Collateral and Credit Issues in Derivatives Pricing*

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ABSTRACT

Regulatory changes are increasing the importance of collateral agreements and credit issues in over-the-counter derivatives transactions. This paper considers the nature of derivatives collateral agreements and examines the impact of collateral agreements, two-sided credit risk, funding costs, and bid-offer spreads on the valuation of derivatives portfolios.

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1. Introduction

Forty years have passed since Black and Scholes (1973) and Merton (1973) published their path-breaking model concerned with the valuation European stock options. Few models in finance and economics have been as influential as Black−Scholes−Merton. The model has been extended to value many different types of derivatives on many different underlying assets and has found a wide array of applications.

One assumption made by Black−Scholes−Merton is that the two sides to a derivatives transaction are certain to honor their commitments. It is now recognized that default risk is an important consideration in the bilaterally-cleared over-the-counter market. Many researchers have written in this area. An excellent summary of their work is provided by Gregory (2012).

Banks now take account of counterparty default risk by calculating a credit valuation adjustment (CVA) for each counterparty. The CVA is the expected loss arising from a default by the counterparty. The bank's own credit risk is considered by a debt (or debit) value adjustment (DVA). The DVA is the expected gain that will be experienced by the bank in the event that it defaults on its portfolio of derivatives with a counterparty. CVA and DVA are calculated and hedged in the same way as derivatives by many banks. The calculations typically involve computationally time-consuming Monte Carlo procedures and are discussed by, for example, Canabarro and Duffie (2003), Picault (2005), and Hull and White (2012a).

The size of CVA and DVA adjustments can be substantial. For example, JPMorgan-Chase reported a DVA gain (a change in its DVA) of $1.4 billion resulting from widening of the JPMorgan's credit spreads in 2011. This increased the bank's net income by about 15% and contributed 2% to its reported return on equity.

One result of the 2008 credit crisis is a requirement that most standardized over-the-counter derivatives be cleared through central clearing parties (CCPs). A CCP operates similarly to an exchange clearing house and requires the two sides to a derivatives transaction to post both initial margin and variation margin.[1] The objective of this requirement is to ensure that there will sufficient collateral to cover losses in the event of a default by a major bank.

The CCP requirement just mentioned will reduce the number of derivatives transactions cleared bilaterally.[2] In addition, the Basel Committee for Banking Supervision has proposed regulations requiring derivatives entered into between financial institutions to be well collateralized when they are not cleared centrally.[3] [4] One result of these measures is likely to be a reduction in CVA and DVA. However, there are a number of exceptions to the regulations. For example, they will not apply to transactions with non-financial end users and to some foreign exchange transactions. CVA and DVA will therefore continue to be important adjustments for some counterparties.

As the new regulations reduce the effect of counterparty credit risk, they will increase the use of collateral. The likely effect of this change on financial stability is discussed by Cecchetti *et al* (2009), Singh and Aitken (2009), Duffie and Zhu (2011), and Heller and Vause (2012). In this paper, we examine the effect that collateralization has on the valuation of derivatives. If the interest paid on cash collateral is the risk-free rate, no valuation adjustment for collateral agreements is necessary. However, if the interest rate is different from the risk-free rate, both parties to the transaction should make adjustments to their valuations (in addition to CVA and DVA) to reflect the present value of expected gains or losses from the difference between the collateral rate of interest and the risk-free rate. We will refer to the present value of expected loss as the collateral rate adjustment (CRA) so that the net value of a bank's derivatives portfolio with a counterparty after all the adjustments is the no-default value minus CVA plus DVA minus CRA.

The main focus of this paper is to show how the arguments used to calculate the no-default value of a derivative can be extended to incorporate changing contractual arrangements. Our approach

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[1] See Group of Twenty (2009)

[2] In some of its analyses, the International Monetary Fund (2010) assumes that three-quarters of interest rate swaps, two-thirds of credit default swaps, and one-third of other OTC derivatives will be cleared centrally.

[3] See Basel Committee on Banking Supervision (2012).

[4] Agreements specifying the collateral that must be posted in the non-centrally cleared over-the-counter market are usually included in a credit support annex (CSA) of an International Swaps and Derivatives Association master agreement between two market participants.

is different from earlier research in that we consider collateralization as well as credit risk. We examine both equilibrium and no-arbitrage arguments and show that they lead to identical results. We explain how counterparty credit risk can be hedged and discuss whether a similar approach can be used by a bank to hedge its own credit risk. The expected recovery rates on the derivative for the bank and its counterparty are allowed to be different from that on their other liabilities .

