THE IRONY IN THE DERIVATIVES DISCOUNTING

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Abstract. A simple and fundamental question in derivatives pricing is the way (contingent) cash-flows should be discounted. As cash can not be invested at Libor the curve is probably not the right discounting curve, even for Libor derivatives. The impact on derivative pricing of changing the discounting curve is discussed. The pricing formulas for vanilla products are revisited in the *funding framework* described.

1. Introduction

This article is about modelling of interest rate derivatives. Nevertheless it does not introduce any new model, does not price complex instruments and does not present any powerful numerical technique. It focuses on the simple and, in the belief of the author, fundamental problem of the way (contingent) derivatives cash-flows are discounted. This question seems not to have received a lot of attention in the literature.

In finance different discounting curves exists. They are used or to price different financial instruments or to price the same instruments issued by different obligors.

In the latter case the explanation is clear in appearance, different issuers have different *credit or default risk*. The risk difference is represented by the spread (or difference) between the rates. The rate does not only represent the discounting or the actualisation of cash-flow but also the risk of non payment. The cash-flow are contingent to the ability (or willingness) of the issuer to pay. The yield of a defaultable bond is a conventional number that summarized the price of the bond into a uniformized convention. It mixes the discounting and default risk parts of the valuation. The complex task of modelling the valuation of contingent claims is replaced by the simpler task of discounting cash-flows. The simplification is only in the appearance, the valuation is done using incorrect method (discounting applicable to non-contingent cash-flows) with an incorrect input (the rate with a *spread*). The value of the spread is unexplained. The ignorance on the method is replaced by the ignorance on the data.

In the first case the reason for the existence of different rates is even less clear. Take the example of the swaps, one of the most liquid interest rate instruments. It exists in two types: Libor interest rate swap (IRS) and the overnight indexed swap[1] (OIS). In practice the first one is often valued by discounting cash-flows using a Libor/swap curve and the second using a Libid-like curve (Libor - 12.5bps). If the two swaps are traded with the same counterpart there is no reason to treat the cash-flows coming from those two operations in different ways. Nevertheless in practice they are discounted with different rates.

The standard derivative arbitrage free theory consider only a *unique* curve at which one is supposed to be able to borrow and invest. In practice several of those unique curves are used together to value instruments in the same portfolio. This creates a fundamental arbitrage possibility. Using

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[1] In its standard version, due to the two days lag in settlement the OIS is not a simple linear derivatives. The way to take the timing adjustment into account is described in [2]. The impact in practice is negligible and not mention further in this paper

Elie Ayache [1] terminology, this is a fundamental *irony*. To use *arbitrage free* pricing formula at the instrument level one uses several curves at the same time, creating de facto an *arbitrage at the portfolio level*.

This article is devoted to the development of a coherent framework where only one curve is used. Some of the questions asked are: How should the different derivatives be priced in a coherent framework? Do the derivative prices depend on who trade them? Are the FRA and swap rates fair for all participants? Which curve should be used?

Quoting Cantor

> In mathematics, the art of posing a question is more important than the art of solving one.
>
> *Georg Cantor, 1867*

It is our belief that the questions posed here are important and I hope that they are well posed.

Let's start by the last question. The answer is more practical than theoretical. The option to be priced is not isolated in a separate entity. It's part of a trading desk, it-self part of a bank (or other institution). The borrowing/investing problem should not be considered in isolation but globally. The banks are usually on the same side of the cash (cash rich or cash borrower) in a consistent way. Simplifying, the saving banks are permanently cash-rich from their customers' savings while the investment bank make investments and need to borrow cash also on a permanent basis. The cash needs of the derivative book are quite modest (hopefully) with respect to the general cash of the banks. The cash entering in the replication arguments for derivatives are marginal. An option desk of a cash rich bank will never borrow money, the bank is simply investing less cash. In an investment bank the desk never invest cash, the bank is borrowing less. Consequently using only one curve for borrowing and investing derivatives cash is justified. Different institutions will use different curves. The investment bank uses the rate at which it can borrow. The cash rich bank will use a rate at which it can invest in a liquid way large amount of cash in the inter-bank market. This rate will be referred a *Libid* in this article. As a convention in the examples the investment rate is supposed to be at Libor - 12.5 bps. In the article sequel this unique curve is called *funding* curve whatever side of the cash the option trader is, even if for half of the world it should be called *investment* curve. The use of a unique curve to discount all the derivatives cash-flows will be referred as the *funding framework*.

