A LIBOR Market Model with Stochastic Basis

Fabio Mercurio
OTC Derivatives and Structured Notes
Bloomberg LP, New York*

Abstract

We extend the LIBOR market model to accommodate the new market practice of using different forward and discount curves in the pricing of interest-rate derivatives. Our extension is based on modeling the joint evolution of forward rates belonging to the OIS curve and corresponding spreads with FRA rates for different tenors. We consider stochastic-volatility dynamics and address the related caplet and swaption pricing problems. We conclude the article with an example of calibration to real market data.

An extended version of this article can be downloaded at:
`http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1563685`

1 Introduction

Until the 2007 credit crunch, market quotes of interest rates consistently followed classic no-arbitrage rules. For instance, a floating rate bond where rates are set in advance and paid in arrears, was worth par at inception, irrespectively of the underlying tenor. Also, a forward rate agreement (FRA) could be replicated by long and short positions in two deposits, with the implied forward rate differing only slightly from the corresponding quantity obtained through OIS rates.

When August 2007 arrived, the market had to face an unprecedented scenario. Interest rates that until then had been almost equivalent, suddenly became unrelated, with the degree of incompatibility that worsened as time passed by. For instance, the forward rate implied by two deposits, the corresponding FRA rate and the forward rate implied by the corresponding OIS rates became substantially different, and started to be quoted with large, non-negligible spreads.

*Stimulating discussions with Peter Carr, Liuren Wu, Antonio Castagna, Raffaele Giura and Massimo Morini are gratefully acknowledged. The author also thanks Nabyl Belgrade, Marco Bianchetti and Riccardo Rebonato for helpful comments. Needless to say, all errors are the author's responsibility.

In fact, differences between rates referring to the same time interval have always been present in the market. For instance, deposit rates and OIS rates for the same maturity would closely track each other, but keeping a distance (spread) of few basis points. Likewise, swap rates with same maturity, but based on LIBOR rates with different tenors, would be quoted at a non-zero (basis) spread. All these spreads were generally regarded as negligible and, in fact, often assumed to be zero when constructing zero-coupon curves or pricing interest-rate derivatives.

To comply with the new market features, as far as yield curves are concerned, practitioners seem to agree on an empirical approach. For each given contract, they select a specific discount curve, which they use to calculate the net present value (NPV) of the contract's future payments, consistently with the contract's features and the counterparty in question. They then build as many forward LIBOR curves as given market tenors (*i.e.* 1m, 3m, 6m, 1y), see *e.g.* Ametrano and Bianchetti (2009). With this approach, future cash flows are generated by the curves associated with the underlying rate tenors and their NPV is calculated through the selected discount curve.

The assumption of distinct discount and forward curves, for a same currency and in absence of counterparty risk, immediately invalidates the classic pricing principles, which were built on the cornerstone of a single, and fully consistent, zero-coupon curve, containing all relevant information about the (risk-neutral) projection of future rates and the NPV calculation of associated pay-outs. A new model paradigm is thus needed to accommodate this market practice of using multiple interest-rate curves.

In this article, we will extend the LIBOR market model (LMM) to the multi-curve setting by modeling the basis between OIS and FRA rates, which is consistent with the market practice of building (forward) LIBOR curves at a spread over the OIS one. Remarkably, introducing a stochastic basis can add flexibility to the model, without compromising its tractability, as we will show by deriving closed-form formulas for cap and swaption prices and by considering an example of calibration to market caplets.[1] Moreover, no market data on OIS or basis volatilities is needed for calibration purposes. In fact, OIS rates and basis spreads can be viewed as factors driving the evolution of LIBOR rates, similarly to those short-rate models where the instantaneous rate is defined as the sum of two (or more) additive factors. Such factors do not need specific options to be calibrated to but their parameters can be fitted to market quotes of standard (LIBOR-based) caps and swaptions.

We will assume that the discount curve coincides with that stripped from OIS swap rates. Since OIS rates can be regarded as the best available proxy for risk-neutral rates, this is equivalent to assume zero counterparty risk in the valuation of derivatives, the market plain-vanilla instruments (swaps, caps, swaptions) in particular. This assumption is reasonable due to the current practice of underwriting collateral agreements to mitigate, possibly eliminate, the counterparty risk affecting a given transaction between banks. As-

[1] A similar approach has been recently proposed by Fujii et al. (2009b) who model stochastic basis spreads in a HJM framework both in single- and multi-currency cases, but without providing examples of dynamics or explicit formulas for the main calibration instruments. An alternative route is chosen by Henrard (2009) who hints at the modeling of basis swap spreads, but without addressing typical issues of a market model, such as the modeling of joint dynamics or the pricing of plain-vanilla derivatives.

suming OIS discounting amounts to assume that the interest rate earned by the collateral is the overnight rate.

Our risk-neutral valuation can also be viewed as the necessary initial step for a sensible valuation of deals affected by counterparty risk, which may be in part, but not completely, immunized by the collateral agreement in place. To this end, one may first obtain risk-neutral parameters by calibrating his/her model to the relevant market data and then apply suitable corrections to the risk-neutral prices of contracts that are characterized by collateral rates different than overnight rates. We refer to Johannes and Sundaresan (2007), Fujii et al. (2009a, 2009b) and Piterbarg (2010), for the derivation and description of the pricing formulas holding in case of general collateral rates.

2 Assumptions and definitions

We assume we are given a single discount curve to be used in the calculation of all NPVs. This curve is assumed to coincide with the OIS zero-coupon curve, which is in turn assumed to be stripped from market OIS swap rates and defined for every possible maturity T:[2] $T \mapsto P_D(0,T) = P^{\mathrm{OIS}}(0,T)$, where $P_D(t,T)$ denotes the discount factor (zero-coupon bond) at time t for maturity T. The subscript D stands for "discount curve".

