# THE IRONY IN THE DERIVATIVES DISCOUNTING PART II: THE CRISIS

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Abstract. Libor derivative pricing has changed with the crisis; Libor is not anymore one unambiguous curve as a large basis has appeared between different Libor tenors. A previous approach to derivative discounting is reviewed at the light of those changes. The valuation of so called linear derivatives, the yield curve construction and the valuation of vanilla options is analyzed.


## 1. Introduction

In the article Henrard (2007) we asked how the cash-flows in Libor[1] derivatives should be discounted. The question was then viewed as a trivial and unquestionable matter. Moreover asking such a question was non-politically correct as the derivatives values enter in the market-to-market accounting of banks and the matter had legal implication. Ironically the article, entitled *The irony in derivative discounting*, was published in July 2007 just one month before the Libor/OIS spread (and other spreads) went wild.

> In mathematics, the art of posing a question is more important than the art of solving one.
>
> *Georg Cantor, 1867*

In the introduction of the above mentioned article we quoted Cantor, a quote repeated here, and expressed our belief that the question asked then was important and our hope it was well posed. After the crisis trigger, the importance of correct curve selection appears even more clearly. Unfortunately it appears also that, against our hope, our question was not perfectly posed. We did no imagine that Libor would become an ambiguous notion with large basis spreads between tenors. The present article, written as the part II of derivatives discounting, proposes a solution to these new challenges.

The importance of the discounting question is attested by the numerous recent related literature (e.g. Kijima et al. (2009), Ametrano and Bianchetti (2009), Chibane and Sheldon (2009), Bianchetti (2009), Mercurio (2009), Morini (2009)) and the efforts in most banks to adjust their systems to the new reality. Some research explain the reason the derivatives based on Libor with different tenors should be priced differently through credit risk analysis like in Morini (2009). Here we will not try to explain the reason behind the differences and take them as a starting point. We try to propose a coherent valuation framework for the derivatives based on different Libor tenors. Those frameworks have received several names: *two curves, one price*, *derivative tenor curves*, *discounting-estimation*, *discounting-forecast* or *discounting-forward*. To our knowledge no theoretical fundamental simple analysis of the framework has been proposed yet; this is what we try to achieve here.

In that new context all the standard approaches to derivatives pricing and in particular the valuation of so called linear products like interest rate swaps (IRS), forward rate agreements (FRA), OIS and futures need to be rethought

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In memoriam Jacques Henrard, who gave me a taste for questions, specially the non-politically correct ones.

[1]In this note we use the term *Libor* as a generic term for all rates fixing with a similar rule and in particular include under that term the *Euribor* (EUR) and *Tibor* (JPY).

carefully. Some extra assumptions on the relation between the different curves need to be added. Even the notions of yield curves and discount factors need to be handled carefully.

Our framework is based on a unique curve used to discount all the cash-flows related to the derivatives, whatever the tenor they are related to. Here we are interested mainly in derivatives, but obviously all cash-flows (from derivatives, bonds, deposits, etc.) should be treated in the same way. In the first article we called that curve the *funding curve*, in reference to the need for derivative desks to fund themselves to obtain cash. The most used name today, and probably a better choice, is the *discounting curve*. We will also use the latter name in the sequel.

The choice of that discounting curve is by itself an open question. Different people will choose different curves. One possibility is to relate it to the rate that can be achieved for the remaining cash of the desk. For that reason the OIS curve is often selected, as the funding/investment is often done on an overnight basis. In time of liquidity shortage, this may be a low rate, not taking into account the real funding cost. A curve based on Libor may be more reasonable; in that case one should select one (and only one) of the Libor tenors. The impact of the discounting curve on IRS is not major (often less than 10% of the delta). What is certainly important is to have a coherent curve with no jump from one choice to another.

In this article we suppose that the discounting curve is given. Our job will start with the construction of the curves used to *estimate* the the Libor's (in a sense to be explained latter). Those curves are often called *estimation*, *forecast* or *forward curves*. Here the plural should be used as there is a different curve for each Libor tenor. In practice the most frequently used tenors are one, three, six and twelve months; in theory one could have a curve for each tenor (two, four, etc months). For those curves we will also use the terms *yield*, *discount factor* and *forward rate*, even if those terms may have a different meaning in this context. In particular they will never be used for discounting[2].

In the next section we set the main hypothesis used for the framework. Then we price the simple derivatives (IRS, FRA, futures). The term simple has to be understood with a pinch of salt. In this framework a FRA is really a contingent claim and a curve hypothesis is needed to price it. Of course the futures require a so called convexity adjustment and thus a model.

The framework could be extended to value cross currency products in a coherent multi-currency framework. In that case the discounting curves in the different currencies should be linked through cross-currency basis. This is not elaborated here.

Hopefully this time our questions are well posed. To justify our attempt on asking similar question again, we quote A. Einstein

> The important thing is not to stop questioning.
>
> *Albert Einstein*

## 2. Estimation hypothesis and linear products

As hinted by the title, we will consider the *discounting* of cash-flows. Our first hypothesis relates to it; its name **D** refers to it.