Hull's book [4], one of the most popular textbook is used as an example of the way the literature treats the question. In the section on the *type of rates* in the *interest rates* chapter (p. 76) the existence of both Libor and Libid is acknowledged, the latter with a description similar to our cash rich bank investment. When it comes to valuing the first derivatives (Section 4.7: Forward Rate Agreements) the explanation is "the assumption underlying the contract is that the borrowing or lending would normally be done at Libor". This is a misleading statement. The truth is that the contract *settlement amount* is computed with a *Libor fixing*. There is *no actual borrowing or lending* and there is *no assumption* in the contract, only a clear (contingent) settlement formula. When entering the (OTC) transaction each party is free to do *any assumption* they like on the rates. As will be proved later the parties can chose (independently) more realistic rates for them than Libor and still agree on the FRA rate at the transaction time. When the rates move the assessment of the trade value may differ between them.

In the examples the case where the funding rate is below Libor is considered. The results obtained are valid also in the other case using simply a negative spread. The comments are also made with the below Libor case in mind.

The idea in developing the paper is to use it with the funding curve as described above. The results could be used in a different way. For each counterpart the rate at which the trader is ready to lend to the counterpart could be used. This would be a way to incorporate credit risk into pricing. This would be useful for the asset swap pricing (Section 3.3) when the initial cash-flow is a large payment to the counterpart. Such a usage would be an irony (see above) on our stated goal of using only one curve for discounting.

2. Spreads

The same financial reality can be express in different ways, with different conventions. Traditionally each currency and instrument type has its own convention. In the noticeable case of the USD IRS, the same instrument in the same currency has two simultaneous conventions (annual ACT/360 and semi-annual 30/360). A spread in one convention is not equal to the spread in another convention. The translation is often not static and depends of the rate level it-self. Consequently it is not possible to use one number as a fixed spread and use it for all conventions without adjustment.

Let's look a the standard money market simple rate and see if it is possible to use a constant spread in a coherently way. The market quoted Libor rate is denoted L. The funding rate is F and the spread S. The basic spread equation is

$$L = F + S. \tag{1}$$

Is it possible to use a constant spread S in a consistent way?

To see that this is not the case, take two consecutive periods of length δ. The same rate F_1 is used for the two periods. The rate over the total period is F_2 and the relation between them is $1 + 2\delta F_2 = (1 + \delta F_1)^2$. Then the relation on Libor is

$$\begin{aligned} 1 + 2\delta F_2 + 2\delta S_2 &= 1 + 2\delta L_2 = (1 + \delta F_1 + \delta S_1)^2 \\ &= (1 + \delta F_1)^2 + 2\delta S_1 \left(1 + \delta F_1 - \frac{1}{2}\delta S_1\right) \end{aligned}$$

This means that

$$S_2 = S_1 \left(1 + \delta F_1 = \frac{1}{2}\delta S_1\right)$$

To the first order in δ the spread are the same but nevertheless not constant with a dependency on the rate level. A constant spread can not be used in the simple interest in a coherent way. Not all current rates and forward can be modelled with the same spread.