One controversial issue related to credit risk is whether it is necessary for a bank (or other market participant) to adjust valuations to reflect funding costs. In practice, many banks do make what is known as a funding value adjustment (FVA). It is argued that if a bank funds a derivatives portfolio by issuing bonds, it should ensure that a hedged derivative position earns the rate of interest on the bonds rather than the risk-free rate. This argument has no theoretical validity, but because it is widely used we will review it.

For ease of exposition, the analysis is developed for a single derivative on a non-dividend-paying stock where interest rates are constant. But it can be extended so that it is applicable to derivatives and derivatives portfolios dependent on many underlying market variables. (This is important because CVA and DVA must usually be calculated for the portfolio of outstanding derivatives between two sides, not on a transaction-by-transaction basis.)

Section 2 of this paper reviews the Black–Scholes (1973) and Merton (1973) arguments. Section 3 shows how they can be extended so that the credit risk of both sides to the transaction is taken into account in both an equilibrium and a no-arbitrage context. Section 4 uses the result of section 3 to consider the impact of collateral agreements on valuation. Section 5 considers the funding value adjustment. Section 6 extends the analysis further to show how bid-offer spreads can be determined. Conclusions are in Section 7.

2. The Basic BSM Argument

We start by considering the Black and Scholes (1973) and Merton (1973) arguments for valuing a derivative dependent on a non-dividend-paying stock when interest rates are constant. The valuation arguments developed by Black and Scholes involve the capital asset pricing model (CAPM). Suppose that f is the price of the derivative and S is the price of the stock. The process assumed for the stock is

$$dS = \mu S dt + \sigma S dz$$

where μ is the expected return on the stock, σ is its volatility, and dz is a Wiener process. An application of Ito's lemma shows that the price of the derivative satisfies

$$\begin{aligned} df &= \mu_f f dt + \sigma S \frac{\partial f}{\partial S} dz \\ \mu_f &= \frac{1}{f}\left[\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right] \end{aligned} \tag{1}$$

If the continuous time CAPM applies, the expected stock return is

$$\mu = r + \beta_S\left(\mu_M - r\right)$$

where r is the rate of return on a riskless asset, μ_M is the expected return on the market portfolio, and β_S is the stock's beta. Based on the definition of beta and the processes followed by the stock and the derivative, the (instantaneous) beta of the derivative is

$$\beta_f = \frac{S}{f}\frac{\partial f}{\partial S}\beta_S$$

Applying the CAPM to the derivative

$$\mu_f = r + \beta_f\left(\mu_M - r\right) = r + \frac{S}{f}\frac{\partial f}{\partial S}\left(\mu - r\right) \tag{2}$$

Combining equations (1) and (2) leads to the well-known Black−Scholes−Merton (BSM) differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{3}$$

This analysis, which appeared in the original article by Black and Scholes, shows that the derivative price that satisfies equation (3) is the price at which it earns an expected rate of return that is commensurate with its risk. This derivation of equation (3) does not involve any hedging arguments.

We will refer to the solution of differential equation (3) as the no-default value of the derivative, f_{nd}. In general, the solution is given by discounting expected risk-neutral payoffs on the derivative back to the present using the riskless rate. For a European-style derivative maturing at time *T*, the solution is

$$f_{\text{nd}}\left(S_t, t\right) = e^{-r(T-t)} E_r\left[f\left(S_T, T\right)\right] \tag{4}$$

where S_t is the stock price at time *t* and E_r is the expectation taken over all paths that the stock price may follow when the stock's expected return is *r*.

An alternative derivation of the differential equation, suggested by Merton (1973), leads to the trading strategy used by banks to manage market risk in a derivatives portfolio. Consider the case in which a bank has sold a derivative. The short position in the derivative exposes the bank to market risk due to uncertain changes in the price of the underlying stock. This market risk is hedged by buying enough shares to hedge out the market risk of the derivative. The resulting portfolio consists of a short position in the derivative and a position of $\partial f / \partial S$ in the stock. The portfolio value is Π

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

An application of Ito's lemma to this portfolio shows that the change in the portfolio value is

$$d\Pi = -\frac{\partial f}{\partial t} dt - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt \tag{5}$$

Since this portfolio is riskless, it should earn the same rate of return as other riskless assets, *r*, so that

$$d\Pi = r\Pi dt = \left[-rf + rS\frac{\partial f}{\partial S} \right] dt \tag{6}$$

Combining equations (5) and (6) also leads to the BSM differential equation (3). The solution to this differential equation is the price of the derivative that satisfies the no-arbitrage condition. Merton's analysis forms the basis on which banks have operated their derivatives businesses for many years, buying and selling the derivatives and using delta hedging to manage the risk.

