)\,\mathrm{d}Z_x(t)$$

where

$$\nu_x(t) := \begin{cases} \sigma_x(t) \displaystyle\sum_{l=j+1}^{x} \frac{\rho_{x,l}^{i,D} \tau_l^D \sigma_l^D(t) F_l^D(0)}{1 + \tau_l^D F_l^D(0)} & \text{if } r < x \\ 0 & \text{if } r = x \\ -\sigma_x(t) \displaystyle\sum_{l=x+1}^{j} \frac{\rho_{x,l}^{i,D} \tau_l^D \sigma_l^D(t) F_l^D(0)}{1 + \tau_l^D F_l^D(0)} & \text{if } r > x \end{cases}$$

Since ν_x is deterministic, L_x^i follows (approximately) a geometric Brownian motion. The expectations (26) are thus straightforward to calculate. We get:

$$\begin{aligned} E_D^{T_r^i}\left[L_x^i(T_j^i)\right] &= L_x^i(0) \exp\left\{\int_0^{T_j^i} \nu_x(t)\,\mathrm{d}t\right\}, \quad x = k, h \\ E_D^{T_r^i}\left[\left(L_x^i(T_j^i)\right)^2\right] &= (L_x^i(0))^2 \exp\left\{\int_0^{T_j^i} [2\nu_x(t) + (\sigma_x(t))^2]\,\mathrm{d}t\right\}, \quad x = k, h \\ E_D^{T_r^i}\left[L_k^i(T_j^i) L_h^i(T_j^i)\right] &= L_k^i(0) L_h^i(0) \exp\left\{\int_0^{T_j^i} [\nu_k(t) + \nu_h(t) + \rho_{k,h}\sigma_k(t)\sigma_h(t)]\,\mathrm{d}t\right\} \end{aligned}$$

Term	Strike	ATM	1.00 %	1.50 %	2.00 %	3.00 %	3.50 %	4.00 %	5.00 %	6.00 %	7.00 %	8.00 %
1 YR	1.91	79.45	99.20	85.75	77.55	67.90	64.85	62.40	58.90	56.60	54.95	53.7
2 YR	2.09	71.20	94.45	80.55	71.85	60.75	56.90	53.90	49.55	46.60	44.50	42.9
3 YR	2.41	60.85	88.60	75.10	66.35	54.70	50.65	47.40	42.70	39.60	37.45	35.9
4 YR	2.71	52.60	82.25	69.45	61.00	49.55	45.50	42.35	37.90	35.05	33.15	31.8
5 YR	2.94	47.50	77.40	65.40	57.35	46.45	42.65	39.70	35.55	32.90	31.20	30.0
6 YR	3.12	44.20	73.80	62.35	54.65	44.20	40.60	37.85	33.90	31.45	29.85	28.7
7 YR	3.26	41.60	69.75	59.20	52.05	42.35	39.05	36.45	32.90	30.65	29.20	28.2
8 YR	3.36	39.75	66.25	56.45	49.80	40.75	37.70	35.35	32.05	30.00	28.75	27.9
9 YR	3.43	38.25	63.40	54.20	47.90	39.40	36.55	34.35	31.30	29.45	28.30	27.5
10 YR	3.48	37.30	61.40	52.55	46.60	38.50	35.80	33.70	30.85	29.10	28.05	27.3
12 YR	3.23	36.00	58.65	50.35	44.75	37.25	34.75	32.85	30.25	28.70	27.70	27.1
15 YR	3.24	35.00	56.25	48.35	43.10	36.10	33.85	32.10	29.70	28.30	27.45	26.9
20 YR	3.20	34.30	54.20	46.65	41.65	35.15	33.05	31.45	29.25	28.00	27.25	26.8
25 YR	3.18	33.90	53.20	45.75	40.85	34.60	32.60	31.05	29.05	27.85	27.15	26.7

Figure 4: USD cap volatilities as of November 25, 2008. Strikes from 1%to 8%. Source: Bloomberg.

so that (26) becomes:

$$\mathrm{Corr}_D^{T_r^i}\big(L_k^i(T_j^i), L_h^i(T_j^i)\big) = \frac{\exp\Big\{\int_0^{T_j^i} \rho_{k,h}\sigma_k(t)\sigma_h(t)\,\mathrm{d}t\Big\} - 1}{\sqrt{\exp\Big\{\int_0^{T_j^i}(\sigma_k(t))^2\,\mathrm{d}t\Big\} - 1}\sqrt{\exp\Big\{\int_0^{T_j^i}(\sigma_h(t))^2\,\mathrm{d}t\Big\} - 1}}$$

Not surprisingly, this expression coincides with that we get in the single-curve case. In fact, the drift rates μ_x, $x = h, k$, which are the only terms that change when moving from single- to two-curve LMM, do not enter the analytical approximation for terminal correlations.