The situation is different in the continuously compounded zero-coupon framework. Let's denote $\bar{F}$ and $\bar{L}$ the continuously compounded equivalent of F and L:

$$(1 + \delta F) = \exp(\delta \bar{F}) \quad \text{and} \quad (1 + \delta L) = \exp(\delta \bar{L})$$

In this context it is showed that a constant spread in spot and forward rates is coherent. Take three dates $t_0 < t_1 < t_2$ and the corresponding rates $F_{i,j}$ and $L_{i,j}$ between the dates t_i and t_j. Then one has

$$\begin{aligned} \exp(\bar{L}_{0,2}(t_2 - t_0)) &= \exp((F_{0,2} + S)(t_2 - t_0)) \\ &= \exp((F_{0,1} + S)(t_1 - t_0)) \exp((F_{1,2} + S)(t_2 - t_1)) \\ &= \exp(\bar{L}_{0,1}(t_1 - t_0)) \exp(\bar{L}_{1,2}(t_2 - t_1)) \end{aligned}$$

For the rest of this article the spreads used will always be understood in the continuously compounded convention. The spread will be constant for all maturities.

Let's take a flat market rate curve at 5% level. If the three month spread is 12.5 bps in the simple interest, the continuously compounded spread is 12.3476 bps. If this last spread is taken constant for all maturities, the six month simple interest spread is 12.6524 bps. The difference is too small to be notice in practice as the rates are usually quoted in 0.5 or 0.25 of a basis point.

The constant spread in continuously compounded rates instead of simple interest like the market convention may seems unnatural. Nevertheless it is a coherent approach and the practical difference is below or at the level of the market precision.

3. Linear derivatives

The term *linear* is more a market jargon than a mathematical reality. All instruments are of course linear function of them-self, but this is not a very useful characteristic. The instruments presented in this section depend more or less linearly of rates.

For all the instruments the same notation will be used. The real price (in the sense of this article, i.e. in the funding framework) will be denoted with the instrument name in upper case which is function of a reference rate (strike).

The price of a FRA with reference rate K is denoted

$$\mathrm{FRA}(K).$$

The pricing formula for the same instrument using a discounting curve C in the standard framework is

$$\overline{\mathrm{FRA}}(C, K).$$

In the funding framework there is no need to indicate the discounting curve as there is no ambiguity. Only the funding curve makes sense to discount cash-flows. In the standard framework several curve are used. Like in the credit risk case they are used to apply the discounting formula but this is a way to use a simple technique to solve a non-simple problem. The output is manipulated by changing the input. It can also be related to the Black formula where the (implied) volatility is changed to obtain the (correct) market price. The option/pricing framework becomes simply a way to compute the price and not a model to explain it anymore.

3.1. **FRA.** The first instrument investigated in this framework is a Forward Rate Agreement (FRA). Consider buying a FRA at a rate K_L between the dates t_0 and t_1 and fixing in θ. The rate L_θ is the Libor rate fixing in θ for the period t_0-t_1. The accrual factor between t_0 and t_1 is δ. The FRA consists in receiving in t_0 the difference of rates $L_\theta - K_L$ multiplied by the accrual factor δ and discounted from maturity t_1 to t_0 at the fixing rate L_θ. The pay-off paid in t_0 is $\delta(L_\theta - K_L)/(1 + \delta L_\theta)$.

The curve to be used in the contingent claim pricing is the funding curve. The discount factors or zero-coupon prices viewed from t for maturity u are denoted $P(t, u)$.

In the standard arbitrage free derivative pricing framework, the price in 0 of the FRA described above is

$$N_0 \,\mathrm{E}^{\mathbb{N}}\left[N_{t_1}^{-1} \delta(L_\theta - K_L)(1 + \delta L_\theta)^{-1} P(\theta, t_1)\right]$$

for a numeraire N and associated measure $\mathbb{N}$.

The price is the pay-off fixed in θ, paid in t_0 and invested (at the cost of funding) to t_1. The payment in t_1 is chosen like in the standard approach to FRA or caplet pricing. The $P(., t_1)$ numeraire will be used to simplify the computation. In that numeraire the funding rate F associated to L is a martingale.