> **D:** The instrument paying one unit in $u$ is an asset for each $u$. It's value in $t$ is denoted $P^D(t,u)$.

With this curve we are able to price fixed cash-flows.

Our goal is to price Libor related derivatives and in particular IRS. Thus we need hypothesis saying at least that those instruments exist in our framework. We call a ($j$-Libor) floating coupon an instrument that pays in $t_2$ the Libor fixing for the tenor $j$ on the period $[t_1, t_2]$ as fixed in $t_0$ ($0 \leq t_0 \leq t_1$) multiplied by the conventional year fraction. The lag between $t_0$ and $t_1$ is called the spot lag and is usually two business days[3]. The difference between $t_2$ and $t_1$ is $j$ months (in the appropriate convention).

As the month addition $t + j$ months will be often used we will use the notation abuse $t + j$ for that date.

> **L:** The value of a (Libor) floating coupon is an asset for each tenor and each fixing date.

[2]Except for FRA where the discounting using the Libor fixing rate is part of the contract description.
[3]This is the case in EUR and USD. In GBP the lag is 0 day.

This hypothesis is implicit in most of the above quoted papers. We prefer to state it explicitly as this is not the consequence of the existence of the discounting curve.

In this article all asset prices are *continuous in time*.

Once we have assumed that the instrument is an asset, we can give a name to its price. We do it indirectly through a curve $P^j$.

**Definition 1.** *The* forward curve $P^j$ *is the continuous function such that,* $P^j(t,t) = 1$, $P^j(t,T)$ *is an arbitrary function for* $t < T < \text{Spot}(t) + j$ *and, for* $T \geq \text{Spot}(t)$,

$$P^D(t, T+j)\left(\frac{P^j(t,T)}{P^j(t,T+j)} - 1\right) \quad (1)$$

*is the price in* $t$ *of the floating coupon with start date* $T$ *and maturity date* $T + j$.

The reason behind the definition is to obtain the usual formulas involving forward rate computation and discounting. The same terms *discounting curve* and *forward rate* will still be used even if they represent different realities.

> La mathématique est l'art de donner le même nom à des choses différentes.
>
> *Henri Poincaré.*

Note that the definition itself is arbitrary. One could fix any $j$ month period (not only the first one) and deduce the rest of the curve from there. Or one could even take an arbitrary decomposition of the $j$ months interval in sub-intervals and distribute those sub-intervals arbitrarily on the real axis in such a way that, modulo $j$ months, they recompose the initial $j$ months period. One could also change the value of $P^j(t,t)$.

We will come back to the curve construction in the next section. Note that our definition of estimation curve is similar to the one of Chibane and Sheldon (2009) but different from the one of Kijima et al. (2009), Ametrano and Bianchetti (2009), Mercurio (2009) and Bianchetti (2009).

In Mercurio (2009) the curve definition is different from our but he barely uses those curves. He defines market models on quantities equivalent to our $F^j$ defined below. In some sense the quantities $L_j$ defined in his Remark 1 are discretized versions of our curves $P^j$.

We insist that the formula above on $P^j$ is a definition. We will relate those numbers to some financial intuition later, but for the moment $P^j$ is simply a function. If the floating coupon has a price, such a function exists for each tenor $j$. This curve is not unique but once the first $j$ months period is fixed, $P^j(t,T)$ is well defined and unique. At this stage the only link between the curve and a market rate is that the Libor rate fixing in $t_0$ for the period $j$ is

$$L^j_{t_0} = \frac{1}{\delta}\left(\frac{P^j(t_0, \text{Spot}(t_0))}{P^j(t_0, \text{Spot}(t_0)+j)} - 1\right) \quad (2)$$

where $\delta$ is the year fraction. To obtain this equality the price continuity was used.

### 2.1. Interest Rate Swap.

With that hypothesis, the computation of vanilla interest rate swap (IRS) prices is straightforward. The hypothesis was selected for that reason. An IRS is described by a set of fixed coupons or cash-flows $c_i$ at dates $\tilde{t}_i$ $(1 \leq i \leq \tilde{n})$. For those flows, the discounting is used. It also contains a set of floating coupons or cash-flows over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ $(1 \leq i \leq n)$[4]. The value of a (fixed rate) receiver IRS is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i)\left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1\right). \quad (3)$$

In the one curve pricing approach, the IRS are usually priced through either the *Libor forward* approach or the *cash-flow equivalent* approach. The *Libor forward* approach consists in estimating the forward Libor rate from the discount factors and discounting the result from payment date to today. To keep that intuition, we define the Libor forward rate in our framework as the figure we have to use to keep the same formula.