7 Introducing stochastic volatility

A vast literature has been written on the single-curve LMM, especially on its lognormal version whose advantages are well known, see *e.g.* Brigo and Mercurio (2006). The advantages are in fact shared by our two-curve extension, which thus allows for: i) caplet prices consistent with Black's formula, leading to an automatic calibration to (at-the-money) caplet volatilities; ii) efficient explicit approximation for swaption volatilities; iii) exact (cascade) calibration of at-the-money swaption volatilities; iv) closed-form approximation for terminal correlations; v) deterministic future implied volatilities.[23]

As is also well known, however, the lognormal LMM has the main drawback of producing constant implied volatilities for any given maturity. To properly account for the typical smile shapes observed in the market, see Figures 4 and 5, one then has to relax the constant-volatility assumption for the dynamics of the FRA rates.

The most popular extensions of the LMM are based on modeling stochastic volatility as in Heston (1993) or in Hagan et al. (2002). In the first category fall the LMMs by Andersen

[23]Points i), ii) and iv) have been shown before, v) follows immediately, whereas iii) can be proved exactly as in Brigo and Mercurio (2006).

USD SKEWS

TERM	-200bps	-100bps	-50bps	-25bps	25bps	50bps	100bps	200bps
3M2Y	139.77	31.84	13.18	6.07	-5.20	-9.65	-16.60	-24.19
3M5Y	50.97	17.82	7.79	3.66	-3.24	-6.10	-10.77	-16.51
3M10Y	38.76	14.40	6.40	3.02	-2.72	-5.15	-9.19	-14.41
3M20Y	38.48	14.09	6.23	2.93	-2.64	-4.98	-8.90	-14.12
3M30Y	38.28	13.88	6.11	2.88	-2.58	-4.87	-8.71	-13.88
1Y2Y	52.38	16.96	7.26	3.37	-2.91	-5.40	-9.20	-13.01
1Y5Y	33.07	12.45	5.55	2.63	-2.36	-4.46	-7.95	-12.27
1Y10Y	30.43	11.26	4.91	2.28	-1.98	-3.64	-6.11	-8.21
1Y20Y	31.44	11.55	5.04	2.35	-2.04	-3.77	-6.37	-8.73
1Y30Y	30.37	11.12	4.87	2.28	-2.01	-3.76	-6.52	-9.61
5Y2Y	16.79	6.56	2.93	1.38	-1.25	-2.35	-4.14	-6.31
5Y5Y	18.87	7.49	3.38	1.60	-1.46	-2.76	-4.93	-7.63
5Y10Y	23.44	8.85	3.85	1.78	-1.52	-2.77	-4.48	-5.46
5Y20Y	24.07	8.96	3.91	1.82	-1.57	-2.87	-4.76	-6.20
5Y30Y	23.37	8.69	3.80	1.77	-1.55	-2.86	-4.83	-6.60
10Y2Y	17.79	6.70	2.94	1.37	-1.20	-2.22	-3.77	-5.22
10Y5Y	21.37	7.86	3.39	1.56	-1.32	-2.38	-3.79	-4.48
10Y10Y	24.01	8.75	3.74	1.71	-1.41	-2.50	-3.80	-3.92
10Y20Y	23.76	8.50	3.60	1.64	-1.33	-2.33	-3.47	-3.41
10Y30Y	23.58	8.19	3.42	1.54	-1.23	-2.15	-3.15	-3.02

Figure 5: USD swaption volatilities as of November 25, 2008. Strikes and vols are expressed as differences from the respective ATM values. Source: Bloomberg.

and Andreasen (2002), Piterbarg (2005), Wu and Zhang (2006) and Zhu (2007). In the second, commonly referred to as SABR, the LMMs by Henry-Labordère (2007), Rebonato (2007), Hagan and Lesniewski (2008) and Mercurio and Morini (2009a, 2009b).[24]

In this section, we deal with general stochastic-volatility dynamics and show how to perform the relevant measure changes in our two-curve setting. To this end, we will follow the same procedure as in the lognormal LMM. Here, we will show again all calculations, not only for the sake of details, but because the considered case deserves a thorough analysis due to its general features.