The link between the Libor rate and the funding is given by

$$(1 + \delta L_t) = \exp(\delta \bar{L}_t) = \exp(\delta \bar{F}_t) \exp(\delta S) = \beta(1 + \delta F_t)$$

or

$$L_t = \frac{1}{\delta}\left(\beta(1 + \delta F_t) - 1\right)$$

where $\beta = \exp(\delta S)$. Let $K_F = \beta^{-1} K_L - (1 - \beta^{-1})/\delta$. The difference between the Libor rate and the FRA rate is

$$L_\theta - K_L = \beta(F_\theta - K_F).$$

The price of the FRA, using the $P(., t_1)$ numeraire is

$$P(0, t_1)\,\mathrm{E}^1\left[\delta(L_\theta - K_L)(1 + \delta L_\theta)^{-1}(1 + \delta F_\theta)\right] = P(0, t_1)\delta\,\mathrm{E}^1[F_\theta - K_F] = P(0, t_1)\delta(F_0 - K_F).$$

The last equality is obtain using the fact that F_t is a martingale in the $P(., t_1)$ numeraire.

Theorem 1. *Suppose a framework where the funding curve has a constant continuously compounded spread to Libor equal to S. A FRA with rate K_L between the dates t_0 and t_1 and fixing in θ is considered. The accrual factor between t_0 and t_1 is δ. The price of the FRA is given by*

$$\mathrm{FRA}(K_L) = P(0, t_1)\delta(F_0 - K_F)$$

where F_0 is the forward rate in the funding curve and $K_F = \beta^{-1} K_L - (1 - \beta^{-1})/\delta$ with $\beta = \exp(\delta S)$.

Note that when $K_L = L_0$, the reference funding rate is $K_F = F_0$. In other term at inception if the FRA is traded at the *fair* Libor rate, it is also traded at the fair funding rate. The term fair is synonymous both of forward rate and rate that give the instrument a zero value. The instrument is designed to have a zero value at the forward rate when the Libor fixing is used for discounting. It is not obvious a priori that the same is true when another discounting curve is used. The funding FRA rate K_F to be used is not simply a shift of the Libor FRA rate K_L. The transformation involve a shift that uses the spread and the accrual factor and a multiplication.

The real price of the FRA using the funding curve can be written as function of the Libor FRA price

$$\text{FRA}(K_L) = \exp(t_0 S)\overline{\text{FRA}}(\text{Libor}, K_L).$$

The sensitivity of the FRA to changes of rates is larger that the one obtained in the Libor framework. The difference come from the fact that the funding rate are lower than Libor and consequently the value of a basis point in larger. The exact meaning of a basis point has to be adapted to the instrument but the result will be similar for all instruments.

3.2. **IRS.** An IRS, in its standard version is an exchange of fixed cash-flows against a stream of floating cash-flows. The IRS analysed is a payer swap. The fixed cash-flows are negative and the floating are positive. The floating side is fixed against Libor, corresponding to the Libor of the period[2] and is paid at the end of it.

The set of dates where the fixed and floating cash-flows take place are denoted t_i $(1 \leq i \leq n)$. The fixed cash-flows are $\delta_i K_L^i$. The subscript L is used as a reminder that the swap is linked to Libor. In practice the frequency of the floating payment is often the double or the quadruple of the frequency of the fixed payment. In this case some of the K_L^i are 0. The set of fixed cash flows are value by discounting with the funding rate. Its value is

$$\sum_{i=1}^{n} \delta_i K_L^i P(0, t_i).$$

The fixing of the floating payments take place in $0 \leq s_i \leq t_i$ $(0 \leq i \leq n-1)$. The rate fixed in s_i is paid for the period $[t_i, t_{i+1}]$. The value of the floating payment, using the $P(., t_{i+1})$ numeraire is

$$\text{SWAPLET} = P(0, t_{i+1})\,\text{E}^{i+1}[\delta_i L_{s_i}^i] = P(0, t_{i+1})\delta_i \left(\beta_i F_0^i + (\beta_i - 1)/\delta_i\right)$$

where $\beta_i = \exp(\delta_i S)$ with δ_i the accrual factor between t_i and t_{i+1}. The value can be rewritten using the standard Libor discounting approach

$$\text{SWAPLET} = \exp(t_{i+1} S)\overline{\text{SWAPLET}}(\text{Libor}).$$

The relationship between IRS and FRA is now analysed. For this analysis a swap with fixed and floating payment of the same frequency is used. In the Libor world, one has

$$\overline{\text{IRS}}(\text{Libor}, K_L) = \sum_{i=1}^{n} \overline{\text{FRA}}(\text{Libor}, K_L^i).$$

The swap is equivalent to a portfolio of FRA's.