**Definition 2.** *The* Libor forward *rate over the period* $[t_{i-1}, t_i]$ *is given at time* $t$ *by*

$$F^j_t(t_{i-1}, t_i) = \frac{1}{\delta_i}\left(\frac{P^j(t, t_{i-1})}{P^j(t, t_i)} - 1\right). \quad (4)$$

With that definition the IRS price is

$$\sum_{i=1}^{\tilde{n}} c_i P^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P^D(t, t_i)\delta_i F^j_t(t_{i-1}, t_i).$$

[4] In practice, due to week-ends and holidays, the periods used for the fixings can be slightly different from the payment dates. We will not make that distinction here.

Note the fundamental difference between $L_t^j$ and $F_t^j$. The object $L$ is, by hypothesis **L**, a fundamental element of our economy; the second one is defined by the above equality. Note also that our definitions of $L$ and $F$ are opposite to Mercurio (2009); our $F_t^j(t_{i-1}, t_i)$ is equivalent to his $L_i^j(t)$ and our $L_{t_0}^j$ corresponds to his $F^j$. The definitions of $F$ and $L$ coincide for $t_{i-1} = \text{Spot}(t_0)$:

$$L_{t_0}^j = F_{t_0}^j(\text{Spot}(t_0), \text{Spot}(t_0) + j).$$

The *cash-flow equivalent* approach consists in replacing the (receiving) floating leg by receiving the notional at the period start and paying the notional at the period end. We would like to have a similar result in our new framework. To this end we define

$$\beta_t^j(u, u+j) = \frac{P^j(t,u)}{P^j(t,u+j)} \frac{P^D(t,u+j)}{P^D(t,u)}. \quad (5)$$

With that definition, a floating coupon price is

$$\begin{aligned} & P^D(t,t_i)\left(\frac{P^j(t,t_{i-1})}{P^j(t,t_i)} - 1\right) \\ &= P^D(t,t_i)\left(\beta_t^j(t_{i-1},t_i)\frac{P^D(t,t_{i-1})}{P^D(t,t_i)} - 1\right) \\ &= \beta_t^j(t_{i-1},t_i)P^D(t,t_{i-1}) - P^D(t,t_i). \end{aligned}$$

This last value is equal to the value of receiving $\beta_t^j$ notional at the period start and paying the notional at the period end.

If the forward discounting rate $F_t^D(t_{i-1}, t_i)$ is defined in the standard way, the floating coupon price can also be written as

$$P^D(t,t_i)\delta_i\left(\beta_t^j F_t^D(t_{i-1},t_i) + \frac{1}{\delta_i}(\beta_t^j - 1)\right)$$

which is the formula proposed in Henrard (2007).

A consequence of the hypothesis **L** and the definition of $\beta_t^j$ is that $\beta$ is a martingale in the $P^D(., t_{i-1})$ numeraire. The Libor coupon value is $\beta_t^j(t_{i-1},t_i)P^D(t,t_{i-1}) - P^D(t,t_i)$. The coupon is an asset due to **L** and so its value divided by the numeraire $P^D(t,t_{i-1})$ is a martingale. The second term is also an asset, hence its rebased value is also a martingale. The last term is thus also a martingale and its value is $\beta_t^j P^D(t,t_{i-1})/P^D(t,t_{i-1}) = \beta_t^j$. This proves that $\beta_t^j$ is a martingale under the $P^D(., t_{i-1})$-measure.

Like in the one curve framework, we can define a forward swap rate. This is the rate for which the vanilla IRS price is 0:

$$S_t^j = \frac{\sum_{i=1}^n \delta_i F_t^j(t_{i-1},t_i)P^D(t,t_i)}{\sum_{i=1}^{\tilde{n}} \tilde{\delta}_i P^D(t,\tilde{t}_i)}.$$

2.2. **Forward Rate Agreement.** A Forward Rate Agreement (FRA) is an instrument linked to a $j$-month period, a fixing date $t_0$ and a fixed rate $K$. At the fixing date $t_0$, the Libor rate $L_{t_0}^j$ is recorded. The contractual payment in $t_1 = \text{Spot}(t_0)$ (the start date) is

$$\frac{\delta(L_{t_0}^j - K)}{1 + \delta L_{t_0}^j}.$$

The origin of the formula is the difference between the Libor fixing and the fixed rate discounted at the Libor rate. The rate is not paid at the end of its period but at the start and is discounted by itself; it is not directly a floating coupon as we defined it above. In that sense, our definition of a FRA is in line with real FRA term sheet and different from the one of Ametrano and Bianchetti (2009), Bianchetti (2009), Chibane and Sheldon (2009) and Mercurio (2009). We will need a new hypothesis that indicates that this instrument is an asset in our economy. In a general contingent claim formula, its price would be

$$N_0 \, \mathrm{E}\left[N_{t_1}^{-1}\frac{\delta(L_{t_0}^j - K)}{1 + \delta L_{t_0}^j}\right].$$

Here we can not use the usual trick of postponing the payment to $t_2$ by multiplying by $1+\delta L_{t_0}^j$ and selecting $P^j(., t_2)$ as the numeraire to simplify the formula. The reason is that an investment is not done at the libor rate but at the discounting rate. Note that $F_t^j$ is the ratio of an asset (floating coupon) and $P^D(., t_i)$. In the $P^D(., t_i)$ numeraire measure, $F_t^j$ is a martingale.