We assume that we are given a set of times $\mathcal{T} = \{0 < T_0^i, \ldots, T_M^i\}$ compatible with curve i, and model FRA rates dynamics under the spot LIBOR measure $Q_D^{\mathcal{T}}$. Following (21), we assume that the $Q_D^{\mathcal{T}}$-dynamics of each $L_k^i(t)$ is given by

$$\mathrm{d}L_k^i(t) = \sum_{h=\beta(t)}^{k} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\, \mathrm{d}\langle L_k^i, F_h^D\rangle_t + \phi_k(t, L_k^i(t))\psi_t(V_k(t))\, \mathrm{d}Z_k^d(t), \tag{27}$$

where ϕ_k is a deterministic function of time and rate, ψ_t is a deterministic function of time and volatility, the stochastic volatility V_k is an adapted process, and $Z^d = \{Z_1^d, \ldots, Z_M^d\}$ is again an M-dimensional $Q_D^{\mathcal{T}}$-Brownian motion with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\ldots,M}$.

We then assume that the volatility process V_k evolves according to:

$$\mathrm{d}V_k(t) = a_k(t, V_k(t))\, \mathrm{d}t + b_k(t, V_k(t))\, \mathrm{d}W_k^d(t), \tag{28}$$

where a_k and b_k are deterministic functions of time and volatility and $W^d = \{W_1^d, \ldots, W_M^d\}$ is an M-dimensional $Q_D^{\mathcal{T}}$-Brownian motion correlated with Z^d.

[24]Further extensions can be considered by adding jumps or Levy-driven noises, see *e.g.* Eberlein and Özkan, F. (2005).

Dynamics (27) and (28) are general enough to include, as a particular case, all the models we mentioned above, provided that the spot LIBOR measure is chosen as reference for defining the basic volatility dynamics. For example, Henry-Labordère (2007) and Mercurio and Morini (2009a, 2009b) assume the following one-factor stochastic volatility dynamics

$$\begin{aligned} \mathrm{d}L_k^i(t) &= \cdots \mathrm{d}t + \sigma_k [L_k^i(t)]^\beta V(t)\,\mathrm{d}Z_k^d(t) \\ \mathrm{d}V(t) &= \nu V(t)\,\mathrm{d}W^d(t), \end{aligned} \tag{29}$$

which amounts to choosing:

$$\begin{aligned} \phi_k(t,x) &= \sigma_k x^\beta \\ \psi_t(x) &= x \\ a_k(t,x) &= 0 \\ b_k(t,x) &= \nu x \end{aligned}$$

where σ_k, β and ν are positive constants.

As we already explained in the lognormal case, the definition of a consistent LMM also requires the specification of the dynamics of the "discount" forward rates F_k^D and their related correlations. These dynamics, however, are equivalent to those in the single-curve case, and as such here omitted for brevity. Remarks on their possible specification will be provided at the end of the section.

7.1 Moving to a forward measure

The standard change-of-numeraire technique, see also Section 6, implies that, when moving from measure $Q_D^{\mathcal{T}}$ to measure $Q_D^{T_j^i}$, the drift of a given (continuous) process X changes according to

$$\begin{aligned} \mathbf{Drift}(X; Q_D^{T_j^i}) &= \mathbf{Drift}(X; Q_D^{\mathcal{T}}) - \frac{\mathrm{d}\langle X, \ln(B_D^{\mathcal{T}}/P_D(\cdot, T_j^i))\rangle_t}{\mathrm{d}t} \\ &= \mathbf{Drift}(X; Q_D^{\mathcal{T}}) - \sum_{h=\beta(t)}^{j} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \frac{\mathrm{d}\langle X, F_h^D\rangle_t}{\mathrm{d}t} \end{aligned}$$

In particular, for the FRA rate L_k^i, we have

$$\begin{aligned} \mathbf{Drift}(L_k^i; Q_D^{T_j^i}) &= \mathbf{Drift}(L_k^i; Q_D^{\mathcal{T}}) - \sum_{h=\beta(t)}^{j} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \frac{\mathrm{d}\langle L_k^i, F_h^D\rangle_t}{\mathrm{d}t} \\ &= \sum_{h=\beta(t)}^{k} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \frac{\mathrm{d}\langle L_k^i, F_h^D\rangle_t}{\mathrm{d}t} - \sum_{h=\beta(t)}^{j} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \frac{\mathrm{d}\langle L_k^i, F_h^D\rangle_t}{\mathrm{d}t} \\ &= (1_{\{k>j\}} - 1_{\{j>k\}}) \sum_{h=\min(j,k)+1}^{\max(j,k)} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \frac{\mathrm{d}\langle L_k^i, F_h^D\rangle_t}{\mathrm{d}t} \end{aligned}$$

We thus have the following.