In the following, as in Kijima et al. (2009), the pricing measures we will consider are those associated with the discount curve. This is also consistent with the results of Fujii et al. (2009a, 2009b) and Piterbarg (2010), since we assume CSA agreements where the collateral rate to be paid equals the (assumed risk-free) overnight rate.

We introduce the following definition.

Definition 1 *Consider times t, T_1 and T_2, $t \le T_1 < T_2$. The time-t FRA rate $\mathbf{FRA}(t;T_1,T_2)$ is defined as the fixed rate to be exchanged at time T_2 for the LIBOR rate $L(T_1,T_2)$ so that the swap has zero value at time t.*

Denoting by Q_D^T the T-forward measure with numeraire the zero-coupon bond $P_D(t,T)$, by (risk-adjusted) no-arbitrage pricing, we immediately have

$$\mathbf{FRA}(t;T_1,T_2) = E_D^{T_2}\big[L(T_1,T_2)|\mathcal{F}_t\big], \tag{1}$$

where E_D^T denotes expectation under Q_D^T and $\mathcal{F}_t$ denotes the "information" available in the market at time t.

In the classic single-curve valuation, *i.e.* when the LIBOR curve corresponding to tenor $T_2 - T_1$ coincides with the discount curve, it is well known that the FRA rate $\mathbf{FRA}(t;T_1,T_2)$

[2] The OIS curve can be stripped from OIS swap rates using standard (single-curve) bootstrapping methods. For the EUR market, EONIA swaps are quoted up to 30 years, so that the stripping procedure presents no new issues. Different is the case of other currencies, even major ones like USD or JPY, where OIS rates are quoted only up to a relatively short maturity. In such cases, one has to resort to alternative constructions, by modeling, for instance, the spread between OIS (forward) rates and corresponding (forward) LIBOR rates or by adding quotes of cross-currency swaps.

coincides with the forward rate

$$F_D(t; T_1, T_2) := \frac{1}{T_2 - T_1}\left[\frac{P_D(t, T_1)}{P_D(t, T_2)} - 1\right]. \tag{2}$$

In our dual-curve setting, however, this does not hold any more, since the simply-compounded rates defined by the discount curve are different, in general, from the corresponding LIBOR fixings.

2.1 The pricing of an interest rate swap (IRS)

Let us consider a set of times $T_a, \ldots, T_b$ compatible with a given tenor,[3] and an IRS where the floating leg pays at each time T_k the LIBOR rate $L(T_{k-1}, T_k)$ set at the previous time T_{k-1}, $k = a+1, \ldots, b$, and the fixed leg pays the fixed rate K at times $T^S_{c+1}, \ldots, T^S_d$.

Under our assumptions on the discount curve, the swap valuation is straightforward.[4] Applying Definition 1 and setting

$$L_k(t) := \mathbf{FRA}(t; T_{k-1}, T_k) = E_D^{T_k}\left[L(T_{k-1}, T_k)|\mathcal{F}_t\right],$$

the IRS time-t value, to the fixed-rate payer, is given by

$$\mathbf{IRS}(t, K; T_a, \ldots, T_b, T^S_{c+1}, \ldots, T^S_d) = \sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t) - K \sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j).$$

where τ_k and τ^S_j denote, respectively, the floating-leg year fraction for the interval $(T_{k-1}, T_k]$ and the fixed-leg year fraction for the interval $(T^S_{j-1}, T^S_j]$.

The corresponding forward swap rate, that is the fixed rate K that makes the IRS value equal to zero at time t, is then defined by

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t)}{\sum_{j=c+1}^{d} \tau^S_j P_D(t, T^S_j)}. \tag{3}$$

3 Extending the LMM

As is well known, the classic (single-curve) LMMs are based on modeling the joint evolution of a set of consecutive forward LIBOR rates, as defined by a given time structure. Forward LIBOR rates are "building blocks" of the modeled yield curve, and their dynamics can be conveniently used to generate future LIBOR rates and discount factors defining swap rates. When moving to a multi-curve setting, we immediately face two complications. The first is the existence of several yield curves (one discount curve and as many forward curves as

[3]For instance, if the tenor is three months, the times T_k must be three-month spaced.

[4]Details of the derivation can be found, for instance, in Chibane and Sheldon (2009), Henrard (2009), Kijima et al. (2009) and Mercurio (2009).

market tenors), which multiplies the number of building blocks (the "old" forward rates) that one needs to jointly model. The second is the impossibility to apply the old definitions, which were based on the equivalence between forward LIBOR rates and the corresponding ones defined by the discount curve.

The former issue can be trivially addressed by adding extra dimensions to the vector of modeled rates, and by suitably modeling their instantaneous covariance structure. The second, instead, is less straightforward, requiring a new definition of forward rates, which needs to be compatible with the existence of different curves for discounting and for projecting future LIBORs.

A natural extension of the definition of forward rate to a multi-curve setting is given by the FRA rate above. In fact, FRA rates reduce to "old" forward rates when the particular case of a single-curve framework is assumed. Moreover, they have the property to coincide with the corresponding LIBOR rates at their reset times, $\mathbf{FRA}(T_1; T_1, T_2) = L(T_1, T_2)$, and the advantage to be martingales, by definition, under the corresponding forward measures. Finally, by (3), swap rates can be written as a (stochastic) linear combination of FRA rates, with coefficients solely depending on discount-curve zero-coupon bonds. Notice also that the time-0 value of FRA rates $L_k(0)$ can easily be bootstrapped from market data of swap rates by iterative application of formula (3) with $t = 0$, see *e.g.* Chibane and Sheldon (2009), Henrard (2009), Fujii et al. (2009a) and Mercurio (2010b).