In the funding world,

$$\text{IRS}(K_L) = \sum_{i=0}^{n-1} P(0, t_{i+1})\delta_i \left(\beta_i F_0^i + (\beta_i - 1)/\delta_i - K_L^i\right).$$

Using the notation $K_F^i = \beta_i^{-1} K_L^i - (1 - \beta_i^{-1})/\delta_i$ the formula is

$$\text{IRS}(K_L) = \sum_{i=0}^{n-1} \beta_i\, \text{FRA}(K_L^i).$$

[2] In practice the payment period can be slightly different from the fixing period due the payment conventions. This difference is ignored here.

The swap is still a portfolio of FRA's but with a different notional. To replace a swap one needs FRA's with larger notional. The weights of the different FRA's are only spread and accrual factor of the period dependent. The swap can be replicated staticly with FRA's. The weights of the different FRA's are equal when the periods are equal.

Let $\theta \leq t_0$. The value in θ of the swap is

$$\begin{aligned}\sum_{i=1}^{n}\beta_i\delta_{i-1}(F_\theta^{i-1}-K_F^i)P(\theta,t_i) &= \sum_{i=1}^{n}\beta_i(\delta_{i-1}K_F^i+(1+\delta_{i-1}F_\theta^{i-1})-1)P(\theta,t_i)\\ &= \sum_{i=1}^{n}\beta_i(-(1+\delta_{i-1}K_F^i)P(\theta,t_i)+P(\theta,t_{i-1}))\end{aligned}$$

The last equality is obtain using the fact that $(1+\delta_{i-1}F_\theta^i)P(\theta,t_i)=P(\theta,t_{i-1})$. In the case where all the β's are equal the formula is

$$\beta\left(\sum_{i=1}^{n}-K_F^iP(\theta,t_i)+P(\theta,t_0)-P(\theta,t_n)\right).$$

This correspond to the standard cash-flow equivalent valuation of a swap multiplied by the common adjustment factor β. In practice the β_i are very similar but not equal.

Theorem 2. *In the funding framework the value in θ of a payer swap with payment dates t_i and fixed rates K_L^i is*

$$\mathrm{IRS}(K_L)=\sum_{i=1}^{n}\beta_i(-(1+\delta_{i-1}K_F^i)P(\theta,t_i)+P(\theta,t_{i-1})).$$

Using the standard notation K_F^i where $(1+\delta_{i-1}K_F^i)\beta_{i-1}=(1+\delta_{i-1}K_L^i)$ the value of the swap is 0 when the fixed rates K_L^i are the *forward rates* in the *Libor* curve. This is equivalent to the rates K_F^i are the forward rates in the *funding* curve.

When the δ_i (and the β_i) are not all equal, the transformation of a unique K_L used for all fixed payments into a unique K_F does not work. The zero-value forward Libor rate swap has not a zero value in the funding world.

Another difference can come from the frequencies. In general the frequency of the fixed and floating payments are not the same. The floating payments are more frequent that the fixed one. This is true for all major currencies except for the AUD. This generate an *arbitrage* opportunity. The swap fixed rate is computed in such a way that the forward rates for the floating periods have the same present value using the Libor curve. Or said in the opposite way that the floating payments reinvested at Libor generate a final payment equal to the fixed payment.