Proposition 6 *The dynamics of each L^i_k and V_k under the forward measure $Q_D^{T^i_j}$ are*

$$
\begin{aligned}
\mathrm{d}L^i_k(t) &= (1_{\{k>j\}} - 1_{\{j>k\}}) \sum_{h=\min(j,k)+1}^{\max(j,k)} \frac{\tau^D_h}{1+\tau^D_h F^D_h(t)}\,\mathrm{d}\langle L^i_k, F^D_h\rangle_t + \phi_k(t, L^i_k(t))\psi_t(V_k(t))\,\mathrm{d}Z^j_k(t) \\
\mathrm{d}V_k(t) &= a_k(t, V_k(t))\,\mathrm{d}t - \sum_{h=\beta(t)}^{j} \frac{\tau^D_h}{1+\tau^D_h F^D_h(t)}\,\mathrm{d}\langle V_k, F^D_h\rangle_t + b_k(t, V_k(t))\,\mathrm{d}W^j_k(t)
\end{aligned}
\tag{30}
$$

In particular, under $Q_D^{T^i_k}$:

$$
\begin{aligned}
\mathrm{d}L^i_k(t) &= \phi_k(t, L^i_k(t))\psi_t(V_k(t))\,\mathrm{d}Z^k_k(t) \\
\mathrm{d}V_k(t) &= a_k(t, V_k(t))\,\mathrm{d}t - \sum_{h=\beta(t)}^{k} \frac{\tau^D_h}{1+\tau^D_h F^D_h(t)}\,\mathrm{d}\langle V_k, F^D_h\rangle_t + b_k(t, V_k(t))\,\mathrm{d}W^k_k(t)
\end{aligned}
\tag{31}
$$

where, for each j, $Z^j = \{Z^j_1, \dots, Z^j_M\}$ is an M-dimensional $Q_D^{T^i_j}$-Brownian motion with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\dots,M}$, and $W^j = \{W^j_1, \dots, W^j_M\}$ is also a $Q_D^{T^i_j}$-Brownian motion.

We can easily check that in the deterministic volatility case, $a_k \equiv b_k \equiv 0$, with lognormal rates, dynamics (30) reduce to the lognormal LMM dynamics (20).

7.2 Moving to a swap measure

Let us denote by $T^S_{c+1}, \dots, T^S_d$, the fixed-leg payment time of a given forward swap rate, with corresponding year fractions $\tau^S_{c+1}, \dots, \tau^S_d$, and assume that each T^S_j belongs to $\mathcal{T} = \{T^i_0, \dots, T^i_M\}$. Then, for each j, there exists an index i_j such that $T^S_j = T^i_{i_j}$.

Consider the annuity term

$$
C^{c,d}_D(t) = \sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j) = \sum_{j=c+1}^{d} \tau^S_j P_D(t, T^i_{i_j}),
$$

which is the numeraire associated to the swap measure $Q^{c,d}_D$.

When moving from measure $Q^{\mathcal{T}}_D$ to $Q^{c,d}_D$, the drift of process X changes according to

$$
\begin{aligned}
\mathbf{Drift}(X; Q^{c,d}_D) &= \mathbf{Drift}(X; Q^{\mathcal{T}}_D) - \frac{\mathrm{d}\langle X, \ln(B^{\mathcal{T}}_D / C^{c,d}_D)\rangle_t}{\mathrm{d}t} \\
&= \mathbf{Drift}(X; Q^{\mathcal{T}}_D) + \frac{\mathrm{d}\langle X, \ln\Big(\sum_{j=c+1}^{d} \tau^S_j \frac{P_D(\cdot, T^S_j)}{P_D(\cdot, T^i_{\beta(t)-1})}\Big)\rangle_t}{\mathrm{d}t}
\end{aligned}
$$