3. Incorporation of Default Risk in Derivative Pricing

We now consider how credit risk affects derivative pricing. For ease of exposition, we consider three cases: the derivative is an asset of the bank (always has a positive value from the bank's point of view), the derivative is a liability of the bank (always has a negative value from the bank's point of view), and the derivative is neither an asset nor a liability of the bank (the value is positive in some situations and negative in others). We assume for the moment that no collateral is posted.

3.1 The Derivative is an Asset of the Bank

Suppose first that the derivative is an asset of the bank so that it always has a positive value to the bank. If the counterparty may default, the process for the derivative given in equation (1) can be modified to include the possibility of default so that it becomes

$$df = \left(\mu_f f \right) dt + \sigma S \frac{\partial f}{\partial S} dz - \gamma_C f dq \tag{7}$$

where dq denotes a jump process that represents the event of default. The probability that a jump occurs in the next interval of length dt is $\lambda_C\, dt$ where λ_C is the counterparty's hazard rate. The size of the jump is one and γ_C is the expected proportional reduction in the value of the derivative in the event of a counterparty default. (For ease of exposition, we assume that the default risk is not systematic. Equivalently we could define λ_C as the risk-neutral hazard rate.) In this case

$$E\left[\frac{df}{f}\right]=\left(\mu_f-\gamma_C\lambda_C\right)dt$$

Combining this with equation (2) results in a modified form of the original BSM differential equation

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=\left(r+\gamma_C\lambda_C\right)f$$

Suppose that the counterparty has an outstanding discount bond. The process followed by the price of this bond is

$$dB_C=r_CB_Cdt-\eta_CB_Cdq$$

where r_C is the instantaneous return earned by the bondholder as long as the bond does not default and η_C is the expected proportional reduction in the value of the bond in the event of a default. Since the default risk is not systematic

$$E\left[\frac{dB_C}{B_C}\right]=\left(r_C-\eta_C\lambda_C\right)dt=rdt$$

This allows us to express the hazard rate in terms of the interest rate on the debt.

$$\lambda_C=\frac{r_C-r}{\eta_C}$$

The differential equation then becomes

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=r_C^*f \qquad (8)$$

where

$$r_C^*=r+\left(r_C-r\right)\frac{\gamma_C}{\eta_C}$$

In the particular case where the bond and the derivative are treated in the same way in the event of default $\gamma_C = \eta_C$ so that $r_C^* = r_C$. We can regard r_C^* as an adjusted rate of interest on the bond to allow for differences between the recovery rates on the bond and the derivative.

Equation (8) can also be derived using Merton's hedging arguments. If the counterparty defaults, the bank will suffer a loss. In this case, the bank hedges the counterparty credit risk by shorting an appropriate amount of the discount bond issued by the counterparty.

The resulting hedge portfolio consists of a long position in the derivative, a position of $-\partial f / \partial S$ in the stock and a short position in *n* units of the discount bond where *n* is chosen so that $nB_C \eta = \gamma f$. The portfolio value, Π, is

$$\Pi = f - \frac{\partial f}{\partial S} S - nB_C = f\left(1 - \frac{\gamma_C}{\eta_C}\right) - \frac{\partial f}{\partial S}$$

An application of Ito's lemma to this portfolio shows that, as long as there is no default, the change in the portfolio value is

$$d\Pi = \frac{\partial f}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} dt - r_C \frac{\gamma_C}{\eta_C} f dt \tag{9}$$

The last term in equation (9) is the interest that must be paid on the short position in the counterparty's debt. Since this portfolio is riskless, it should earn the same rate of return, *r*, as other riskless assets so that

$$d\Pi = r\Pi dt = rf\left(1 - \frac{\gamma_C}{\eta_C}\right) dt - rS \frac{\partial f}{\partial S} dt \tag{10}$$