Note that $P^j$ can not be an asset. If it was the case we would have, for any numeraire $N$,

$$\begin{aligned} P^D(0,t) &= N_0 \, \mathrm{E}\left[N_t^{-1}P^D(t,t)\right] \\ &= N_0 \, \mathrm{E}\left[N_t^{-1}1\right] \\ &= N_0 \, \mathrm{E}\left[N_t^{-1}P^j(t,t)\right] = P^j(0,t). \end{aligned}$$

The two curves $P^D$ and $P^j$ would be identical, which is a contradiction to our stated goal of decoupling them.

We need extra assumptions. The goal of this article is to obtain a relatively simple coherent and practical approach to Libor derivatives pricing.

Our next hypothesis is that the spread between the curves, as defined through the quantity $\beta_t^j$, is independent of the curves.

**SI:** The multiplicative coefficient between discount factor ratios, $\beta_t^j(u, u+j)$, defined in Equation (5) is independent of the ratio $P^D(t,u)/P^D(t,u+j)$.

For the pricing of options, we will impose a stricter hypopthesis.

**S0:** The multiplicative coefficient between discount factor ratios, $\beta_t^j(u, u+j)$, defined in Equation (5) constant through time: $\beta_t^j(u,v) = \beta_0^j(u,v)$ for all $t$ and $u$.

With the independence hypothesis, the adjustment $\beta_j^t$ is a martingale for any $P^D(.,u)$-measure or for the cash account measure. The change of numeraire on the independent variable does not affect the martingale property of $\beta_t^j$.

The hypothesis **S0** can be viewed as the equivalent of the constant continuously compounded spread used in Henrard (2007). Here the spread is not constant across maturities but deterministic and given by its initial values. The spread is $\ln(\beta^j(u,v))/(u-v)$. In that sense this framework is a direct extension of the one developed in the previous work adapted to the current market situation where the spreads are not equal for all maturities.

The hypothesis **S0** is equivalent[5] to the hypothesis that $\beta_t^j$ is deterministic. The equivalence can be obtained easily through a martingale argument.

We have

$$1+\delta L_{t_0}^j(t_1,t_2) = \frac{P^j(t_0,t_1)}{P^j(t_0,t_2)} = \beta^j(t_1,t_2)\frac{P^D(t_0,t_1)}{P^D(t_0,t_2)}.$$

**Theorem 1.** *Under hypothesis* ***D****,* ***L*** *and* ***SI*** *the price in $t$ of the $j$ months FRA with fixing date $t_0$, and rate $K$ is*

$$P^D(t,t_1)\frac{\delta(F_t^j - K)}{1+\delta F_t^j} = P^D(t,t_2)\frac{\delta(F_t^j - K)}{\beta^j(t_1,t_2)}.$$

*where* $t_1 = \text{Spot}(t_0)$ *and* $t_2 = \text{Spot}(t_0) + j$.

The result can be obtained trough simple manipulation of the above coefficients and the independent hypothesis on $\beta$.

The above result relies on hypothesis **SI**. Even if the floating coupon value is given in **L** this is not enough to price the FRA; an extra hypotheses is necessary. The FRA discounting with the Libor rate between start date and end date creates an adjustment represented by the coefficient $\beta^j$.

Note that, as already mentioned in Henrard (2007), the IRS is not anymore a portfolio of FRA with same notional.

The above formula is used by some systems decoupling discounting and forward curves. The hypothesis **SI** justifying such a formula is seldom provided.

A FRA is at-the-money when the rate $K$ is such that the instrument value is 0. Using the above result, it is the case when

$$K = F_0^j = \frac{1}{\delta}\left(\beta^j(t_1,t_2)\frac{P^D(0,t_1)}{P^D(0,t_2)} - 1\right).$$

This is essentially the FRA fair rate obtained in Mercurio (2009) with

$$\beta^j = \frac{1}{R + (1-R)\,\mathrm{E}[Q(t_1,t_2)]}.$$

The adjustment factors $\beta^j$ can be linked to credit related parameters as recovery rate and default probability. It is also linked to the quantity $K_L(t,t_1,t_2)$ defined in (Kijima et al., 2009, Equation (4.7)). In their article the spread is linked to some model parameters, in our case it is fitted to the market curves.

2.3. **Libor futures.** A general pricing formula for eurodollar futures in the Gaussian HJM model was proposed in Henrard (2005a). The formula extended a previous result proposed in Kirikos and Novak (1997). The formula was briefly extended to the discounting framework in Henrard (2007). We now study the instrument under our new hypothesis.

The futures are liquid only for the three month Libor up to two or three year. To a lesser extend some one month futures are available on the shorter part of the curve.