We need to calculate

$$
\begin{aligned}
&\mathrm{d}\ln\Big(\sum_{j=c+1}^{d}\tau_j^S\frac{P_D(t,T_j^S)}{P_D(t,T^i_{\beta(t)-1})}\Big)\\
&=\cdots\mathrm{d}t+\frac{P_D(t,T^i_{\beta(t)-1})}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{j=c+1}^{d}\tau_j^S\,\mathrm{d}\,\frac{P_D(t,T_j^S)}{P_D(t,T^i_{\beta(t)-1})}\\
&=\cdots\mathrm{d}t+\frac{P_D(t,T^i_{\beta(t)-1})}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{j=c+1}^{d}\tau_j^S\,\mathrm{d}\,\frac{P_D(t,T^i_{i_j})}{P_D(t,T^i_{\beta(t)-1})}\\
&=\cdots\mathrm{d}t+\frac{P_D(t,T^i_{\beta(t)-1})}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{j=c+1}^{d}\tau_j^S\,\mathrm{d}\prod_{h=\beta(t)}^{i_j}\frac{1}{1+\tau_h^D F_h^D(t)}\\
&=\cdots\mathrm{d}t+\frac{P_D(t,T^i_{\beta(t)-1})}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{j=c+1}^{d}\tau_j^S\Big(-\sum_{h=\beta(t)}^{i_j}\frac{P_D(t,T^i_{i_j})}{P_D(t,T^i_{\beta(t)-1})}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\Big)\mathrm{d}F_h^D(t)\\
&=\cdots\mathrm{d}t-\sum_{j=c+1}^{d}\frac{\tau_j^S P_D(t,T^i_{i_j})}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{h=\beta(t)}^{i_j}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\mathrm{d}F_h^D(t)\\
&=\cdots\mathrm{d}t-\sum_{j=c+1}^{d}\frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{h=\beta(t)}^{i_j}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\mathrm{d}F_h^D(t)
\end{aligned}
$$

Hence, the instantaneous covariation between X and the log of the numeraires ratio is given by

$$
\mathrm{d}\langle X,\ln\Big(\sum_{j=c+1}^{d}\tau_j^S\frac{P_D(\cdot,T_j^S)}{P_D(\cdot,T^i_{\beta(t)-1})}\Big)\rangle_t=-\sum_{j=c+1}^{d}\frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{h=\beta(t)}^{i_j}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\mathrm{d}\langle X,F_h^D\rangle_t
$$

which, for $X=L_k^i$, leads to

$$
\mathbf{Drift}(L_k^i;Q_D^{c,d})\,\mathrm{d}t=\mathbf{Drift}(L_k^i;Q_D^{\mathcal{T}})\,\mathrm{d}t+\mathrm{d}\langle X,\ln\Big(\sum_{j=c+1}^{d}\tau_j^S\frac{P_D(\cdot,T_j^S)}{P_D(\cdot,T^i_{\beta(t)-1})}\Big)\rangle_t
$$

$$
=\sum_{h=\beta(t)}^{k}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle L_k^i,F_h^D\rangle_t-\sum_{j=c+1}^{d}\frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{h=\beta(t)}^{i_j}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\mathrm{d}\langle L_k^i,F_h^D\rangle_t
$$

Noticing that

$$
\sum_{j=c+1}^{d}\frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}=1,
$$

so that

$$
\sum_{h=\beta(t)}^{k}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle L_k^i,F_h^D\rangle_t=\sum_{j=c+1}^{d}\frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)}\sum_{h=\beta(t)}^{k}\frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle L_k^i,F_h^D\rangle_t,
$$

we immediately have the following.

Proposition 7 *The dynamics of each L_k^i and V_k under the swap measure $Q_D^{c,d}$ are*

$$
\begin{aligned}
\mathrm{d}L_k^i(t) = &\sum_{j=c+1}^{d} \frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)} (1_{\{k>i_j\}} - 1_{\{i_j>k\}}) \sum_{h=\min(i_j,k)+1}^{\max(i_j,k)} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \mathrm{d}\langle L_k^i, F_h^D\rangle_t \\
&+ \phi_k(t, L_k^i(t))\psi_t(V_k(t))\, \mathrm{d}Z_k^{c,d}(t) \\
\mathrm{d}V_k(t) = &a_k(t,V_k(t))\,\mathrm{d}t - \sum_{j=c+1}^{d} \frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)} \sum_{h=\beta(t)}^{i_j} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \mathrm{d}\langle V_k, F_h^D\rangle_t \\
&+ b_k(t,V_k(t))\,\mathrm{d}W_k^{c,d}(t),
\end{aligned}
\tag{32}
$$

where, for each j, $Z^{c,d} = \{Z_1^{c,d}, \dots, Z_M^{c,d}\}$ is an M-dimensional $Q_D^{c,d}$-Brownian motion with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\dots,M}$, and $W^{c,d} = \{W_1^{c,d}, \dots, W_M^{c,d}\}$ is also a $Q_D^{T_j^i}$-Brownian motion.