Consider a *receiver* swap with only one fixed payment. The receiver jargon means that the fixed coupon is received and the floating coupons are paid. The trader pays the floating coupons first, without receiving anything. It has to borrow to made the payment. The borrowing is done at the funding level. This initial payments are compensated at the end by a unique fixed payment. The fixed payment is computed using the Libor curve. It is like if the trader was borrowing the coupon at funding and investing it at Libor. The borrowing and investing have to be done correctly to ensure there is no market risk involved and that the profit is guaranteed.

The arbitrage opportunity involving a Libor swap with two floating periods and one fixed payment is described in details. In practice this could be a six month fixed versus three month Libor IRS. Four dates are involved. The trade date is t_0, the start date of the swap is t_1 and t_2, t_3 are the two payment dates. Let L_s^i be the Libor rates for the two sub-periods and L_s the rate for the full period. The equivalent funding rates are F_s^i and F_s. The time s can be 0 for the rate viewed from t_0 or $s=1,2$ for the rates that fixes before the two sub-periods. The arbitrage involves four cash transactions, two forward traded in t_0 and two spot traded on the fixing dates. The cash-flows are reported in Table 1.

Time	**IRS**		**Cash spot**		**Cash forward**	
	Fixed	Float				
t_1				$-\beta_1$		β_1
t_2		$-\delta_1 L_1^1$	$-\beta_2$	$1+\delta_1 L_1^1$	$\beta_2 - 1$	
t_3	δL_0	$-\delta_2 L_2^2$	$1+\delta_2 L_2^2$		$-(\beta_2-1)(1+\delta_2 F_0^2)$	$-\beta_1(1+\delta F_0)$

Table 1. Cash-flow of a IRS arbitrage

Let D_s^i be the difference between investing at Libor and borrowing at the funding rate during the period i, i.e.

$$D_s^i = \delta_i(L_s^i - F_s^i).$$

The transactions are build in such a way that the floating cash-flows are perfectly hedged and the total cash-flows at dates t_0 and t_1 are null. The resulting cash-flows take place in t_2 and is perfectly determined on the trade date. Its value is

$$\delta_1 L_0^1 D_0^2.$$

It can be interpreted as if the trader is earning the difference of interest between Libor and funding on the rate valid for the first period. The arbitrage comes from the frequency difference.

As an example take a six month IRS with two three-month sub-periods, a spread of 12.5 bps and a Libor rate of 5%. The resulting profit is around 0.25.5%.0.25.12.5bps $\simeq$ 0.04bps.

A similar arbitrage can be done with more floating periods for one fixed period. This is done for four periods. The previous notation is extended to $D_0^{1\cdots i}$ the difference for investing in the i first sub-periods, $L^{1\cdots i}$ and $\delta_{1\cdots i}$ the Libor rates and accrued factors over these sub-periods. The arbitrage creates a final cash-flow of

$$D_0^4 \delta_{123} L_0^{123} + D_0^3(1+\delta_4 F_o^4)\delta_{12} L_0^{12} + D_0^2(1+\delta_{34} F_0^{34})\delta_1 L_0^1.$$

With a one year IRS with four three-month periods and the same rates as before the profit is around 0.23 bps.

The arbitrage is strongly dependent on the spread being constant over tenors and time. The spread is used at each start date and at each fixing date. In practice the arbitrage profit is too small to protect against the potential spread change. It is not a real arbitrage but at best a *statistical arbitrage*. On average over different spread changes it may possible to monetise this amount. Moreover the bid-offer for IRS is usually around one basis point in rate, above the arbitrage opportunity described.

In the set-up of the arbitrage, for each fixing date two cash transactions are required: one spot lending and one forward borrowing. The notional of those transactions is similar to the swap notional. Apart from the fact that forward deposit transaction are not common, the multiple borrowing and lending could have a serious impact in term of credit risk.

The magnitude of the difference between the *funding* and the *Libor* framework can be seen in Figure 1. The example is a five year swap with an initial Libor curve flat at 5%. Two swap are analysed. One has a annual fixed coupon payment against quarterly floating payments. The second one is quarterly for both legs with the initial fixed rate equal to the forward Libor rate on each period.