The future fixing date is denoted $t_0$. The fixing is on the Libor rate between $t_1 = \text{Spot}(t_0)$ and $t_2 = t_1 + j$. The accrual factor for the period $[t_1,t_2]$ is $\delta$, the fixing is linked to the yield curve by

$$1+\delta L_{t_0}^j = \frac{P^j(t_0,t_1)}{P^j(t_0,t_2)}.$$

[5]Another possible hypothesis proposed by Tanaka (2009) would be that the ratio $P^j(t,u)/P^D(t,u)$ is deterministic. This hypothesis implies the one we propose. That ratio can not be constant (it tends to one as $t$ tends to $u$) We prefer to deal with constant quantities.

The futures price is $\Phi^j_t$. On the fixing date, the relation between the price and the rate is

$$\Phi^j_{t_0} = 1 - L^j_{t_0}.$$

The futures margining is done on the futures price (multiplied by the notional and divided by 4).

The exact notation for the HJM one-factor model used here is the one of Henrard (2005a).

**Theorem 2.** *Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the HJM one-factor model on the discount curve under the hypotheses $\boldsymbol{D}$, $\boldsymbol{L}$ and $\boldsymbol{SI}$, the price of the futures fixing on $t_0$ for the period $[t_1, t_2]$ with accrual factor $\delta$ is given by*

$$\begin{aligned}(6)\qquad \Phi^j_t &= 1 - \frac{1}{\delta}\left(\frac{P^j(t,t_1)}{P^j(t,t_2)}\gamma(t) - 1\right)\\ &= 1 - \gamma(t)F^j_t + \frac{1}{\delta}(1-\gamma(t))\end{aligned}$$

*where*

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s,t_2)(\nu(s,t_2)-\nu(s,t_1))ds\right).$$

*Proof.* Using the generic pricing future price process theorem (Hunt and Kennedy, 2004, Theorem 12.6),

$$\Phi^j_t = \mathrm{E}_{\mathbb{N}}\left[1 - L^j_{t_0}\middle| \mathcal{F}_t\right].$$

In $L^j_{t_0}$, the only non-constant part is the ratio of $j$-discount factors which is, up to $\beta^j_{t_0}$ the ratio of $D$-discount factors. Using (Henrard, 2005a, Lemma 1) twice, we obtain

$$\begin{aligned}&\frac{P^D(t_0,t_1)}{P^D(t_0,t_2)}\\ &\quad= \frac{P^D(t,t_1)}{P^D(t,t_2)}\exp\left(-\frac{1}{2}\int_t^{t_0}\nu^2(s,t_1)-\nu^2(s,t_2)ds\right.\\ &\qquad\left.+\int_t^{t_0}\nu(s,t_1)-\nu(s,t_2)dW_s\right).\end{aligned}$$

Only the second integral contains a stochastic part. This integral is normally distributed with variance $\int_t^{t_0}(\nu(s,t_1)-\nu(s,t_2))^2ds$. So the expected discount factors ratio value is reduced to

$$\begin{aligned}\frac{P^D(t,t_1)}{P^D(t,t_2)}\quad &\exp\left(-\frac{1}{2}\int_t^{t_0}\nu^2(s,t_1)-\nu^2(s,t_2)ds\right.\\ &\qquad\left.+\int_t^{t_0}(\nu(s,t_1)-\nu(s,t_2))^2ds.\right)\end{aligned}$$

The coefficient $\beta^j_{t_0}$ is independent and a martingale, hence we have the announced result. □

## 3. Curve construction

Our goal in this section is to construct the curves $P^j(0,.)$. The three instruments detailed above are the most used to construct the yield curves $P^j(0,.)$. As in the previous section, we suppose that the discounting curve $P^D(0,.)$ is given.

Note that in Mercurio (2009) the curve construction is not discussed and in Kijima et al. (2009) the model used imposes a specific parametrized shape to the spread between curves.

As described in the definition of $P^j$, we take $P^j(0,0) = 1$ and for $P^j(0,t)$ an arbitrary continuous function for $t < \text{Spot}(0) + j$ such that

$$L^j_0 = \frac{1}{\delta}\left(\frac{P^j(0,\text{Spot}(0))}{P^j(0,\text{Spot}(0)+j)} - 1\right)$$

i.e. the curve matches the current fixing. In theory the curve is completely arbitrary on that interval and that curve interval has no impact. In practice, as the theoretically possible dates are not all used in the construction and some interpolation is used, this part may have an impact. This is discussed later.

### 3.1. FRA.

The FRA are less liquid than futures, short swaps and deposits. For that reason they were seldom used in yield curves construction before the crisis. The reason they come back to fashion now is that they are the only way to obtain information on the short part of the yield curve for tenors different than three months, the latter being obtained by futures.