A consistent extension of a LMM to the multi-curve case can then be obtained by modeling the joint dynamics of FRA rates with different tenors and of forward rates belonging to the discount curve.[5] This extension was first proposed by Mercurio (2009, 2010a), who considered lognormal dynamics for given-tenor FRA rates, and then added stochastic volatility to their evolution. We here follow a different approach, and explicitly model the basis between OIS and FRA rates, defining the joint evolution of different tenors at the same time. This is also inspired by the historical pattern of the (forward) basis, which we show in Fig. 1 by plotting the difference between 6(month)x12(month) FRA rates and respective forward EONIA rates. Such a difference was fairly constant and small before August 2007, but since then it started to move stochastically, with positive and no-longer negligible values.

4 Model dynamics and derivative pricing

Let us consider a time structure $\mathcal{T} = \{0 < T_0, \ldots, T_M\}$ and different tenors $x_1 < x_2 < \cdots < x_n$ with associated time structures $\mathcal{T}^{x_i} = \{0 < T_0^{x_i}, \ldots, T_{M_{x_i}}^{x_i}\}$. We assume that each x_i is a multiple of the preceding tenor x_{i-1}, and that $\mathcal{T}^{x_n} \subset \mathcal{T}^{x_{n-1}} \subset \cdots \subset \mathcal{T}^{x_1} = \mathcal{T}$. We

[5]The reason for modeling OIS rates in addition to FRA rates is twofold. First, by assumption, our pricing measures are related to the discount (*i.e.* OIS) curve. Since the associated numeraires are portfolios of zero-coupon bonds $P_D(t, T)$, the FRA-rate drift corrections implied by a measure change will depend on the (instantaneous) covariation between FRA rates and corresponding OIS forward rates. Second, swap rates explicitly depend on zero-coupon bonds $P_D(t, T)$, and, clearly, can only be simulated if the relevant OIS forward rates are simulated too.

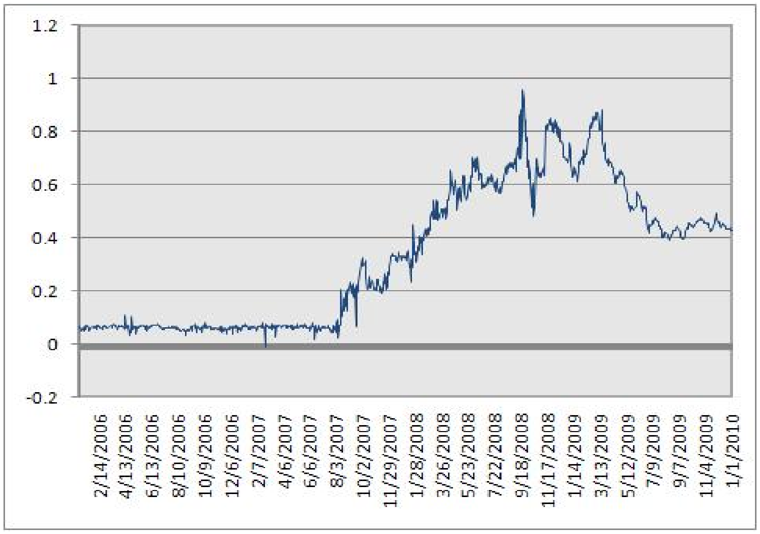

Figure 1: Basis between 6x12 forward EONIA rates and 6x12 FRA rates, from 2 Jan, 2006 to 2 Jan, 2010, EUR market. Source: Bloomberg.

then denote by $Q_D^{\mathcal{T}}$ the spot LIBOR measure associated to times $\mathcal{T}$, whose numeraire is the discretely-rebalanced bank account $B_D^{\mathcal{T}}$:

$$B_D^{\mathcal{T}}(t) = \frac{P_D(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P_D(T_{j-1}, T_j)},$$

where $\beta(t) = m$ if $T_{m-2} < t \leq T_{m-1}$, $m \geq 1$.

Forward OIS rates are defined, for each tenor $x \in \{x_1, \ldots, x_n\}$, by

$$F_k^x(t) := F_D(t; T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^x}\left[\frac{P_D(t, T_{k-1}^x)}{P_D(t, T_k^x)} - 1\right], \quad k = 1, \ldots, M_x, \tag{4}$$

where τ_k^x is the year fraction for the interval $(T_{k-1}^x, T_k^x]$, and basis spreads are defined by

$$S_k^x(t) = \mathbf{FRA}(t, T_{k-1}^x, T_k^x) - F_k^x(t) = L_k^x(t) - F_k^x(t), \quad k = 1, \ldots, M_x. \tag{5}$$

By definition, both L_k^x and F_k^x are martingales under the forward measure $Q_D^{T_k^x}$, and hence their difference S_k^x is a $Q_D^{T_k^x}$-martingale, too.

The joint evolution of rates and spreads must be provided under a common probability measure. To this end, we assume that, under $Q_D^{\mathcal{T}}$, the OIS forward rates $F_k^{x_1}$, $k = 1, \ldots, M_1$, follow "shifted-lognormal" stochastic-volatility processes

$$\mathrm{d}F_k^{x_1}(t) = \sigma_k^{x_1}(t) V^F(t)\Big[\frac{1}{\tau_k^{x_1}} + F_k^{x_1}(t)\Big]\left[V^F(t) \sum_{h=\beta(t)}^{k} \rho_{h,k}\sigma_h^{x_1}(t)\,\mathrm{d}t + \mathrm{d}Z_k^{\mathcal{T}}(t)\right] \tag{6}$$

where, for each k, $\sigma_k^{x_1}$ is a deterministic function, $Z^{\mathcal{T}} = \{Z_1^{\mathcal{T}}, \ldots, Z_{M_1}^{\mathcal{T}}\}$ is an M_1-dimensional $Q_D^{\mathcal{T}}$-Brownian motion with instantaneous correlation matrix $(\rho_{k,j})_{k,j=1,\ldots,M_1}$.

