Short-Rate Pricing after the Liquidity and Credit Shocks: Including the Basis *

Chris Kenyon†

March 10, 2010, Draft 1.10

Abstract

The basis between swaps referencing funded fixings and swaps referencing overnight-collateralized fixings (e.g. 6 month Euribor vs 6 month Eonia) has increased in importance with the 2007-9 liquidity and credit crises. This basis means that new pricing models for fixed income staples like caps, floors and swaptions are required. Recently new formulae have been proposed using market models. Here we present the equivalent pricing in a short-rate framework which is important for applications involving credit, like CVA, where this is often more natural.

Keywords: credit crisis, liquidity crisis, forward curve, discount curve, basis swaps, bootstrapping, swaps, swaptions, counterparty risk, CVA, multi-curve term structure modeling, closed form formulas.

1 Introduction

The 2007-9 liquidity and credit crises included large basis spreads opening up between overnight-collateralized instruments (e.g. 6M Eonia) and non-collateralized fixings[1] (e.g. 6M Euribor). Deposits of the same maturity as

---

[1]In Fixed Income "fixing" is a noun, it means the value at which a rate is fixed once a day for contract settlement, e.g. "Euribor® is the rate at which Euro interbank term deposits are offered by one prime bank to another prime bank within the EMU zone, and is published at 11:00 a.m. (CET) for spot value (T+2).".

fixings follow the same pattern. Yield curves built from instruments referencing these different instrument types demonstrate a significant basis (e.g. building from Eonia swaps vs building from Euribor swaps). This means that new formulae are required for pricing previously-standard instruments, e.g. swaps, caps, and swaptions. Recent work offers different approaches [CS09, KTW09, Hen09, Mer09, Mor09, AB10] using short-rate, HJM and BGM settings with or without some fx analogy. However, within the short-rate setup, analytic swaption pricing formulae are lacking. Thus calibration to volatility data is akward. Whilst [KTW09] use a short rate setting for multiple bond qualities they did not provide direct analytic swaption pricing (they price via a Gram-Charlier expansion on the bond price distribution). Short-rate settings can be more appropriate when credit risk needs to be included.

We provide direct pseudo-analytic swaption pricing formulae within a short-rate setup including the basis between overnight-collateralized instruments (e.g. 6M Eonia) and non-collateralized fixings (e.g. 6M Euribor). We use a discounting curve to represent riskless investment and a separate curve for market expectations of non-collateralized fixings. We call this second curve the fixing curve. Our approach differs from [KTW09] in that they use discounting and basis curves whereas we use discounting and fixing curves. We follow [AB10] in using an explicit fx argument to deal with potential arbitrage considerations. Our approach can also be generalized as more data becomes available, especially with respect to Eonia-type swaptions (currently missing from the market). Thus we propose a short-rate solution for the observed basis enabling modeling of standard instruments and calibration to swaptions.

A short-rate setup, for the observed basis between discounting and fixing, is potentially useful for a variety of problems, e.g. CVA (credit valuation adjustment) where forward-rate setups are less natural, as in [BPP09]. Thus this paper complements existing forward-rate multi-curve setups [Hen09, Mer09, AB10] and extends [KTW09].

2   Model

We put ourselves in the European context by using Euribor and Eonia. The equations are general and this naming is only for convenience.

Discount Curve We use a riskless curve as our discount curve. For a concrete example we take Eonia as the riskless discount curve. We aim to price standard fixed income instruments (e.g. swaps, swaptions)

subject to collateral agreements, thus the yield curve for similarly collateralized instruments is appropriate for discounting *whatever fixings they reference.*

Fixing Curve We calibrate a second curve, or "fixing curve" to reproduce fixings and market expected fixings (when used together with the discount curve). For a concrete example consider the 6 month Euribor fixing and market-quoted standard EUR swaps. This follows the concept in [AB10, KTW09] of having a discount curve and a separate curve that, together with the discount curve, reproduces market expectations of fixings and instruments based on them e.g. forwards and swaps.

N.B. a FRA (Forward Rate Agreement) payment is not the same as a swap payment for the same fixing because a FRA is paid in advance (discounted with the fixing by definition), whereas a standard swap fixing is paid in arrears.