A set of FRA of the given tenor $j$ is selected. For $j = 1$ one often prefers to use directly short swaps and no FRA is used. For $j = 3$, only one or two FRA's are used, up to the first future. For $j = 6$ one would take the FRA's 1x7, 2x8, . . . 12x18. For $j = 12$ one would use the FRA's 1x13, 2x14, . . . 12x24.

The FRA are sorted in an increasing maturity order. Their start and end dates are $s_i$ and $e_i$. For each of those FRA's, the market quoted fair rates $K_i$ for which the FRA has a zero market value is known. Using the result on the valuation of such an instrument, the value is 0 when $K_i = F_0^j(s_i, e_i)$. The equality gives the value of $P^j(0, e_i)$ based on the known value $P^j(0, s_i)$.

Note that for the curve construction purpose, the FRA can be consider as a floating coupon. This is only true because the FRA value is 0 and the discounting has no impact.

3.2. **Futures.** The futures are used only for $j = 3$. They could be used also for $j = 1$ where some short futures are quoted. There is usually less liquidity on those instruments and the existence of short swaps make them less useful.

The mechanism is similar to the one for FRA's. A set of futures with increasing maturity is selected. The HJM model parameters are supposed to be obtained (calibrated) already. The market price of the future $\Phi_0^j$ gives the value of $F_0^j$ through Equation (6), this allows to obtain the curve $P^j$ up to the future maturity like in the FRA case.

3.3. **IRS.** For longer maturities IRS are used. The IRS are usually quoted directly with only one tenor. To obtain the IRS with other tenors one has to use *basis swaps*, swap that exchange floating coupons in one tenor against floating coupon with another tenor. In EUR the basis swap are often quoted for a pair of IRS fixed vs floating with different fixed rates. The spread being the difference between the fixed rates. The IRS's are sorted in increasing maturities and the curve is obtained up to their maturity in such a way that the value of the swap with market coupon as given by Equation (3) is 0. Interpolation is used when necessary.

3.4. **The arbitrary curve.** As mentioned above, the curve up to the first tenor is arbitrary. The intuition behind it is that the *forward curve* is used only to compute forwards and thus only the ratio between two values of $P^j$ is important, never a single value. This is true only to a certain extend; not all the points of the curve are constructed directly, some are interpolated.

Consider as an example the curve with $j = 3$. Suppose you construct the curve with an arbitrary curve up to 3 months and then the curve up to two years with futures (this is an approach often used). Here are two cases that you could encounter if you are *too arbitrary* on the selection. Take the case were there is one month to the start of the next future. Take an arbitrary low spot rate and a one month forward rate very high. In that case, the curve is constructed with the 3 month futures from that high rate and all the rates are also high. This is not a problem for your futures, they are all well priced as the ratios of discount factors on those points are correct. How is a one year versus 3 month IRS priced in that context? The one year rate is interpolated between the 10 months and 13 month rates coming from the future, both of them too high. So the 12 month rate used for the forward is too high, the spot rate is too low and your one year rate is too high. Even if all futures are correctly priced, the swap which is interpolated is not. Even if one adds the one year swap in the curve construction one could create a similar problem. Take now the same initial arbitrary curve with a very low two months rate. Use the futures and the swap to construct the curve and now try to price a forward swap starting in 2 months and with one year maturity; this swap could be used as underlying of a swaption. It is not too difficult to see that the forward rate would also be overestimated. For that reason we suggest to use financially meaningful numbers for the arbitrary values, for example the rate obtained from shorter fixing. In our case, the three month curve would be constructed with one and two month Libor deposits to establish the arbitrary part.

The arbitrariness of $P^j(0, t)$ over the first period is not broached in Chibane and Sheldon (2009) and Mercurio (2009).

## 4. Delta

Once a curve is constructed and financial instruments are priced with it, the next step is to compute the risks associated, the first of which is the *delta's*. By this we mean the change of value of a financial instrument when the rate used to construct the curve is moved by a small amount (usually one basis point).

In the discounting context, to parody the titles of Henrard (2005b) and Bianchetti (2009), the title of this section could be: *One price, two* *curves and three delta's*. Due to the two curves, there are two sets of financial instruments playing a role in the curve construction and as such two deltas. But when one looks more in details there are three ways the rates act. The first one is the direct influence of the discounting rate on the price, the second one is the market instruments rates used in the forward curve and the third one is the indirect impact of the change of the discounting curve on the forward rate construction. The forward curve is constructed using as input different instruments and the discounting curve. The three impacts can be evaluated separately.

For the simple instruments mentioned above, the impact of the discounting curve is relatively small. The total impact is null for at-the-money swaps.

In the standard one curve approach the curve is usually only constructed with deposits and swaps. When it is constructed also with futures, the information is often translated into standard tenor deposits and swaps rates (by interpolation) to obtain a standardized curve construction from which a standardized delta can be obtained. Here the situation is different; by definition, there is no *deposit forward rate*. One is using FRA, but 3M FRA and 6M FRA rate sensitivities have not the same meaning and can not be added. A standardized delta can not be obtained through FRA's and futures. A way around the problem is to create *artificial deposits for the forward curve*. Those terms are clearly antinomic. What we mean by that is to create artificial market-like rates that applied to formulas similar to deposit formulas create a curve with the same discount factors as the forward curve. From the output curve $P^j$ one creates artificial standardized input instruments. In this way the deltas computed are somehow compatible.