Combining equations (9) and (10) leads to the differential equation (8). The solution is obtained by discounting risk-neutral payoffs at r_C^* For a European-style derivative providing a payoff at time *T*

$$f(S_t, t) = e^{-r_C^*(T-t)} E_r \left[f(S_T, T) \right]$$

which from equation (4) becomes

$$f(S_t, t) = f_{\mathrm{nd}} e^{-\left(r_C^* - r\right)(T-t)}$$

The variable $r_C^* - r$ can be regarded as the "loss rate" on the derivative. Because CVA is the reduction in the value of the derivative arising from the possibility of a counterparty default it follows that

$$\mathrm{CVA} = f_{\mathrm{nd}} - f_{\mathrm{nd}} e^{-(r_C^* - r)(T-t)}$$

3.2 The Derivative is a Liability of the Bank

Now consider the case in which the derivative is a liability of the bank so that it always has a positive value to the counterparty and a negative value to the bank. The Black−Scholes CAPM derivation of the value of the derivative is essentially unchanged except: the value of the derivative is negative, the hazard rate is λ_B, the bank's hazard rate, and the relevant borrowing rate is that of the bank, r_B. The resulting differential equation is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r_B^* f \tag{11}$$

where

$$r_B^* = r + \left(r_B - r\right)\frac{\gamma_B}{\eta_B}$$

Here γ_B and η_B are the expected proportional impact on the value of its derivative and its bonds of a default by the bank. (Because the derivative is a liability the bank gains γ_B times the value of the derivative from its own default.)

The Merton hedge portfolio argument is also essentially the same except it is now necessary to purchase some of the bank's own debt to hedge the change in derivative value in the event of default. Some authors are uncomfortable with this. Burgard and Kjaer (2011, 2012) and Lu and Juan (2011) argue that it is not possible to use the proceeds from the sale of the derivative to buy the bank's own debt and cast doubt on equation (11). Their argument can be countered in several ways. First we can treat the analysis as a thought experiment of what is required to eliminate all risks from the derivative portfolio so that the hedged portfolio can be compared with a riskless asset. Since the value of the derivative is the same whether we hedge it or not, the results of the thought experiment can be used to determine the value even if it is not possible to buy the banks own debt to hedge the default risk.[5]

An alternative explanation is that it is not necessary to buy the bank's own debt. If the proceeds from the sale of the derivative can be used as a source of bank financing, the amount of funding the bank requires from external sources is reduced. This reduction in external borrowing reduces the interest payments to external sources. As a result the proceeds from the sale of the derivative earn an effective rate of r_B and equation (11) follows. The final counter is that the Black−Scholes CAPM derivation which produces the same differential equation does not rely on buying or selling the bank's debt. It merely uses the yield on debt to infer the bank's hazard rate.

The general solution to equation (11) for any derivative involves determining the expected risk-neutral payoffs at all times and discounting these at r_B^*. For a European-style derivative providing a payoff at time T, the solution is

$$f\left(S_t, t\right) = e^{-r_B^*(T-t)} E_r\left[f\left(S_T, T\right)\right]$$

If the derivative is always a liability, the credit adjusted value can be determined by merely adjusting the rate used to discount the expected payoffs. Analogously to the asset case

$$f = f_{\text{nd}} + \text{DVA}$$

where

$$\text{DVA} = f_{\text{nd}} e^{-(r_B^* - r)(T-t)} - f_{\text{nd}}$$

[5] In this context, it is worth noting that delta hedging and other derivatives hedging strategies rarely work perfectly in practice. Nevertheless, it is assumed that they do work perfectly when hedging arguments are used to value derivatives.

Note that DVA is positive because f_{nd} is negative.

3.3 The Derivative can be an Asset or a Liability

Finally, consider the case in which the value of the derivative may be positive or negative. Both the Black−Scholes CAPM and the Merton riskless hedging arguments are exactly the same except the argument that applies at any particular time depends on whether the value of the derivative is positive or negative at that time. Whenever the derivative value is positive, equation (8) applies; whenever the value is negative, equation (11) applies. This means that

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=r_C^*\max\left(f,0\right)+r_B^*\min\left(f,0\right) \tag{12}$$

In general:[6]

$$f\left(S_t,t\right)=f_{\text{nd}}-\text{CVA}+\text{DVA}$$

There are three special cases in which a simple discount rate adjustment can be used to value the derivative. The first two are the cases already discussed in which f_{nd} is always positive or always negative. The third special case is the situation in which $r_C^*=r_B^*=r^*$. In this case, equation (12) simplifies to