We use a short-rate model for the discounting curve D, and for the fixing curve f, with short rates $r_D(t)$ and $r_f(t)$ respectively[2]. Although the short rates are both denominated in the same currency, say EUR, they represent theoretically different investment qualities (risks), as in [KTW09], thus there is an exchange rate between them. This setup also mirrors that of [AB10] to preserve no-arbitrage between alternative investment opportunities.

In contrast to [KTW09], the second (fixing) curve f does not actually represent a practical investment opportunity because there are no "market-fixing-curve-bonds" available: every bond comes from some source. There is no market-consensus-risk-bank to issue such bonds, although the actual fixings do exist and are published. However, we keep this setup for intellectual no-arbitrage consistency. It will turn out that the quality-exchange-rate drops out and has no affect in actual pricing.

2.1 Standard Instruments

The setup for standard instruments (swaps and swaptions) is very similar to [KTW09, AB10, Mer09], our innovations are in the next section where we show how to price them within a specific short rate setup.

2.1.1 Swaps

Swaps in this dual curve set up are similar to differential floating for fixed swaps (aka floating for fixed quanto swaps) in [BM06] (page 623). To price

---

[2]We need d later for an index, hence the use of D for the discounting curve.
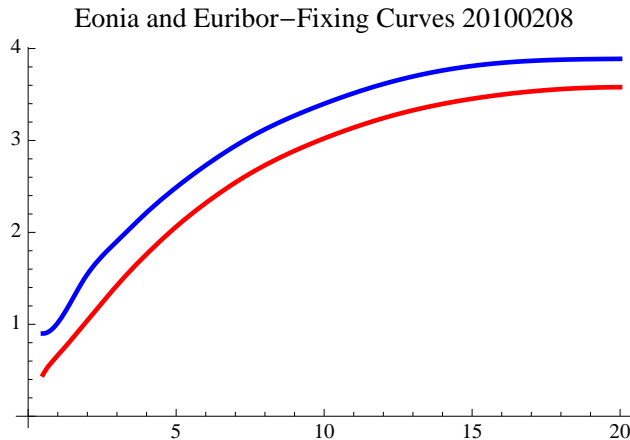
Figure 1: Discounting (lower) and fixing (upper) curves. Note that a conventional Euribor curve (not shown) would be very close to the fixing curve.

swaps we need to define the simply compounded (fixing curve) interest rate that the swaps fix on, $L_k(t)$, at T_{k-1} for the interest rate from T_{k-1} to T_k, i.e. tenor τ. We define this rate in terms of the fixing curve:

$$L_k(t) := \frac{1}{\tau} \left(\frac{P_f(t, T_{k-1})}{P_f(t, T_k)} - 1 \right).$$

Using the terminology of [Mer09] and the equations above, we have for the the floating leg of a swap:

$$\mathbf{FL}(t : T_a, \ldots, T_b) = \sum_{k=a+1}^{b} \mathbf{FL}(t : T_{k-1}, T_k) = \sum_{k=a+1}^{b} \tau_k P_D(t, T_k) L_k(t)$$

N.B. Since separate curves are used for discounting and fixing there is no reason for a floating leg together with repayment of par at the end to price at par. A riskless floating rate bond should price to par, but that bond would not get Euribor coupons but Eonia coupons. This has previously been pointed out by [Mer09, KTW09].

From these definitions we can bootstrap both discounting and fixing curves from, in euroland, an Eonia discount curve (built conventionally) and Euribor deposits and swaps. Note that we will need the dynamics of $r_*(t)$ under the $T-$forward measure for swaption pricing.

Figure 1 shows the discount curve (Eonia) and the fixing curve bootstrapped from Euribor swaps and 6M deposit. Note that a conventional Euribor curve (not shown) would be very close to the fixing curve.

4

2.1.2 Swaptions

For a European swaption strike K and maturity T, directly following on from the formulae for a swap above, we have :

$$\mathbf{ES}[\omega, T, K, a, b, c, d]$$
$$= P_D(0, T)\,\mathbb{E}^T \left\{ \left[\omega \left(\sum_{i=a+1}^{b} \tau_i^{\text{fix}} P_D(T, T_i) L_i(T) - \sum_{i=c+1}^{d} \tau_i K P_D(T, T_i) \right) \right]^+ \right\}$$

where $\omega = 1$ for a payer swaption (and -1 for a receiver swaption), and the $a, b, c, d, \tau_i^{\text{fix}}, \tau_i$ take care of the different tenors and payment frequencies of the two sides.