Even inside one of those curves, the compatibility is not certain. Take for example the 3 months curve. The 6 months point could be represented by an (artificial) 6 months deposit or by an (artificial) 6 months swap versus 3 month Libor. If the market rates of the two instruments are well selected, the curve constructed will be the same. Nevertheless the delta computed with those two approaches will *not be the same*.

Even if the curve construction and deltas definitions are somehow arbitrary, one can obtain some meaningful deltas. The next (non trivial) question is how to use them. In the traditional one curve approach one would use swaps and deposits used to construct the curve in decreasing maturity order to hedge the deltas. An at-the-money instrument of correct notional is entered into in such a way that the exposure of the longest maturity is cancelled. Once this is done one works inductively down to the overnight point to cancel all exposures.

In the two curves approach (or multi-curves if one trades more than one tenor) one has not one but two deltas for each maturity. One can use swaps to cancel the forward curve exposure at the expense of leaving a (small) discounting exposure. The discounting exposure could be cancelled by lending or borrowing some cash to a counterpart. Often derivative desks don't have the authority (nor the willingness) to lend or borrow long term, so the exposures would remain unhedged or compensated by exposure on other curves.

In some cases an *hybrid* approach is used for pricing and delta computation. The (precise) two curves approach described above is used for simple instruments and a less precise one curve approach is used for more complex instruments. The unique curve (that can be one of the forward curves) is used in the traditional way. The level at which the split operates can vary from vanilla swaption to exotic instruments. That hybrid approach is dangerous. The one curve exposure so computed is added to some two curve exposure in such a way that one does not know what the figures actually represent. Such an hybrid approach could lead to cases where the apparent exposure is 0 for all tenors while the actual exposure is globally 0 but there is a large basis position.

Table 1 gives some examples of deltas for swaps and swaptions. The swaptions are discussed in the next section. The delta is split in two for each instrument: the discount curve (both direct and indirect effects) and the forward curve. The swaps studied are annual versus three month Libor 6Mx5Y swaps with coupon between 1 and 5%. As can be seen the split is largely dependent on the deal moneyness. The discounting curve delta is usually between -7% and +7% of the total delta. Larger or smaller coupons would give more extreme splits. Note that the at-the-money swaps (2.91% in our example) have a 0 delta with respect to the discounting curve.

The swaptions used are receiver swaptions; they are priced with an extended Vasicek (Hull-White) model. Their deltas represents a fraction of the swap deltas. As we selected extreme strikes, the fraction is almost 0% on one side and almost 100% on the other. The ratio is not equal between the discounting and the forward parts. A swaption can not be perfectly (delta) hedged with swaps only; some fixed cash-flows (deposits) may be needed. This is the usual duality between *in-the-model* and *out-of-the-model* hedging. As we have a model with one stochastic curve and a deterministic spread, it should be possible to hedge all instruments based on that curve. In the delta computation the curves are moved separately, out of the model. This is how different instruments (swap and swaptions) have non-similar deltas.

| | **Cpn 1%** | | **Cpn 2%** | | **Cpn 3%** | | **Cpn 4%** | | **Cpn 5%** | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Dsc | Fwd | Dsc | Fwd | Dsc | Fwd | Dsc | Fwd | Dsc | Fwd |
| Swap (EUR) | 0.28 | -4.57 | 0.13 | -4.57 | -0.02 | -4.57 | -0.18 | -4.57 | -0.33 | -4.57 |
| Swap rel. | -6.63 | 106.63 | -2.93 | 102.93 | 0.52 | 99.48 | 3.75 | 96.25 | 6.78 | 93.22 |
| Swaption (EUR) | -0.00 | -0.01 | -0.00 | -0.43 | -0.05 | -2.51 | -0.18 | -4.29 | -0.33 | -4.56 |
| Tot swpt/swap | | 0.29 | | 9.67 | | 55.66 | | 94.19 | | 99.86 |
| Swpt/swap | -0.01 | 0.27 | -2.84 | 9.31 | 212.05 | 54.84 | 100.41 | 93.95 | 99.99 | 99.85 |

Figures in %, except when indicated otherwise. Figures in EUR are for a 10,000 EUR notional and a one basis point shift.

TABLE 1. Delta for swaps and swaptions.

## 5. Libor contingent claims - Cash-flow equivalent

In this section a general approach to price Libor related contingent claims is proposed under the hypothesis **S0** of a deterministic spread. The approach was already used for FRA and futures in the previous sections.