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=r^*f \tag{13}$$

The value of the derivative can be obtained by discounting risk-neutral payoffs at r^*. When a payoff is expected at a single time *T*:

$$f(S_t,t)=f_{\text{nd}}e^{-\left(r^*-r\right)\left(T-t\right)}$$

and

[6] Note that some adjustment for the possibility that both parties will default during the life of the derivatives portfolio may be necessary in the calculation of CVA and DVA. See, for example, Brigo and Morini (2011).

$$\mathrm{CVA}-\mathrm{DVA}=f_{\mathrm{nd}}-f_{\mathrm{nd}}e^{-\left(r^{*}-r\right)(T-t)}$$

4. Impact of Collateral Agreements

Up to now we have assumed that no collateral is posted. In practice, it is now common to include a credit support annex (CSA) in the agreement covering transactions that are cleared bilaterally. The CSA typically provides formulas governing the amount of collateral that is required by each side at any given time. It is important to recognize that the laws governing derivatives collateral are different from the laws governing physical assets when they are used as security for a loan. When there is a default on a loan, the seizure of collateral by creditors is subject to delays and must be authorized by a bankruptcy court. The collateral posted for derivatives positions is usually in the form of cash or liquid securities. When there is an early termination following an event of default, collateral posted by the defaulting party is under the control of the non-defaulting party and can immediately be used to compensate the non-defaulting party for its losses.

The ISDA master agreement typically allows the non-defaulting party to claim the cost of replacing the outstanding transactions. This cost is the mid-market value of the transactions at the time of the default, adjusted for the bid-offer spreads that the non-defaulting party would incur when trading with third parties to carry out the replacement. Suppose first that the cost of replacing the transactions is positive (i.e., the portfolio of outstanding transactions, after bid-offer spread adjustments, has a positive value to the non-defaulting party). If the replacement cost is less than the collateral that has been posted, the non-defaulting party incurs no loss and is required to return any excess collateral. If it is greater than the posted collateral, the non-defaulting party is an unsecured creditor for the collateral shortfall. If the cost is negative (i.e., the portfolio of outstanding transactions has a positive value to the defaulting party after bid-offer spread adjustments), there are also two situations to consider. If the collateral posted by the non-defaulting party is greater than the value of the transactions to the non-defaulting party, the non-defaulting party is an unsecured creditor for the excess collateral. If the collateral posted by the non-defaulting party is less than the value of the transactions to the defaulting party, the non-defaulting party is required to pay the amount of the collateral shortfall to the defaulting party.

We start with a simple somewhat idealized situation. We suppose that each side has to post cash collateral equal to max(*X*, 0) where *X* is the mid-market no-default value of the outstanding derivatives to the other side. Collateral is posted continuously right up to the time of default. Furthermore, we assume that the bid-offer spread adjustments referred to above do not apply. We suppose that the interest paid on the collateral exceeds the risk-free rate by a spread *s*, which may be positive, negative, or zero.

Clearly, in the case where *s* is zero, collateral arrangements do not affect the value of the derivative. Because we assume that collateral is posted right up to the time of default and there are no bid-offer spread adjustments, no losses are incurred by either side in the event of a default and equation (3) applies. In practice, *s* is sometimes non-zero (i.e., the interest paid on cash collateral is different from the risk-free rate). The collateralized case is then same as the no-default case considered in Section 1 except that that $r+s$ is paid on the value of the derivative at all times.

First, consider the equilibrium approach to valuation. The bank has an investment *f* in the derivative and borrowings of *f* on which an interest of $r+s$ is paid. (If *f* is negative, the derivative is a liability and the "borrowings" are an investment earning $r+s$.) As a result, the effective rate of return on the derivative in equation (1) becomes

$$\mu_f = \frac{1}{f}\left[\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right] - (r+s)$$

The equilibrium expected return equals the equilibrium return on the derivative less the equilibrium return on the collateral. The latter is the risk-free rate so that

$$\mu_f = \left[r + \beta_f\left(\mu_M - r\right)\right] - r = \frac{S}{f}\frac{\partial f}{\partial S}(\mu - r)$$

Combining these last two equations leads to the differential equation (3) being modified to

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=\left(r+s\right)f \tag{14}$$

Now consider the hedging analysis. The change in the value of the hedge portfolio given by equation (5) becomes

$$d\Pi=-\frac{\partial f}{\partial t}dt-\frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}dt+\left(r+s\right)fdt$$

Equation (6) becomes

$$d\Pi=r\Pi dt=\left[-rf+rS\frac{\partial f}{\partial S}\right]dt+rfdt$$

In the first of these equations, the last term reflects the income on the collateral part of the hedged portfolio. In the second equation, the last term reflects the equilibrium expected return on the collateral part of the hedged portfolio. Combining these two equations also leads to equation (14).