The stochastic volatility V^F is assumed to be a process common to all OIS forward rates, and, for simplicity, instantaneously uncorrelated with every $Z_k^{\mathcal{T}}$, with $V^F(0) = 1$. In practice, one can assume that V^F follows SABR or Heston (1993) dynamics or consider the limit case of a deterministic evolution. The discussion that follows, however, needs no dynamics specification and can be based on a general volatility process.

The dynamics of forward rates F_k^x, for tenors $x \in \{x_2, \ldots, x_n\}$, can be obtained, through Ito's lemma, by noting that F_k^x can be written in terms of "smaller" rates $F_k^{x_1}$ as follows:

$$\prod_{h=i_{k-1}+1}^{i_k} [1 + \tau_h^{x_1} F_h^{x_1}(t)] = 1 + \tau_k^x F_k^x(t), \tag{7}$$

for some indices i_{k-1} and i_k.

As far as spread dynamics are concerned, we assume, for each tenor $x \in \{x_1, \ldots, x_n\}$, the following one-factor models

$$S_k^x(t) = S_k^x(0)\mathcal{M}^x(t), \quad k = 1, \ldots, M_x, \tag{8}$$

where, for each x, $\mathcal{M}^x$ is a (continuous and) positive $Q_D^{\mathcal{T}}$-martingale independent of rates F_k^x and of the stochastic volatility V^F. Clearly, $\mathcal{M}^x(0) = 1$.

The spreads S_k^x are thus positive to be consistent with typical market patterns, see *e.g.* Fig. 1. Being martingales under the respective forward measures, spreads are also martingales under $Q_D^{\mathcal{T}}$, thanks to their independence from OIS rates, and their dynamics do not change when moving to different forward or swap measures. The martingales $\mathcal{M}^{x_1}, \ldots, \mathcal{M}^{x_n}$ can be (instantaneously) correlated with one another, to capture relative movements between curves associated with different tenors.

A convenient choice in terms of model flexibility and tractability is to assume that spreads S_k^x (equivalently, $\mathcal{M}^x$) follow stochastic-volatility processes whose option prices are known in closed form. This will be the case of our explicit example below.

4.1 Rate dynamics under the associated forward measure

When moving from measure $Q_D^{\mathcal{T}}$ to measure $Q_D^{T_k^x}$, the drift of a (continuous) process X changes according to

$$\begin{aligned}\mathbf{Drift}(X; Q_D^{T_k^x}) &= \mathbf{Drift}(X; Q_D^{\mathcal{T}}) + \frac{\mathrm{d}\langle X, \ln(P_D(\cdot, T_k^x)/B_D^{\mathcal{T}}(\cdot))\rangle_t}{\mathrm{d}t} \\ &= \mathbf{Drift}(X; Q_D^{\mathcal{T}}) + \frac{\mathrm{d}\langle X, \ln(P_D(\cdot, T_k^x)/P_D(\cdot, T_{\beta(t)-1}))\rangle_t}{\mathrm{d}t}\end{aligned} \tag{9}$$

where $\langle X, Y\rangle_t$ denotes instantaneous covariation between processes X and Y at time t.

Applying Ito's lemma to (7), and (9) to (6), we get, for each $x \in \{x_1, \ldots, x_n\}$,

$$\mathrm{d}F_k^x(t) = \sigma_k^x(t) V^F(t) \Big[\frac{1}{\tau_k^x} + F_k^x(t)\Big] \mathrm{d}Z_k^{k,x}(t) \tag{10}$$

where σ_k^x, $x \in \{x_2, \dots, x_n\}$, is a deterministic function, whose value is determined by volatilities $\sigma_h^{x_1}$ and correlations $\rho_{h,k}$, and $Z_k^{k,x}$ is a $Q_D^{T_k^x}$-Brownian motion whose instantaneous correlation with $Z_h^{h,x}$ is inherited from the instantaneous covariance structure of rates $F_h^{x_1}$. Since, V^F is assumed to be instantaneously uncorrelated with every F_k^x, its $Q_D^{T_k^x}$-dynamics will be the same as that defined under $Q_D^{\mathcal{T}}$.

From (10), we notice that (6) are the simplest stochastic-volatility dynamics that are consistent across different tenors. This means, for example, that if three-month rates follow shifted-lognormal processes with common stochastic volatility, the same type of dynamics is also followed by six-month rates under the respective forward measures. Our choice of dynamics (6) is motivated by this feature, which allows us to price simultaneously in closed-form caps and swaptions with different underlying tenors.

Should one be interested in modeling one specific tenor x only, more general dynamics of OIS rates F_k^x can be considered. We refer to Mercurio (2010b) for some examples and further details.

4.2 Caplet pricing

For each tenor $x \in \{x_1, \dots, x_n\}$, let us denote by $L^x(T_{k-1}^x, T_k^x)$ the x-tenor LIBOR rate set at time T_{k-1}^x with maturity T_k^x, and consider the associated strike-K caplet, which pays out at time T_k^x

$$\tau_k^x [L^x(T_{k-1}^x, T_k^x) - K]^+ = \tau_k^x [L_k^x(T_{k-1}^x) - K]^+. \tag{11}$$

Our assumptions on the discount curve imply that the caplet price at time t is given by

$$\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[L_k^x(T_{k-1}^x) - K]^+ | \mathcal{F}_t\right\} \tag{12}$$