Note that because the floating leg does not price to par we must include the specification of the floating coupons. These floating coupons are the market expectation of the relevant fixings $L_i(T)$.

2.2 Short-Rate Pricing

We present semi-analytic pricing (i.e. simulation and formulae) for swaptions under general affine short-rate models and pseudo-analytic (i.e. integration and formulae) pricing in the gaussian case.

For full analytic tractability we use a 1-factor Hull-White type model for the discount curve (e.g. Eonia), expressed as a G1++ type model (analogous to the terminology in [BM06]) for the discounting short rate $r_D(t)$ under the bank-account numeraire for D:

$$
\begin{aligned}
r_D(t) &= x(t) + \varphi_D(t), \quad r_D(0) = r_{D,0} \\
dx(t) &= -a_D x(t) dt + \sigma_D dW_1(t)
\end{aligned}
\tag{1}
$$

We also use a one factor model for the fixing curve, i.e. to fit the market-expected fixings, and we label this short rate $r_f(t)$ under the bank-account numeraire for f.

$$
\begin{aligned}
r_f(t) &= y(t) + \varphi_f(t), \quad r_f(0) = r_{f,0} \\
dy(t) &= -a_f y(t) dt + \sigma_f dW_2(t)
\end{aligned}
\tag{2}
$$

where $(W_1(t), W_2(t))$ is a two-dimensional Brownian motion with instantaneous correlation ρ. We also have a quality-exchange-rate X with process under $\mathbb{Q}_f$ for the quantity of f-quality investment required to obtain one unit of D-quality investment:

$$dX(t) = (r_D(t) - r_f(t))X(t)dt + \nu X(t)dW_X(t)$$

where W_X has zero correlation with $W_1(t)$ and with $W_2(t)$. This implies that Equation 2 is unchanged under the measure change from $\mathbb{Q}_f$ to $\mathbb{Q}_D$.

This is a very parsimonious representation relative to two G2++ models. It is, however, sufficient since we only wish to calibrate to Eonia discount rates and Euribor swaptions. N.B. neither Eonia caps/floors nor swaptions are liquid at present (early 2010). It is tempting to use 3m swaptions and 6m swaptions to get implied euribor swaption volatility, however, there is a significant difference in their liquidity so this is problematic.

2.2.1 Swaption Pricing

For a European swaption strike K and maturity T we have for any affine short rate model (i.e. bond prices available as $A()e^{B()}$):

$$\mathbf{ES}[\omega, T, K, a, b, c, d]$$

$$= P_D(0,T)\,\mathbb{E}^T\left\{\left[\omega\left(\sum_{i=a+1}^{b}\tau_i^{\text{fix}}P_D(T,T_i)L_i(T) - \sum_{i=c+1}^{d}\tau_i K P_D(T,T_i)\right)\right]^+\right\}$$

$$= P_D(0,T)\,\mathbb{E}^T\left\{\left[\omega\left(\sum_{i=a+1}^{b}P_D(T,T_i)\left(\frac{P_f(T,T_{i-1})}{P_f(T,T_i)} - 1\right) - \sum_{i=c+1}^{d}\tau_i K P_D(T,T_i)\right)\right]^+\right\}$$

$$= P_D(0,T)\int_{\mathbb{R}^2}\left\{\left[\omega\left(\sum_{i=a+1}^{b}A_D(T,T_i)e^{-B_D(T,T_i)x}\left(\frac{A_f(T,T_{i-1})}{A_f(T,T_i)}e^{(-B_f(T,T_{i-1})+B_f(T,T_i))y} - 1\right)\right.\right.\right.$$

$$\left.\left.\left. - \sum_{i=c+1}^{d}\tau_i K A_D(T,T_i)e^{-B_D(T,T_i)x}\right)\right]^+\right\}f(x,y)dxdy \tag{3}$$

Note that up to this point the equations apply to any affine short rate model. However, to go further analytically we require the joint distribution of x and y under the $T-$forward measure, i.e. $f(x,y)$ above. This is available for Gaussian models but not for CIR-type specification. This formulae could be applied in a CIR-type specification by combining simulation up to T with the affine bond formulae used above.