Equation (14) shows that derivatives can be valued by discounting risk-neutral payoffs at *r* + *s*, the interest rate on the collateral. For a European-style derivative providing a payoff at time *T*, the solution to equation (14)

$$f\left(S_t,t\right)=e^{-(r+s)(T-t)}E_r\left[f\left(S_T,T\right)\right]$$

and

$$\text{CRA}=f_{\text{nd}}\left(1-e^{-s(T-t)}\right)$$

In practice, collateral agreements are not as simple as the one we have just considered. An agreement can be one-sided (only one party is required to post collateral) or two-sided (both parties are required to post collateral). Not all collateral is in the form of cash. A collateral agreement usually specifies securities that can be posted in lieu of cash. It also specifies haircuts (i.e., percentage reductions that will be applied to the values of the securities for the purpose of determining their cash equivalent). Returns from the securities that are posted as collateral (income and capital gains) belong to the party that is posting the securities as collateral. The expected return on the securities is by definition their economic return and so no collateral rate adjustment is necessary when securities are posted as collateral. (This is true regardless of the haircut.)

Each day, the collateral already posted is compared with that specified in the agreement. This leads to either some collateral being returned or further collateral being posted. A minimum transfer amount is usually specified to avoid the inconvenience of relatively small daily transfers of collateral being required. Sometimes a threshold, H, is specified. This means that the collateral that has to be posted is $\max(X - H, 0)$. In other circumstances, an independent amount is specified. This is in effect a negative threshold.

The idealized case we considered above is an approximation to a two-sided zero-threshold agreement. This is the type of collateral agreement that has traditionally been used between two derivatives dealers. To a good approximation, CRA can be incorporated into a valuation by discounting at $r + s$ rather than r providing all collateral is posted in the form of cash. In practice, however, the collateral posted between dealers does not usually have to be in the form of cash. Dealers will find it optimal to post cash when $s > 0$ and to post securities when $s < 0$ so that a CRA is necessary only when $s > 0$.

In the idealized case, it can be assumed that CVA and DVA are both zero. But this assumption cannot be made for the two-sided zero-threshold agreements encountered in practice. This is because time typically elapses between the following events:

a) the defaulting party stops posting collateral and stops returning excess collateral; and

b) there is an early termination of outstanding derivatives because of an event of default has occurred

If the derivatives portfolio moves in favor of the non-defaulting party during this period of time, a loss will be incurred and the non-defaulting party will be an unsecured creditor for an amount reflecting the movement. (There is no corresponding gain when the derivatives portfolio moves against the non-defaulting party because the non-defaulting party must return any excess collateral.) Furthermore, the non-defaulting party will also be an unsecured creditor for the bid-offer spread adjustments (referred to earlier) that are made when determining the claim amount.

Our results can be used to reach some conclusions about the relative sizes of CRA and CVA/DVA. The differential equations (8), (11), (13), and (14) have a similar form. Comparing equations (13) and (14), for example, shows that a CRA when the spread is x in a two-sided zero-threshold agreement is the same as the net credit adjustment, CVA−DVA, when no collateral is posted and the credit spreads of each side (adjusted if necessary for differences between the recovery rates on bonds and derivatives) is x.

In practice, when collateral is posted, CVA and DVA are much smaller than in the no-collateral case. As a result, CRA is likely to be considerably larger than the net credit adjustments in the situation we have considered. Furthermore, as new regulations are implemented collateral requirements are likely to become considerably greater than those given by a two-sided zero-threshold agreement. This will also have the effect of increasing CRA and decreasing the net credit adjustment still further.

In general, CRA, like CVA and DVA, must be valued in the same way as a derivative. Monte Carlo simulations must be used to calculate the present value of the expected difference between the interest rate cash flows applicable to the collateral and those that would be applicable to the collateral if the risk-free rate was paid and received. When securities can be posted instead of cash, the extent to which this will happen needs to be estimated.[7]

CRA differs from CVA and DVA in a number of ways. CRA depends on a rate that is contractually defined while CVA and DVA depend on the credit spreads of the two parties and other market variables. When the credit spreads or other market variables change, the CVA and DVA changes result in reportable income. If the rate paid on CRA is the risk-free rate plus a

[7] As indicated earlier, dealers are likely to post cash or securities depending on whether s is greater than or less than zero. Non-financial end users are more likely to post cash collateral in all situations.

contractual spread there will be no income effects as a result of changes in market variables. Income effects only occur is the rate paid on collateral is specified as a fixed rate.