Since $L_k^x(T_{k-1}^x) = F_k^x(T_{k-1}^x) + S_k^x(T_{k-1}^x)$, by the independence of $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$, the density $f_{L_k^x(T_{k-1}^x)}$ is equal to the convolution of densities $f_{F_k^x(T_{k-1}^x)}$ and $f_{S_k^x(T_{k-1}^x)}$, where we denote by f_X the density function of the random variable X under $Q_D^{T_k^x}$, conditional on $\mathcal{F}_t$. We can then write:

$$\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \int_{-\infty}^{+\infty} (l - K)^+ f_{L_k^x(T_{k-1}^x)}(l)\, \mathrm{d}l \tag{13}$$

In general, however, deriving the convolution $f_{L_k^x(T_{k-1}^x)}$ and integrating numerically (13) may not be the most efficient way to calculate the caplet price. In fact, an alternative derivation is based on applying the tower property of conditional expectations:

$$\begin{aligned}\mathbf{Cplt}(t, K; T_{k-1}^x, T_k^x) &= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[F_k^x(T_{k-1}^x) - (K - S_k^x(T_{k-1}^x))]^+ | \mathcal{F}_t\right\} \\ &= \tau_k^x P_D(t, T_k^x) E_D^{T_k^x}\left\{[S_k^x(T_{k-1}^x) - (K - F_k^x(T_{k-1}^x))]^+ | \mathcal{F}_t\right\}\end{aligned} \tag{14}$$

Thanks to the independence of the random variables $F_k^x(T_{k-1}^x)$ and $S_k^x(T_{k-1}^x)$, we equivalently have:

$$\begin{aligned}\mathbf{Cplt}(t,K;T_{k-1}^x,T_k^x) &= \tau_k^x P_D(t,T_k^x)\int_{-\infty}^{+\infty} E_D^{T_k^x}\big\{[F_k^x(T_{k-1}^x)-(K-z)]^+|\mathcal{F}_t\big\} f_{S_k^x(T_{k-1}^x)}(z)\,\mathrm{d}z\\ &= \tau_k^x P_D(t,T_k^x)\int_{-\infty}^{+\infty} E_D^{T_k^x}\big\{[S_k^x(T_{k-1}^x)-(K-z)]^+|\mathcal{F}_t\big\} f_{F_k^x(T_{k-1}^x)}(z)\,\mathrm{d}z\end{aligned} \tag{15}$$

The caplet price (12) can then be calculated in closed form as soon as we explicitly know the densities $f_{S_k^x(T_{k-1}^x)}$ and $f_{F_k^x(T_{k-1}^x)}$ and/or the associated caplet prices. Using either of formulas (15) may be more or less convenient from a numerical point of view depending on the chosen dynamics. An explicit example will be given in Section 5 below.

4.3 Swaption pricing

Let us consider a (payer) swaption, which gives the right to enter at time $T_a^x = T_c^S$ an IRS with payment times for the floating and fixed legs given, respectively, by $T_{a+1}^x,\dots,T_b^x$ and $T_{c+1}^S,\dots,T_d^S$, with $T_b^x = T_d^S$ and where the fixed rate is K. We assume that each T_j^S belongs to $\{T_a^x,\dots,T_b^x\}$.[6] Then, for each j, there exists an index i_j such that $T_j^S = T_{i_j}^x$.

The swaption payoff at time $T_a^x = T_c^S$ is given by

$$[S_{a,b,c,d}(T_a^x)-K]^+\sum_{j=c+1}^{d}\tau_j^S P_D(T_c^S,T_j^S), \tag{16}$$

where $S_{a,b,c,d}(t)$ is defined by (3). Setting

$$C_D^{c,d}(t) = \sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S) = \sum_{j=c+1}^{d}\tau_j^S P_D(t,T_{i_j}^x),$$

the swaption payoff (16) is conveniently priced under the swap measure $Q_D^{c,d}$, whose associated numeraire is the annuity $C_D^{c,d}(t)$. In fact, denoting by $E_D^{c,d}$ expectation under $Q_D^{c,d}$, we have:

$$\mathbf{PS}(t,K;T_a^x,\dots,T_b^x,T_{c+1}^S,\dots,T_d^S) = \sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)\,E_D^{c,d}\big\{\,[S_{a,b,c,d}(T_a^x)-K]^+\,|\mathcal{F}_t\big\} \tag{17}$$

so that, also in a multi-curve environment, pricing a swaption is equivalent to pricing an option on the underlying swap rate.

To calculate the last expectation, we set

$$\omega_k(t) := \frac{\tau_k^x P_D(t,T_k^x)}{\sum_{j=c+1}^{d}\tau_j^S P_D(t,T_j^S)} \tag{18}$$

[6]This assumption is motivated by the measure change, from forward to swap measure, which is needed in the approximation of the swaption price, see *e.g.* Mercurio (2009, 2010a).

and write:[7]

$$S_{a,b,c,d}(t) = \sum_{k=a+1}^{b} \omega_k(t) L_k^x(t) = \sum_{k=a+1}^{b} \omega_k(t) F_k^x(t) + \sum_{k=a+1}^{b} \omega_k(t) S_k^x(t) =: \bar{F}(t) + \bar{S}(t), \quad (19)$$

with the last equality defining processes $\bar{F}$ and $\bar{S}$.

Process $\bar{F}$ is a $Q_D^{c,d}$-martingale, being equal to the classic single-curve forward swap rate that is defined by OIS discount factors, and whose reset and payment times are given by $T_c^S, \dots, T_d^S$. If dynamics (10), which define a standard (single-curve) LMM based on OIS rates, are sufficiently tractable, we can approximate $\bar{F}(t)$ by a driftless stochastic-volatility process, $\tilde{F}(t)$, of the same type as (10). This property holds for the majority of LMMs in the financial literature,[8] so that we can safely assume it also applies to our dynamics (10). In particular, this will be the case of our explicit example below.