For instance, the swap-measure dynamics of the volatility process under the specification (29) is given by

$$
\mathrm{d}V(t) = -\sum_{j=c+1}^{d} \frac{\tau_j^S P_D(t,T_j^S)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t,T_j^S)} \sum_{h=\beta(t)}^{i_j} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)} \mathrm{d}\langle V, F_h^D\rangle_t + \nu V(t)\,\mathrm{d}W^{c,d}(t),
$$

where $W^{c,d}$ now denotes a one-dimensional $Q_D^{T_j^i}$-Brownian motion.

7.3 Specifying the covariations in the drifts

As we already noticed in the lognormal LMM case, FRA rate and volatility dynamics (30) and (32) differ from their single-curve homologues because of the different covariance terms in the drifts. For instance, when $i \equiv D$, the dynamics under the T_j^i-forward measure are

$$
\begin{aligned}
\mathrm{d}F_k(t) &= (1_{\{k>j\}} - 1_{\{j>k\}}) \sum_{h=\min(j,k)+1}^{\max(j,k)} \frac{\tau_h}{1+\tau_h F_h(t)}\, \mathrm{d}\langle F_k, F_h\rangle_t + \phi_k(t, F_k(t))\psi_t(V_k(t))\, \mathrm{d}Z_k^j(t) \\
\mathrm{d}V_k(t) &= a_k(t,V_k(t))\,\mathrm{d}t - \sum_{h=\beta(t)}^{j} \frac{\tau_h}{1+\tau_h F_h(t)}\, \mathrm{d}\langle V_k, F_h\rangle_t + b_k(t,V_k(t))\,\mathrm{d}W_k^j(t)
\end{aligned}
\tag{33}
$$

where, for each h, F_h denotes the common value of F_h^D and L_h^i, and τ_h is the associated year fraction.

Dynamics (33) are fully specified by the instantaneous covariance structure of forward rates and their volatilities. When dealing with two distinct curves, we see from (30) that we must replace the forward rate F_k with the FRA rate L_k^i and the forward rates F_h (second argument in the covariations) with F_h^D (also when $h = k$). These dynamics of FRA rates and volatilities depend on extra quantities, namely the forward rates F_h^D and their instantaneous covariations with them, which therefore need to be modeled, too.

In Section 6, we assumed lognormal dynamics for the forward rates F_h^D, to mimic the evolution of the given FRA rates. In principle, however, we are free to specify the former dynamics almost independently from the latter. For example, a convenient choice is to assume, for each k,

$$\mathrm{d}F_k^D(t) = \cdots \mathrm{d}t + \nu_k(t)[1 + \tau_k^D F_k^D(t)]\mathrm{d}Z_k^D(t),$$

where ν_k's are deterministic functions of time, since it implies that

$$\begin{aligned}
\mathrm{d}L_k^i(t) &= (1_{\{k>j\}} - 1_{\{j>k\}}) \sum_{h=\min(j,k)+1}^{\max(j,k)} \tau_h^D \nu_h(t)\,\mathrm{d}\langle L_k^i, Z_h^D\rangle_t + \phi_k(t, L_k^i(t))\psi_t(V_k(t))\,\mathrm{d}Z_k^j(t) \\
\mathrm{d}V_k(t) &= a_k(t, V_k(t))\,\mathrm{d}t - \sum_{h=\beta(t)}^{j} \tau_h^D \nu_h(t)\,\mathrm{d}\langle V_k, Z_h^D\rangle_t + b_k(t, V_k(t))\,\mathrm{d}W_k^j(t)
\end{aligned}$$

In this case, in fact, the dynamics of L_k^i and V_k depend on rates F_h^D only through their instantaneous correlations with them, so that no simulation of the dynamics of F_h^D is needed in the Monte Carlo pricing of a LIBOR dependent derivative.[25]

Another convenient choice would be to set to zero the instantaneous covariations in (30) and (32), independently of the dynamics of F_h^D. In this case, the dynamics of L_k^i and V_k would remain the same irrespective of the chosen (forward or swap) measure.

However, these choices may not be so realistic. From a practical point of view, yield curves i and D are likely to move in a similar (highly correlated) manner, so that it may be more sensible to assume similar $Q_D^{\mathcal{T}}$-dynamics for rates L_k^i and F_h^D:

$$\begin{aligned}
\mathrm{d}F_k^D(t) &= \sum_{h=\beta(t)}^{k} \frac{\tau_h^D}{1 + \tau_h^D F_h^D(t)}\,\mathrm{d}\langle F_k^D, F_h^D\rangle_t + \phi_k^D(t, F_k^D(t))\psi_t^D(V_k^D(t))\,\mathrm{d}Z_k^{d,D}(t) \\
\mathrm{d}V_k^D(t) &= a_k^D(t, V_k^D(t))\,\mathrm{d}t + b_k^D(t, V_k^D(t))\,\mathrm{d}W_k^{d,D}(t)
\end{aligned}$$