Another important difference is that CRA is additive across transactions and can therefore be calculated on a transaction-by-transaction basis. Defaults have a very small effect on CRA and can safely be ignored when it is calculated. Because of the impact of netting agreements, CVA and DVA are in general not additive across transactions and must be calculated on a portfolio-by-portfolio basis.[8] Furthermore, collateral arrangements cannot be ignored when CVA and DVA are calculated. Indeed, it is crucially necessary to estimate for each simulation trial the collateral that will be available at each of the default times considered.

The net value of the derivative portfolio is

$$f\left(S_t, t\right) = f_{\text{nd}} - \text{CVA} + \text{DVA} - \text{CRA}$$

The fair market value of a new transaction with a counterparty is the incremental impact it will have on $f\left(S_t, t\right)$.[9]

5. Funding Value Adjustment

In addition to CVA, DVA, and CRA, many banks make a funding value adjustment (FVA) to reflect the difference between their funding costs and the risk-free rate. This adjustment has no theoretical validity. Indeed it is well known in corporate finance that funding decisions should be kept separate from investment decisions.[10] The return required on an investment contemplated by a company should reflect the investment's risk not the company's average funding costs. It is

[8] An exception here is that the net credit adjustment (CVA−DVA) is additive when $r_C^* = r_B^*$ and collateral agreements are symmetrical.

[9] This can be calculated efficiently if all Monte Carlo samples for the last calculation of CVA and DVA are stored because it is then only necessary to simulate the new proposed transaction for the scenarios that were considered.

[10] An exception is that the favorable tax treatment of debt in many jurisdictions can create an interaction between funding and investment decisions. See Myers (1974).

therefore correct to set the return on a riskless portfolio equal to the risk-free rate as Black-Scholes-Merton did.

Articles by Burgard and Kjaer (2011, 2012) and Hull and White (2012b) show that it is not appropriate to make FVA adjustments when determining the value of derivatives. The Burgard and Kjaer papers focus on the mechanics of hedging. Hull and White have a more traditional finance valuation perspective. Both sets of authors argue against a funding value adjustment. However their conclusions are slightly different. Burgard and Kjaer's analysis leads them to conclude that an adjustment may be necessary if a bank cannot hedge its own risk. By contrast, Hull and White argue that the fair valuation of derivative products should be related to the economic properties of the contracts and not be affected by whether the underlying risks can be hedged. Section 3.2 discusses this disagreement and lends support to the Hull and White position.

One feature of FVA adjustments that is disturbing is that it is an unsymmetrical adjustment which may lead to a market breakdown. We define the valuation of a derivatives contract between two parties, A and B, as "symmetric" if the value of the portfolio to A is equal and opposite to the value of the portfolio to B. If they agree on the values of market variables and on the appropriate valuation models, the no-default valuation of a derivative (or derivatives portfolio) is symmetric because A's cash flows are the mirror image of B's cash flows and are valued in the same way. CVA and DVA do not interfere with the symmetric property because A's CVA is B's DVA and vice-versa. Because the collateral paid (received) by A equals the collateral received (paid) by B, A's CRA will always be equal and opposite to B's CRA so that the CRA adjustment does not alter the symmetric property. However, FVA does not have the symmetric property. If A's funding cost is different from B's, their FVA adjustments will not be equal and opposite which means they may not be able to agree on a price.

In spite of the arguments showing that the practice of making funding value adjustments has no theoretical basis, the practice persists. The accounting profession now requires banks to make CVA and DVA adjustments to the reported values of their derivatives portfolios.[11] Whether

[11] See Financial Accounting Standards Board (2006, 2007).

accounting standards will allow FVA adjustments remains to be seen. One possible outcome is that FVA adjustments will not be allowed for reporting purposes because they do not reflect the economic “fair value” of a portfolio, but will continue to be used for internal decision making. FVA then becomes part of a private valuation that affects a bank’s appetite for the transaction and is similar to many other considerations such as the nature of transactions involving the same underlying asset that are already on the bank’s books, liquidity considerations, regulatory capital constraints, the bank’s ability to hedge the underlying risks, and so on.

6. Bid−Offer Spreads

In the previous sections we have presented the arguments that lead to the determination of the fair market value of a derivative where the fair market value is the price at which an investment in the derivative results in an expected rate of return on investment that is commensurate with the risk of the investment. However, a financial institution that is buying or selling a derivative as a service to a customer usually wants to earn a rate of return that is greater than the fair return in order to compensate for the service.