We require Equations 2 and 1 in the T_D-forward measure (i.e. zero coupon bond from the discounting curve D as the numeraire). Using standard change-of-numeraire machinery we obtain:

$$dx(t) = \left[-a_D x(t) - \frac{\sigma_D^2}{a_D}(1 - e^{-a_D(T-t)})\right]dt + \sigma_D dW_1^T$$

$$dy(t) = \left[-a_f y(t) - \rho\frac{\sigma_f \sigma_D}{a_D}(1 - e^{-a_D(T-t)})\right]dt + \sigma_f dW_2^T$$

This is because the Radon-Nikodym derivative dQ^T/dQ uses the zero coupon discounting bond, hence only x is involved in the measure change (not y as well).

These have explicit solutions:

$$x(t) = x(s)e^{-a_D(t-s)} - M_x^T(s,t) + \sigma_D \int_s^t e^{-a_D(t-u)}dW_1^T(u)$$

$$y(t) = y(s)e^{-a_f(t-s)} - M_y^T(s,t) + \sigma_f \int_s^t e^{-a_f(t-u)}dW_2^T(u)$$

where

$$M_x^T(s,t) = \int_s^t \left[\frac{\sigma_D^2}{a_D}(1 - e^{-a_D(T-t)}) \right] e^{-a_D(t-u)}du$$

$$M_y^T(s,t) = \int_s^t \left[\rho\frac{\sigma_f\sigma_D}{a_D}(1 - e^{-a_D(T-t)}) \right] e^{-a_f(t-u)}du$$

Now we can express a European Swoption price as:

Theorem 1. *The arbitrage-free price at time $t = 0$ for the above European unit-notional swaption is given by numerically computing the following one-dimensional integral:*

$$\mathbf{ES}(\omega, T, K, a, b, c, d) = -\omega P_D(0,T) \int_{-\infty}^{\infty} \delta(x) \frac{e^{-\frac{1}{2}(\frac{x-\mu_x}{\sigma_x})^2}}{\sigma_x\sqrt{2\pi}}$$
$$\left[\Phi(-\omega h_1(x)) - \sum_{i=a+1}^{b} \lambda_i e^{\kappa_i(x)}\Phi(-\omega h_2(x)) \right] dx$$

where $\omega = 1$ for a payer swaption (and -1 for a receiver swaption),

$$\delta(x) := \sum_{i=a+1}^{b} A_D(T,T_i)e^{-B_D(T,T_i)x} + \sum_{i=c+1}^{d} \tau_i K A_D(T,T_i)e^{-B_D(T,T_i)x}$$

$$h_1(x) := \frac{\bar{y}(x) - \mu_y}{\sigma_y\sqrt{1-\rho_{xy}^2}} - \frac{\rho_{xy}(x - \mu_x)}{\sigma_x\sqrt{1-\rho_{xy}^2}}$$

$$h_2(x) := h_1(x) + \alpha_i \sigma_y\sqrt{1-\rho_{xy}^2}$$

$$\alpha_i := -(-B_f(T,T_{i-1}) + B_f(T,T_i))$$

$$\lambda_i(x) := \frac{1}{\delta(x)}A_D(T,T_i)e^{-B_D(T,T_i)x}\frac{A_f(T,T_{i-1})}{A_f(T,T_i)}$$

$$\kappa_i(x) := -\alpha_i \left[\mu_y - \frac{1}{2}(1 - \rho_{xy}^2)\sigma_y^2\alpha_i + \rho_{xy}\sigma_y\frac{x - \mu_x}{\sigma_x} \right]$$

7

Where $\bar{y} = \bar{y}(x)$ is the unique solution of:

$$\sum_{i=a+1}^{b} A_D(T, T_i) e^{-B_D(T,T_i)x} \left(\frac{A_f(T, T_{i-1})}{A_f(T, T_i)} e^{(-B_f(T,T_{i-1})+B_f(T,T_i))\bar{y}} - 1 \right)$$

$$= \sum_{i=c+1}^{d} \tau_i K A_D(T, T_i) e^{-B_D(T,T_i)x}$$

and

$$\begin{aligned}
\mu_x &:= -M_x^T(0, T) \\
\mu_y &:= -M_y^T(0, T) \\
\sigma_x &:= \sigma_D \sqrt{\frac{1 - e^{-2a_D T}}{2a_D}} \\
\sigma_y &:= \sigma_f \sqrt{\frac{1 - e^{-2a_f T}}{2a_f}} \\
\rho_x &:= \frac{\rho \sigma_D \sigma_f}{(a_D + a_f)\sigma_x \sigma_y} \left[1 - e^{-(a_D+a_f)T} \right]
\end{aligned}$$