where ϕ_k^D, ψ_t^D, a_k^D and b_k^D have the same form of the corresponding functions for the FRA rate L_k^i but with possibly different parameters, and $\{Z_1^{d,D}, \dots, Z_M^{d,D}\}$ and $\{W_1^{d,D}, \dots, W_M^{d,D}\}$ are M-dimensional $Q_D^{\mathcal{T}}$-Brownian motions, (highly) correlated with $\{Z_1^d, \dots, Z_M^d\}$ and $\{W_1^d, \dots, W_M^d\}$, respectively. See also Remark 2.

Moving to a forward or a swap measure will produce, in the dynamics of F_k^D and V_k^D, drift corrections that are equivalent to those we obtain in the single-curve case.

[25] If the derivative's payoff depends on swap rates, forward rates F_h^D need to be simulated anyway.

7.4 Option pricing

All the stochastic-volatility LMMs mentioned above lead to closed-form formulas for caps and swaptions. Our two-curve setting allows for extensions of these formulas under each of these models. However, the derivation procedures will be different, being already different in the basic single-curve case. As an example, we will show how to price caps and swaptions in the Wu and Zhang (2006) model under constant coefficients.

Assuming single-factor stochastic-volatility dynamics of Heston's (1993) type, the SDEs under the forward measure $Q_D^{T_k^i}$ are:

$$
\begin{aligned}
\mathrm{d}L_k^i(t) &= \sigma_k L_k^i(t)\sqrt{V(t)}\,\mathrm{d}Z_k^k(t) \\
\mathrm{d}V(t) &= \kappa(\theta - V(t))\,\mathrm{d}t - \sum_{h=\beta(t)}^{k} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle V, F_h^D\rangle_t + \epsilon\sqrt{V(t)}\,\mathrm{d}W^k(t) \\
\mathrm{d}F_k^D(t) &= \sigma_k^D F_k^D(t)\sqrt{V^D(t)}\,\mathrm{d}Z_k^{k,D}(t) \\
\mathrm{d}V^D(t) &= \kappa^D(\theta^D - V^D(t))\,\mathrm{d}t - \sum_{h=\beta(t)}^{k} \frac{\tau_h^D}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle V^D, F_h^D\rangle_t + \epsilon^D\sqrt{V^D(t)}\,\mathrm{d}W^{k,D}(t)
\end{aligned}
$$

where σ_k, κ, θ, ϵ and σ_k^D, κ^D, θ^D, ϵ^D are positive constants, and $\{Z_1^k,\dots,Z_M^k\}$ and $\{Z_1^{k,D},\dots,Z_M^{k,D}\}$ are M-dimensional $Q_D^{T_k^i}$-Brownian motions, whereas W^k and $W^{k,D}$ are one-dimensional $Q_D^{T_k^i}$-Brownian motions.

In particular, the dynamics of V are more explicitly given by

$$
\mathrm{d}V(t) = \kappa(\theta - V(t))\,\mathrm{d}t - \epsilon\sqrt{V(t)V^D(t)}\sum_{h=\beta(t)}^{k} \frac{\tau_h^D\sigma_h^D F_h^D(t)}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle W^k, Z_h^{k,D}\rangle_t + \epsilon\sqrt{V(t)}\,\mathrm{d}W^k(t)
$$

To derive an analytical formula for the caplet paying $[L_k^i(T_{k-1}^i) - K]^+$ at time T_k^i, we can approximate, in different ways, the volatility's drift term produced by the measure change, see also Mercurio and Moreni (2006). For instance, we may set:

$$
\begin{aligned}
&\epsilon\sqrt{V(t)V^D(t)}\sum_{h=\beta(t)}^{k} \frac{\tau_h^D\sigma_k^D F_k^D(t)}{1+\tau_h^D F_h^D(t)}\,\mathrm{d}\langle W^k, Z_h^{k,D}\rangle_t \\
&\approx \epsilon V(t)\frac{\sqrt{V^D(0)}}{\sqrt{V(0)}}\sum_{h=1}^{k} \frac{\tau_h^D\sigma_h^D F_h^D(0)}{1+\tau_h^D F_h^D(0)}\,\mathrm{d}\langle W^k, Z_h^{k,D}\rangle_t =: \eta V(t)\,\mathrm{d}t
\end{aligned}
$$

where the last equality defines the constant η parameter, so that we can write

$$
\mathrm{d}V(t) = [\kappa\theta - (\kappa+\eta)V(t))]\,\mathrm{d}t + \epsilon\sqrt{V(t)}\,\mathrm{d}W^k(t).
$$

Since the dynamics of V are (approximately) of square-root type also under $Q_D^{T_k^i}$, we can then price the caplet by means of the Heston (1993) option formula.