We first consider the case where there is no credit risk. The derivation based on the CAPM, equation (2) has to be modified to incorporate the excess return that is to be earned on the derivative to give

$$\mu_f = \alpha + r + \beta_f\left(\mu_M - r\right) = \alpha + r + \frac{S}{f}\frac{\partial f}{\partial S}\left(\mu - r\right) \tag{15}$$

where α is the desired excess rate of return. When equation (15) is used, in the absence of counterparty credit risk the resulting differential equation that the price of the option must satisfy is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = \left(r + \alpha\right) f$$

For a European style derivative the solution to this differential equation is

$$f\left(S_t,t\right)=e^{-(r+\alpha)(T-t)}E_r\left[f\left(S_T,T\right)\right]$$

If the bank buys the derivative from the customer (the derivative is an asset for the bank), the bank would like to earn a rate of return that is greater than the fair rate so that $\alpha > 0$. In this case, the value of the derivative is lower than the fair market value, i.e., the bank pays the customer a price that is less than the fair market value. If the bank sells the derivative to the customer (the derivative is a liability for the bank), the derivative is acting as a source of funding for the bank and the bank would like to pay a rate of return on its funding that is less than the fair rate so that $\alpha < 0$. In this case, the value of the derivative is greater than the fair market value, i.e., the customer pays the bank a price that is more than the fair market value. These adjustments to the price result in a bid-offer spread.

If the derivative may be either an asset or a liability to the bank, the bank will want $\alpha = \alpha^+ > 0$ whenever it is an asset and $\alpha = \alpha^- < 0$ whenever it is a liability. In this case the differential equation for the asset price is

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=\left(r+\alpha^+\right)\max\left(f,0\right)+\left(r+\alpha^-\right)\min\left(f,0\right)$$

This must be solved numerically.

These results can also be derived using the Merton hedging argument. In this case equation (6) has to be modified to

$$d\Pi=\left[-\left(r+\alpha\right)f+rS\frac{\partial f}{\partial S}\right]dt$$

That is, the option portion of the hedged portfolio is required to earn a rate of return that is different from the riskless rate while the stock portion still earns the riskless rate.

This analysis can be extended to incorporate credit risk and collateralization.

7. Conclusions

We have shown how the arguments used to produce the no-default value of a derivative can be extended to incorporate collateral agreements as well as credit risk. We have also made the point that funding value adjustments, although they are popular among practitioners, are not appropriate when the fair market value is calculated.

The results have been presented in the context of a single derivative dependent on a single underlying asset providing no income. The underlying asset is assumed to follow a lognormal diffusion process. The results can be generalized. First, they can be extended to apply to a single derivative or a portfolio of derivatives dependent on many underlying assets.[12] Second, alternative processes can be assumed for the underlying asset. Third, multifactor extensions of the continuous time CAPM can be used. A point that it is sometimes overlooked in practice is that is not necessary for a market participant to be able to hedge all the underlying risks. Valuations produced on the assumption that all risks can be hedged are the same as those produced using equilibrium models.

New regulations can be expected to reduce, but not eliminate, CVA and DVA. Basel III regulations require banks to hold market risk capital for the credit spread risks underlying CVA.[13] This is likely to mean that banks continue to devote resources to improving their estimates of CVA (and DVA). However, the management of collateral and the calculation of CRA is likely to assume an increased importance as new regulations increase collateral requirements. The impact on derivatives valuation of situations where the interest paid on cash collateral is different from the risk-free rate can be as large as the impact of CVA and DVA. One simplifying feature of CRA is that, unlike CVA and DVA, it is additive and can be calculated on a transaction-by-transaction basis.

The credit crisis of 2008 has caused derivatives practitioners and their regulators to reconsider many of the ways derivatives are valued and managed. For example, it is now becoming

[12] In practice, netting agreements mean that all derivatives transactions between two parties are treated as a single transaction for the calculation of CVA and DVA.

[13] See Basel Committee on Banking Supervision (2010).

recognized that LIBOR is not the best proxy to use for the risk-free rate when derivatives are valued. Counterparty credit risk and collateral issues in the bilaterally cleared over-the-counter market are receiving more attention and contractual arrangements are changing. This paper shows that the economic arguments underlying derivatives valuation can be extended to incorporate these changing contractual arrangements.

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