The case of process $\bar{S}$ is slightly more involved. In fact, contrary to $\bar{F}$, $\bar{S}$ explicitly depends both on OIS discount factors, defining the weights ω_k, and on basis spreads. However, this issue can easily be addressed by resorting to a standard approximation as far as swaption pricing in a LMM is concerned, that is by freezing the ω_k at their time-0 value, thus removing the dependence of $\bar{S}$ on OIS discount factors:

$$\bar{S}(t) = \sum_{k=a+1}^{b} \omega_k(t) S_k^x(t) = \sum_{k=a+1}^{b} \omega_k(t) S_k^x(0) \mathcal{M}^x(t) \approx \mathcal{M}^x(t) \sum_{k=a+1}^{b} \omega_k(0) S_k^x(0) = \bar{S}(0) \mathcal{M}^x(t)$$

The swaption price (17) can then be expressed as follows:

$$\mathbf{PS}(t, K; T_a^x, \dots, T_b^x, T_{c+1}^S, \dots, T_d^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)\, E_D^{c,d}\big\{ \big[\tilde{F}(T_a^x) + \bar{S}(0)\mathcal{M}^S(T_a^x) - K\big]^+ | \mathcal{F}_t \big\} \quad (20)$$

which can be calculated in the same way as the caplet price (14).

4.4 The pricing of basis swaps

A popular market contract based on different LIBOR tenors, in the same currency, is a basis swap, which is composed of two floating legs where payments set on a given LIBOR tenor are exchanged for payments set on another tenor. For instance, one can receive quarterly the 3-month LIBOR rate and pay semiannually the 6-month LIBOR rate, both set in advance and paid in arrears. The market actively quotes basis swaps, at least for the main tenors (3m vs 6m). These quotes are typically positive, meaning that a positive spread has to be added to the smaller-tenor leg to match the NPV of the larger-tenor leg.

Let us be given two tenors x and y with $x < y$ and the associated time structures $\mathcal{T}^x = \{T_0^x, \dots, T_{M_x}^x\}$ and $\mathcal{T}^y = \{T_0^y, \dots, T_{M_y}^y\}$, $\mathcal{T}^y \subset \mathcal{T}^x$. We assume that $T_{M_x}^x = T_{M_y}^y$.

[7] See also Fujii et al. (2009b) for a similar decomposition.

[8] This is the case, for instance, of the LMMs of Piterbarg (2005), Wu and Zhang (2006), Henry-Labordère (2007), Rebonato (2007) and Mercurio and Morini (2009).

Let us then consider the two floating legs in the basis swap where x-rates are exchanged for y-rates. The x-leg pays at each time T_i^x, $i = 0, \dots, M_x$, the x-LIBOR rate $L^x(T_{i-1}^x, T_i^x)$. Likewise, the y-leg pays at each time T_j^y, $j = 0, \dots, M_y$, the y-LIBOR rate $L^y(T_{j-1}^y, T_j^y)$, where we set $T_{-1}^x = T_{-1}^y := 0$. The NPVs of the two legs at time 0 are:

$$\sum_{k=0}^{M_z} \tau_k^z P_D(0, T_k^z) L_k^z(0), \quad z \in \{x, y\}.$$

As mentioned above, typical market quotes imply that:

$$\sum_{j=0}^{M_y} \tau_j^y P_D(0, T_j^y) S_j^y(0) > \sum_{i=0}^{M_x} \tau_i^x P_D(0, T_i^x) S_i^x(0)$$

This time-0 condition is satisfied by our multi-tenor model (8) by construction. However, there is no guarantee that the corresponding condition at a future time t will also hold true. If we want to preserve the positivity of basis spreads, we then have to constrain the joint evolution of processes $\mathcal{M}^x$ and $\mathcal{M}^y$, for instance by assuming a very high correlation between them, or, in the limit case, by imposing their equality.

5 A specific example of rate and spread dynamics

Dynamics (6) and (8) can be both driven by stochastic volatility. However, for ease of computation, we can choose to model with stochastic volatility either the forward rates F_k^x or the related spreads S_k^x, for each given tenor x, but not both. Electing S_k^x is in fact more convenient since we can have non-zero correlation between spreads and their volatility, yet keeping the volatility dynamics unchanged under different forward or swap measures. This feature is what motivates the following example.

We fix a tenor x and an index k and assume that the corresponding OIS forward rate follows the shifted-lognomal process:

$$\mathrm{d}F_k^x(t) = \sigma_k^x \Big[\frac{1}{\tau_k^x} + F_k^x(t)\Big] \, \mathrm{d}Z_k^{k,x}(t) \tag{21}$$

where σ_k^x is a positive constant. This corresponds to assuming $V^F \equiv 1$ in (6).

The related spread (equivalently, process $\mathcal{M}^x$) is assumed to follow SABR dynamics:

$$\begin{aligned} \mathrm{d}S_k^x(t) &= \big(S_k^x(t)\big)^{\beta_k} V_k^S(t) \, \mathrm{d}Z_k^S(t) \\ \mathrm{d}V_k^S(t) &= \epsilon_k V_k^S(t) \, \mathrm{d}W_k^S(t), \quad V_k^S(0) = \alpha_k, \quad \mathrm{d}Z_k^S(t) \, \mathrm{d}W_k^S(t) = \rho_k \, \mathrm{d}t \end{aligned} \tag{22}$$

where $\alpha_k > 0$, $\beta_k \in (0, 1]$, $\epsilon_k > 0$, $\rho_k \in [-1, 1]$ are constants, and Z_k^S and $W_k^S(t)$ are standard $Q_D^{T_k^x}$-Brownian motions independent of $Z_k^{k,x}$.