The Libor and funding frameworks give different P/L. For a 1% increase coupon in the quarterly swap the value increases less in the Libor framework than in the funding one. This is expected as the value of a basis point is higher in the funding framework where the rates are lower. The annual swap is similar except that for the initial 5% coupon, its value is 0 in the Libor framework and 105 (for one million) in the funding one. The difference corresponds to the arbitrage described above.

The out-of-the-money swaps can correspond to old ATM

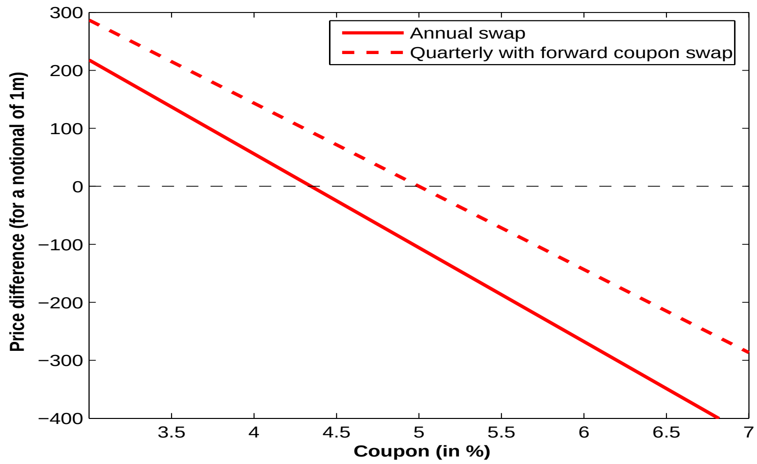

FIGURE 1. Price difference between the *Libor* and *funding* framework for a IRS.

3.3. **Asset swap.** The swap part of an asset swap is not in principle different of a standard IRS. A set of fixed cash-flows is exchanged against a regular stream of floating cash-flows.

The difference resides in the fact that the fixed side of the swap is not a market rate but the coupon of an asset. This means that the convention on the fixed payment may be different from the standard one in the swap market. This is only a minor difference. The main difference is due to the difference in coupon between an at-the-money swap and the asset. Even if the asset is paying a Libor-like yield, the coupon may be very different from the swap rate. For the bond the difference is compensated by a non par price. In consequence the swap will be with non-market coupon compensated by an up-front payment. The difference in yield between the asset and Libor is compensated by the payment of a spread above the Libor floating rate. For the asset swap investor, the package cash-flows is similar to investing into a par floating rate note.

The question for the swap trader is how to price the spread, i.e. how to value a non at-the-money swap. The simplest approach is to discount all cash-flows at Libor rate with the reasoning that *it is a swap and swaps are priced out of Libor*. This is a simplistic approach that does not take into account the cost of funding on the up-front payments and subsequent spreads and difference in coupon. For traders with a cost of funding close to Libor this is not a problem. If the cost of funding is like in our examples at Libor - 12.5 bps the impact can be significant, up to 2 bps in the spread.

A second approach is to compute a par Libor rate using the convention of the bond. The par swap versus the Libor floating leg has a value of 0. All the other fixed cash-flows are value with the funding curve by discounting. This is an hybrid (and consequently not coherent) approach but gives a quite fair and realistic pricing.

A coherent and precise approach is to value the swap as described in the previous section. The difference with the hybrid method is that if the fixed frequency is different from the floating frequency, the real price of the par Libor swap may be different from 0, the price at which it can be treated in the market. The difference being usually below the bid-offer spread.

4. CONTINGENT INSTRUMENTS

Like for the previous section, the term *contingent* is more a market jargon than a clear technical cut-off. The FRA and swaps pay-off are also contingent on the floating rates at different dates.