and

$$\begin{aligned}
A_*(T, T_i) &= \frac{P_*(0, T_i)}{P_*(0, T)} \exp[\frac{1}{2}(V(T, T_i, a_*, \sigma_*) - V(0, T_i, a_*, \sigma_*) + V(0, T, a_*, \sigma_*))] \\
B_*(T, T_i) &= \frac{1 - e^{a_*(T_i - T)}}{a_*} \\
V(T, T_i, a_*, \sigma_*) &= \frac{\sigma_*^2}{a_*^2} \left((T_i - T) + \frac{2}{a_*} e^{-a_*(T_i - T)} \frac{1}{a_*} e^{-2a_*(T_i - T)} - \frac{3}{2a_*} \right)
\end{aligned}$$

where $*$ is either f or D.

Proof. We manipulate Equation 3 of **ES** above into the same form as for Theorem 4.2.3 [BM06] and provide an appropriate new definition of $\bar{y}$ to finish the proof.

The $[]^+$ part of Equation 3 of **ES** is of the form:

$$\sum_i U_i e^{-V_i x} (F_i e^{-G_i y} - 1) - \sum_j M_j e^{N_j x}$$

Expanding:

$$\sum_i U_i e^{-V_i x} F_i e^{-G_i y} - \sum_i U_i e^{-V_i x} - \sum_j M_j e^{N_j x}$$
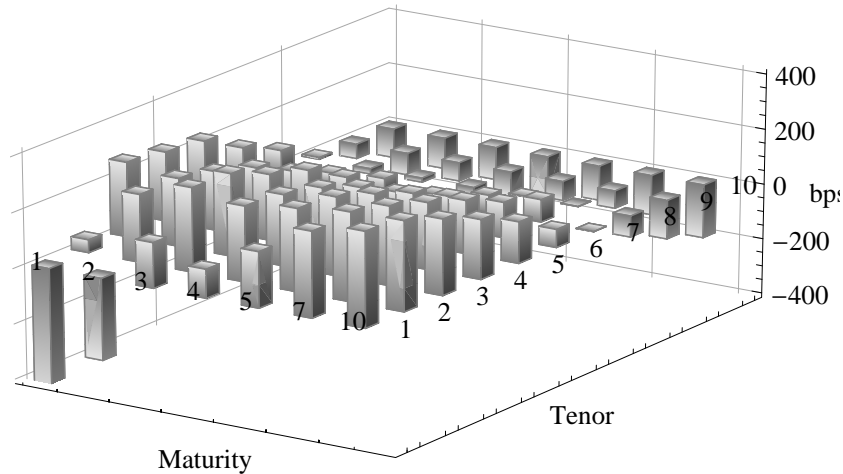
Figure 2: Swaption calibration for G1++/G1++ model, errors in bps.

Now if we divide by the two negative terms we obtain (remembering to keep track of any reversed limits):

$$\frac{1}{\sum_i U_i e^{-V_i x} - \sum_j M_j e^{N_j x}} \left(1 - \sum_i U_i e^{-V_i x} F_i e^{-G_i y} \right)$$

Now define $\bar{y} = \bar{y}(x)$ as the unique solution of:

$$\sum_i U_i e^{-V_i x} (F_i e^{-G_i \bar{y}} - 1) = \sum_j M_j e^{N_j x}$$

So if we freeze x we have the same form as Theorem 4.2.3 of [BM06], and the rest of the proof is immediate. $\qquad\square$

As an example we calibrated to Eonia, Euribor swaps and Euribor swaptions using data asof 8/Feb/10. We find calibration parameters:

a_D	σ_D	a_f	σ_f	ρ
1.0	0.4	0.1	0.0105	-0.95

Swaption calibration with the G1++/G1++ model is similar to that displayed in [BPP09] using a G2++ model, which also uses post-crisis data, see Figure 2.

3 Conclusion

We provide pseudo-analytic pricing for swaptions in a discounting/fixing multi-curve gaussian short-rate setup. Semi-analytic swaption pricing for

general affine short-rate models is possible by combining simulation up to maturity with analytic bond formulae at swaption maturity. Pricing of other standard instruments, e.g. caps, are simplifications of the swaption formulae provided. Structurally the pseudo-analytic calculations are very similar to G2++ pseudo-analytic swaption pricing in [BM06]. This means that the current calculations could be directly extended to a G1++/G2++ setup or G2++/G2++ setup once Eonia swaptions, that are currently not available, become liquid. This work extends [KTW09] who also worked in the short-rate setting but did not provide pseudo-analytic swaption pricing.