The price of the caplet $\tau_k^x[F_k^x(T_{k-1}^x)+S_k^x(T_{k-1}^x)-K]^+$ can be calculated using the second equality in (15). Straightforward algebra leads to:

$$
\begin{aligned}
&\mathbf{Cplt}(t,K;T_{k-1}^x,T_k^x)\\
&=\int_0^{K+\frac{1}{\tau_k^x}} \frac{\mathbf{Cplt}^{\mathrm{SABR}}(t,K+\frac{1}{\tau_k^x}-z;T_{k-1}^x,T_k^x)}{z\sigma_k\sqrt{T_{k-1}^x-t}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{\left(\ln\frac{z}{F_k^x(t)+1/\tau_k^x}+\frac{1}{2}\sigma_k^2(T_{k-1}^x-t)\right)^2}{\sigma_k^2(T_{k-1}^x-t)}\right\}\mathrm{d}z\\
&\quad+\tau_k^x P_D(t,T_k^x)\big(S_k^x(t)-K-1/\tau_k^x\big)\Phi\left(\frac{\ln\frac{F_k^x(t)+1/\tau_k^x}{K+1/\tau_k^x}-\frac{1}{2}\sigma_k^2(T_{k-1}^x-t)}{\sigma_k\sqrt{T_{k-1}^x-t}}\right)\\
&\quad+\tau_k^x P_D(t,T_k^x)\big(F_k^x(t)+\frac{1}{\tau_k^x}\big)\Phi\left(\frac{\ln\frac{F_k^x(t)+1/\tau_k^x}{K+1/\tau_k^x}+\frac{1}{2}\sigma_k^2(T_{k-1}^x-t)}{\sigma_k\sqrt{T_{k-1}^x-t}}\right)
\end{aligned}
\tag{23}
$$

where Φ denotes the standard normal distribution function and

$$
\mathbf{Cplt}^{\mathrm{SABR}}(t,\mathcal{K};T_{k-1}^x,T_k^x)=\tau_k^x P_D(t,T_k^x)\big[F_k^x(t)\Phi(d_1)-\mathcal{K}\Phi(d_2)\big] \tag{24}
$$

with

$$
\begin{aligned}
d_{1,2}&:=\frac{\ln(F_k^x(t)/\mathcal{K})\pm\frac{1}{2}\sigma^{\mathrm{SABR}}(\mathcal{K},F_k^x(t))^2(T_{k-1}^x-t)}{\sigma^{\mathrm{SABR}}(\mathcal{K},F_k^x(t))\sqrt{T_{k-1}^x-t}}\\
\sigma^{\mathrm{SABR}}(\mathcal{K},F)&:=\frac{\alpha_k}{(F\mathcal{K})^{\frac{1-\beta_k}{2}}\left[1+\frac{(1-\beta_k)^2}{24}\ln^2\left(\frac{F}{\mathcal{K}}\right)+\frac{(1-\beta_k)^4}{1920}\ln^4\left(\frac{F}{\mathcal{K}}\right)+\cdots\right]}\frac{\zeta}{x(\zeta)}\\
&\quad\cdot\left\{1+\left[\frac{(1-\beta_k)^2\alpha_k^2}{24(F\mathcal{K})^{1-\beta_k}}+\frac{\rho_k\beta_k\epsilon_k\alpha_k}{4(F\mathcal{K})^{\frac{1-\beta_k}{2}}}+\epsilon_k^2\frac{2-3\rho_k^2}{24}\right]T_{k-1}^x+\cdots\right\}\\
\zeta&:=\frac{\epsilon_k}{\alpha_k}(F\mathcal{K})^{\frac{1-\beta_k}{2}}\ln\left(\frac{F}{\mathcal{K}}\right),\quad x(\zeta):=\ln\left\{\frac{\sqrt{1-2\rho_k\zeta+\zeta^2}+\zeta-\rho_k}{1-\rho_k}\right\}
\end{aligned}
$$

Swaption prices can also be calculated analytically, by following the procedure suggested in the previous Section 4.3.

5.1 An example of calibration to real market data

We finally consider a simple example of calibration to market data of the extended LMM defined by (21) and (22). As already noticed in the introduction, OIS rates and basis spreads can be interpreted as additive factors driving the evolution of FRA rates. As such, their model calibration requires no information on respective volatilities and can be directly performed on LIBOR-based instruments. To this end, we use EUR data as of February 8th, 2010 and calibrate 6-month caplets with reset date at $T_{k-1}^x = 3$ (years), for which $L_k^x(T_{k-1}^x) = 3.07\%$ and $F_k^x(T_{k-1}^x) = 2.50\%$, so that $S_k^x(T_{k-1}^x) = 0.57\%$.

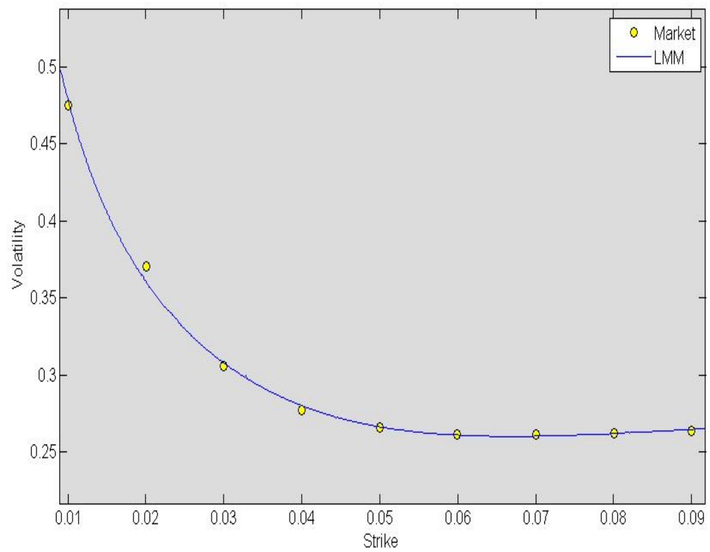

Figure 2: Comparison between market caplet volatilities with the calibrated volatilities implied by the LMM (21) and (22). EUR market data as of February 8th, 2010.