4.1. **Rate futures.** The rate futures are *margined* on a continuous (daily in practice) basis. Let θ be the fixing date and M_θ the future price on fixing date. The price M_0 of the future in 0 is given by (see [5, Theorem 12.6])

$$M_0 = \mathrm{E}^{\mathbb{N}}[M_\theta]$$

In the rate futures case the final price is $1 - L_\theta$ where L_θ is the Libor rate associated to the future (usually three month). The rate futures price in the Libor framework with extended Vasicek model is given in [3] and in the LMM framework in [6]. In the funding framework the fixing price becomes

$$M_\theta = 1 - L_\theta = 1 - \beta F_\theta - (\beta - 1)/\delta.$$

The futures price in 0 is

$$\mathrm{FUT}_0 = \beta \overline{\mathrm{FUT}}_0(\mathrm{Funding}) - (\beta - 1)(1/\delta + 1).$$

4.2. **Cap and floor.** Consider a *caplet* with a strike K_L between the dates t_0 and t_1 and fixing in θ. The accrual factor between t_0 and t_1 is δ.

The pay-off paid in t_1 is

$$\delta \left(L_\theta - K_L\right)^+ = \delta\beta(F_\theta - K_F)^+.$$

The price of the caplet is

$$\mathrm{CAP}(K_L) = P(0, t_1)\delta\,\mathrm{E}^i\left[(L_\theta - K_L)^+\right] = \beta\overline{\mathrm{CAP}}(\mathrm{Funding}, K_F).$$

Any modelling approach (Black, HJM, BGM, general LMM,...) can be used to model the funding curve. In this case the LMM is misnamed as the curved modelled is not Libor but the more real rate for the trader, the rate at which his cash works.

In the standard approach the put-call parity is

$$\overline{\mathrm{FRA}}(C, K) = \overline{\mathrm{CAP}}(C, K) - \overline{\mathrm{FLOOR}}(C, K).$$

In the funding framework

$$\begin{aligned}
\mathrm{FRA}(K_L) &= \overline{\mathrm{FRA}}(\mathrm{Funding}, K_F) = \overline{\mathrm{CAP}}(\mathrm{Funding}, K_F) - \overline{\mathrm{CAP}}(\mathrm{Funding}, K_F) \\
&= \beta^{-1}(\mathrm{CAP}(K_L) - \mathrm{FLOOR}(K_L)).
\end{aligned}$$

The notional in the cap/floor parity with the FRA is not the same. The FRA notional is lower due to the special discounting rule at Libor embedded in the instrument. It means also that even if the model used has always positive interests, a cap with strike 0 is not equal to a FRA with rate 0.

4.3. **Swaption.** Consider a receiver swaption with expiry θ on a swap like the one described in Theorem 2. The price of the swap in θ is equivalent to the price of a set of fixed cash-flows. The cash-flow equivalent here is more involved than in a standard swap because of the presence of different β_i.

In the simplified case, where all the β's are equal the pay-off of the swaption in θ is

$$\beta\left(\sum K_F^i P(\theta, t_i) - P(\theta, t_0) + P(\theta, t_n)\right)^+$$

and the value in 0 is

$$\mathrm{SWAPTION}_0(K_L) = \beta\overline{\mathrm{SWAPTION}}_0(\mathrm{Funding}, K_F).$$

5. Conclusion

A *funding framework* using the actual rates paid (or received) on cash is described. The spread between the fixing (*Libor*) rates and the reference funding rates are supposed constant. In the framework the prices of simple linear and contingent derivatives are analysed. The first (reassuring) conclusion is that when traded at the fair (forward) rate in the Libor framework the linear derivatives have a price in the funding framework which is close to 0. A small arbitrage appears when fixed and floating legs have different frequencies. The value of the derivatives changes faster in the funding framework than in the Libor one. The ratio of changes speed is linked to the spread and the fact that at a lower rate cash-flows have more value.

The real difference does not appear for the rates at which standard trades are done but for the valuation of not at-the-market trades and trades with non-standard cash-flows (like asset swaps). For those transactions the difference in valuation can be significant. Traders with cost of funding significantly different from Libor need to take it into consideration.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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Contents

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