Remark 8 *We notice that an exact formula can be obtained by simply setting to zero the correlation between the Brownian motions W^k and $Z_h^{k,D}$, since in this case the dynamics of V will not change as a consequence of the measure change. This assumption is more innocuous than in the single-curve setting. In fact, the correlation between Brownian motions W^k and Z_k^k can be arbitrary and used for calibration of the market caplet skew, whereas the correlation between W^k and $Z_h^{k,D}$ can be conveniently assumed to be zero without creating any contradiction (unless the whole correlation matrix is not positive definite).*

The pricing of swaptions is, as usual, a bit trickier. In the two-curve case, moreover, we must face the further complication that swap rates depend also on forward rates F_k^D and not only on FRA rates L_k^i. As already done in the lognormal case, we can however freeze the weights ω_k at their time-0 value,

$$S_{a,b,c,d}^i(t) \approx \sum_{k=a+1}^{b} \omega_k(0) L_k^i(t),$$

obtaining the following dynamics under the swap measure $Q_D^{c,d}$:

$$\begin{aligned}
\mathrm{d}S_{a,b,c,d}^i(t) &\approx \sqrt{V(t)} \sum_{k=a+1}^{b} \omega_k(0)\sigma_k L_k^i(t)\, \mathrm{d}Z_k^{c,d}(t) \\
&\approx S_{a,b,c,d}^i(t)\sqrt{V(t)} \sum_{k=a+1}^{b} \omega_k(0)\sigma_k \frac{L_k^i(0)}{S_{a,b,c,d}^i(0)}\, \mathrm{d}Z_k^{c,d}(t) \\
&= \nu S_{a,b,c,d}^i(t)\sqrt{V(t)}\, \mathrm{d}Z_{a,b,c,d}^{c,d}(t)
\end{aligned}$$

for some $Q_D^{c,d}$-Brownian motion $Z_{a,b,c,d}^{c,d}$, and where

$$\nu := \sqrt{\sum_{k=a+1}^{b} \sum_{h=a+1}^{b} \omega_k(0)\omega_h(0)\sigma_k\sigma_h\rho_{h,k} \frac{L_k^i(0)L_h^i(0)}{[S_{a,b,c,d}^i(0)]^2}}\,.$$

Therefore, the dynamics of $S_{a,b,c,d}^i(t)$ under its associated swap measure is (approximately) similar to that of each FRA rate under the corresponding forward measure.

The $Q_D^{c,d}$-dynamics of volatility V is more involved, see equation (32). However, we can again resort to freezing techniques and derive approximated dynamics of square-root type, so as to be able to apply Heston's (1993) option pricing formula also in the swaption case. We notice once again that setting to zero the instantaneous correlation between V and the forward rates F_h^D gives a zero drift correction, meaning that no further approximation is required.

8 Conclusions

We have started by describing the change in value of the market interest rate quotes, which occurred since August 2007. As a major stylized fact, we noticed that once-compatible rates began to diverge sensibly, producing a clear segmentation of market rates. Practitioners tackled the issue by building different yield curves for different rate tenors.

In this article, we have shown how to price the main (linear and plain vanilla) interest rate derivatives under the assumption of two distinct curves for generating future LIBOR rates and for discounting. The pricing formulas for caps and swaptions result in a simple modification of the corresponding Black formulas used by the market in the single-curve setting.

We have then extended the basic lognormal LMM and derived its dynamics under the relevant measure changes. We have concluded by introducing stochastic volatility, considering a general dynamics that contains as a particular case all stochastic volatility LMMs known in the financial literature.

The analysis with two distinct yield curves can be extended to allow for the simultaneous presence of more forward curves, one for each tenor considered. This is fundamental if we have to price a contract that depends on different LIBOR or swap rates. If two curves i and j enters a given payoff at the same time, we just have to assume that either time structure $\{T_0^i, \dots, T_M^i\}$ or $\{T_0^j, \dots, T_M^j\}$ is included in the other. This is usually the case in practice, since the tenors that are typically considered in the market are 1, 3, 6 and 12 months. When pricing payoffs depending on swap rates, a similar assumption has to be made on the times defining the fixed and floating legs of the forward swaps in question.

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