The calibration is performed by minimizing the sum of squared differences between model prices (23) and respective market ones. As we can infer from Fig. 2, our model specification fits the considered market data, in terms of implied volatilities, almost perfectly. In fact, as is typical of the SABR functional form, we have equivalently good fits for different choices of the parameter β_k. In the figure, we show our calibration result corresponding to the choice of $\beta_k = 0.5$.

The main advantage of our extended-LMM example is the freedom to use a non-zero ρ_k to calibrate the negative slope of implied volatilities at the at-the-money, and, at the same time, set to zero the correlation between stochastic volatility and forward OIS rates, so as to keep the same volatility dynamics under different forward and swap measures.[9] Accordingly, the β_k parameter can either be fixed a priori, as in our calibration example, or used to calibrate other market data, like for instance CMS swap spreads. In the single-curve SABR LMM, for instance, we do not have the same flexibility. In fact, one can either use the correlation parameters to fit market skews or keep the same volatility dynamics under different measures.

[9]Clearly, some attention is still required since we need to ensure that the overall correlation matrix is positive semi-definite.

6 Conclusions

In this article, we have shown how to extend the LMM to price interest-rate derivatives under distinct yield curves, used for generating future LIBOR rates and for discounting. To this end, we have chosen to model the joint evolution of OIS forward rates and corresponding basis spreads for different tenors simultaneously, under the assumption that the discount curve coincides with the OIS-based one.

The dynamics we have considered imply the possibility to price in closed-form both caps and swaptions, with procedures that are only slightly more involved than the corresponding ones in the single-curve case. Moreover, modeling different tenors at the same time has the major advantage of allowing for the valuation of derivatives based on multiple tenors, like e.g. basis swaps. Another interesting application concerns the pricing of caps or swaptions with a non-standard underlying tenor, given the market quotes of standard-tenor options, which can be obtained by introducing convenient assumptions on the model parameters.

We have finally considered a simple example of calibration to a market caplet smile. This is to be intended as a preliminary result, since the model robustness and flexibility should be tested on a much broader data set, including swaption smiles and CMS swap spreads.

Another issue that needs further investigation is the modeling of correlations with parametric forms granting the positive definiteness of the overall correlation matrix. To this end, one may try to extend to the multi-curve case the parametrization proposed by Mercurio and Morini (2007a) in a single-curve setting.

References

[1] Ametrano, F. and M. Bianchetti (2009). Bootstrapping the Illiquidity, in *Modelling Interest Rates: Advances for Derivatives Pricing*, edited by F. Mercurio, Risk Books.
volatilities. *Risk*, December, 163-168.

[2] Bianchetti, M. (2009) Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Decoupling Forwarding and Discounting Yield Curves. Working Paper. Available online at:
`http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1334356`

[3] Chibane, M. and Sheldon, G. (2009). Building curves on a good basis. Working Paper, Shinsei Bank. Available online at:
`http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1394267`

[4] Fujii, M., Shimada, Y. and A. Takahashi (2009a). A note on construction of multiple swap curves with and without collateral. CARF Working Paper Series F-154, available online at `http://ssrn.com/abstract=1440633`.

[5] Fujii, M., Shimada, Y. and A. Takahashi (2009b). A Market Model of Interest Rates with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies.

Working Paper, University of Tokyo and Shinsei Bank. Available online at: http://www.e.u-tokyo.ac.jp/cirje/research/dp/2009/2009cf698.pdf

[6] Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002) Managing Smile Risk. *Wilmott magazine*, September, 84-108.

[7] Henrard, M. (2009). The Irony in the Derivatives Discounting Part II: The Crisis. Preprint, Dexia Bank, Brussels.

[8] Henry-Labordère, P. (2007) Combining the SABR and LMM Models. *Risk* October, 102-107.

[9] Heston, S.L. (1993) A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies* 6, 327-343.

[10] Johannes, M. and Sundaresan, S. (2007) The Impact of Collateralization on Swap Rates. *Journal of Finance* 62, 383410.

[11] Kijima, M., K. Tanaka and T. Wong (2009). A Multi-Quality Model of Interest Rates, *Quantitative Finance* 9(2), 133-145.

[12] Mercurio, F. (2009) Interest Rates and The Credit Crunch: New Formulas and Market Models. Available online at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1332205

[13] Mercurio, F. (2010a) Modern LIBOR Market Models: Using Different Curves for Projecting Rates and for Discounting. *International Journal of Theoretical and Applied Finance* 13, 1-25,

[14] Mercurio, F. (2010b) LIBOR Market Models with Stochastic Basis. Available online at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1563685

[15] Mercurio, F., and M. Morini (2007a). A Note on Correlation in Stochastic Volatility Term Structure Models, *Working Paper available at SSRN.com*.

[16] Mercurio, F., and M. Morini (2009). Joining the SABR and LIBOR models together. *Risk*, March, 80-85.

[17] Morini, M (2008). The puzzle in the interest rate curve: counterparty risk? Preprint. Banca IMI, Milan.

[18] Morini, M (2009). Solving the Puzzle in the Interest Rate Market. Available online at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1506046

[19] Piterbarg, V. (2005). Stochastic volatility model with time-dependent skew, *Applied Mathematical Finance*, 12(2), 147-185.

[20] Piterbarg, V. (2010). Funding beyond discounting: collateral agreements and derivatives pricing, *Risk* February, 97-102.

[21] Rebonato, R. (2007) A time Homogeneous, SABR-Consistent Extension of the LMM. *Risk* 20, November, 92-97.

[22] Wu, L. and F. Zhang (2006). LIBOR market model with stochastic volatility, *Journal of Industrial and Management Optimization*, Vol. 2, No. 2, 199227.