To price contingent claims in the discounting–forward framework, we propose to use any standard model on the discount curve. The instruments are based on fixed cash-flows and Libor based cash-flows. For the Libor based cash-flows, the cash-flow equivalent technique described at the end of Section 2.1 will be used.

Let's apply this approach to a swaption. The underlying swap value is given by Equation (3). Using the cash-flow equivalent description, the equation can be replaced by an equation involving only $P^D$:

$$\begin{aligned}\mathrm{IRS} &= \sum_{i=1}^{\tilde{n}} c_i P^D(t,\tilde{t}_i) - \\ &\quad \sum_{i=1}^{n} \left(\beta^j(t_{i-1},t_i)P^D(t,t_{i-1}) - P^D(t,t_i)\right).\end{aligned}$$

Regrouping the terms with same discount factor together and defining $\bar{n}$, $\bar{t}_i$ and $d_i$ appropriately, one has

$$\mathrm{IRS} = \sum_{i=0}^{\bar{n}} d_i P^D(t,\bar{t}_i).$$

In that sense, the swaptions are equivalent to bond options with adjusted coupon and strike price in the discounting only framework. This result is adapted to any model which models the whole term structure and $P^D(.,t_i)$ in particular.

## 6. Other spread hypothesis

Our hypothesis **S0** is one way to link the curves. Other possibilities are presented below.

### 6.1. **Market spread.**

Under this hypothesis, the spread between the market standard tenor swap rate and other forward is known for every fixing. Let $M$ be the *standard market frequency*, the one for which we have market data.

**MR:** The spreads or basis

$$B^j(t_0,t_1) = S^j_{t_0}(t_0,t_1) - S^M_{t_0}(t_0,t_1) \tag{7}$$

are know for every fixing date $t_0$ an every tenor $j$.

Using the martingale property of the swap rate it is easy to verify that a deterministic hypothesis is equivalent to the proposed constant (forward) spread.

If one has a model for $S^M$, the swap rates with other tenors are obtained by a deterministic shift. This is particularly adapted to a Bachelier type model where the volatility would be the same for rates with different tenors. This can be used in a Black-like model under the understanding that if the market rate is log-normal, the others are shifted log-normal.

This model is in practice very close to the one we proposed in the previous sections which can be viewed as an approach with known continuously compounded spreads.

6.2. **Black spread.** The simplest model used to model interest rates derivatives is the Black model on forward rates. The base equation is

$$dS^M_t = \sigma S^M_t dW_t. \tag{8}$$

The volatility $\sigma$ is the one given by the market for the standard tenor and specific expiry, tenor and strike. The similar rate for the forward (non-market convention) curve is supposed to follow a similar equation

$$dS^j_t = \sigma S^j_t dW_t.$$

**B1:** The forward (swap) rate follows a Black equation (between 0 and expiry) with the same Brownian motion than the rate in the market convention.

Under that approach the spread between the rates $S^M - S^j$ is not constant nor deterministic. It is a constant proportion of the rate. The spread grows (and reduces) with the rate:

$$S^j_t = \frac{S^j_0}{S^M_0} S^M_t.$$

With such an hypothesis or a similar on Libor forward rate the pricing of FRA required to construct the curves $P^J$ would be difficult.

Mercurio (2009) proposes such a model for each rate $S^j$ but does not propose a relation between them. This type of model is difficult to calibrate as few options on non market standard tenor are available. If basis spread were to stay large and volatile, such a market may appear.

6.3. **SABR/CEV spread.** The base equations are

$$dS^M_t = \alpha_t (S^M_t)^\beta dW^1_t. \tag{9}$$

$$d\alpha_t = \nu \alpha_t dW^2_t \tag{10}$$

**S1:** The forward swap rate follows a SABR equation (between 0 and expiry) with the same paremeters and Brownian motion than the rate in the market convention.

The result will depend on the $\beta$ parameter, like the computation of the delta in the SABR framework. A $\beta$ close to 1 will give a result close to Black spread and a $\beta$ close to 0 will give a result close to market spread.

In the Black and SABR approaches, the spreads would increase when the rates increase. In the recent crisis, the spreads have increased while the rates were decreasing. This approach would be counterintuitive.

The three approaches mentioned above are convenient only if the objective is only to model one (swap) rate and not the whole term structure. These approaches would not be the most convenient for exotics (Bermuda, callable, etc.).

## 7. Conclusion

A previous article dealt with the concept of discounting for Libor derivatives in presence of constant spread. It is here extended to price Libor derivatives to the current market situation where different Libor tenors imply different swap rates (basis spread) and those spreads are maturity dependent and changing.

Simple hypotheses are introduced to propose a coherent framework. In that framework the value of simple Libor derivatives used to construct yield curves is detailed. The instruments are IRS, FRA and futures.

In a simplified framework a technique to price any Libor contingent claim is sketched. With that technique, the pricing in the multi-curves framework is not technically more difficult than in the one curve approach.

Up to our knowledge this is the first proposal for a coherent and simple method to price Libor derivatives with different reference tenors.

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