There is a significant lack of data in the market at present, in early 2010. Eonia/OIS swaptions are missing for example. We calibrate our model jointly to give Eonia and Euribor volatility, however it is not possible to explicitly test our identification with current market data. Also 6M volatility data does not give full information about 3M volatility. Although futures options are liquid and can help, the long end of the volatility curve is simply missing for non-standard tenors. Without this data pricing options on non-standard tenor fixings (e.g. 3M Euribor, 12M USD Libor) is problematic.

A short-rate setup for multiple yield curves, i.e. discounting and fixing, is potentially useful for a variety of problems, e.g. CVA (credit valuation adjustment) where forward-rate setups are less natural, as in [BPP09]. Thus this paper complements existing market model approaches [Hen09, Mer09, AB10].

Appendix

Here we provide the data used in the model (asof 8th February 2010). Note that the 6M money market instrument is typically a few basis points away from the 6M Euribor fixing, and the money market instrument changes throughout the day. Thus the swap rates given are one particular snapshot. (We ignore fixing lags in this paper, conventionally 2 business days for EUR).

Euribor & Swap Rates	
Tenor	Rate (%)
6M	0.97
2Y	1.586
3Y	1.952
4Y	2.264
5Y	2.532
6Y	2.761
7Y	2.959
8Y	3.122
9Y	3.254
10Y	3.366
12Y	3.546
15Y	3.708
20Y	3.798

Eonia Yields	
Tenor (Years in Act360)	Rate (%)
0.505556	0.45573
1.01389	0.66794
1.52222	0.85542
2.03333	1.05402
3.05278	1.44287
4.06389	1.78273
5.07778	2.08370
6.09167	2.34107
7.10833	2.56461
8.12778	2.75217
9.13889	2.90716
10.15000	3.04098
11.16670	3.15866
12.18060	3.25962
15.2250	3.46469
20.3000	3.57937

Swaption Implied Volatilities (maturity x tenor)										
	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y	51.1	39.2	33.7	30.2	27.6	26.2	25.1	24.4	23.8	23.3
2Y	37.3	30.4	27.5	25.7	24.2	23.3	22.7	22.2	21.8	21.4
3Y	28.8	24.8	23.1	22.	21.1	20.7	20.4	20.2	20.	19.8
4Y	24.1	21.4	20.3	19.6	19.1	19.	18.9	18.8	18.8	18.8
5Y	21.	19.1	18.4	18.	17.7	17.7	17.6	17.6	17.7	17.7
7Y	19.	17.8	17.3	17.1	16.9	16.9	16.8	17.	17.1	17.2
10Y	17.1	16.5	16.3	16.1	16.	16.1	16.1	16.3	16.4	16.6

References

[AB10]   F. Ametrano and M. Bianchetti. Bootstrapping the Illiquidity: Multiple Yield Curves Construction for Market Coherent Forward Rates Estimation. *MODELING INTEREST RATES, Fabio Mercurio, ed., Risk Books, Incisive Media*, 2010.

[BM06]   D. Brigo and F. Mercurio. *Interest Rate Models: Theory and Practice, 2nd Edition.* Springer, 2006.

[BPP09]   D. Brigo, A. Pallavicini, and V. Papatheodorou. Bilateral Counterparty Risk Valuation for Interest-Rate Products: Impact of Volatilities and Correlations. *SSRN eLibrary*, 2009.

[CS09]   M. Chibane and G. Sheldon. Building Curves on a Good Basis. *SSRN eLibrary*, 2009.

[Hen09]   M.P. Henrard. The Irony in the Derivatives Discounting Part II: The Crisis. *SSRN eLibrary*, 2009.

[KTW09]   M. Kijima, K. Tanaka, and T. Wong. A multi-quality model of interest rates. *Quantitative Finance*, 9(2):133–145, 2009.

[Mer09]   F. Mercurio. Interest Rates and The Credit Crunch: New Formulas and Market Models. *SSRN eLibrary*, 2009.

[Mor09]   M. Morini. Solving the Puzzle in the Interest Rate Market. *SSRN eLibrary*, 2009.