# SWAPTIONS: 1 PRICE, 10 DELTAS, AND ... 6 1/2 GAMMAS.

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Abstract. In practice, option pricing models are calibrated using market prices of liquid instruments. Consequently for these instruments, all the models give the *same price*. But the risk implied by them can be widely different. This note compares simple risk measures (first and second order sensitivity to the underlying yield curve) for simple instruments (swaptions). The main conclusion is that the hedging varies widely (up to 10% of the underlying risk) between the models, with the prevailing differentiating factor being the *model dynamic*. The shape of the smile has also an impact but to a lesser extent. Hedging efficiency using historical simulation is analysed. Using data from the last three years, normal-like models perform consistently better.

## 1. Introduction

The standard practice in option pricing is to calibrate models to market prices of liquid instruments. Consequently all models (with enough degrees of freedom) will give the same price for standard options. Nevertheless, due to the different hypotheses implicit in the different models, the *risks* inferred will be different. As the most used risk measure, the first order sensitivity or *out-of-the-model delta* (or simply delta) will be the focus of this paper. The second order sensitivity or out-of-the-model *gamma* is also briefly discussed.

There are certainly several places where it is mentioned that different models will lead to different risk numbers (in particular Rebonato [8]). This note does not present theoretically new developments but emphasizes the practical importance of model choice, even for simple and liquid instruments, for which the price is given by the market. The models studied are widely used in practice and actual figures are given reflecting the extent of the difference in risks.

The first models analysed are the classical Black model for swaption (geometric Brownian motion of the forward swap rate) and its normal version (arithmetic Brownian motion of the forward swap rate). The next is the extended Vasicek (or Hull and White [4]; arithmetic Brownian motion of the continuously compounded rates) and its explicit formula for European swaptions [2]. The last three are stochastic volatility models based on a constant elasticity volatility (CEV) extension of Black model, known as SABR (stochastic, alpha, beta, rho)[1]. We use three versions of this model, one with the elasticity parameter $\beta$ equal to 0 (normal), one with 1 (log-normal) and one with no correlation between rates and volatility ($\rho = 0$). For the Black, normal and Hull-White models we also compute the theoretical in-the-model delta equivalent and use the sensitivity of the underlying swap multiplied by this theoretical number. Moreover for the Black model, we compute the total delta and the delta coming from changing only the forward rate (no adjustment of the annuity or present value of a basis point). More details on the numerical procedures used are given in the Appendix. Troughout the paper the difference between normal-like or log-normal-like models is referred to as the *market* or *model dynamic*. It represents the fundamental market movements implied by each model's stochastic differential equation. The impact of model dynamic and market data (rate, smile, volatility level) are analysed separately. It is evident that the model dynamic is the dominant of the two impacts.

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The different risk figures time series are provided in Section 4 for three models (Black, Hull-White and SABR with $\beta = 0$). The graphs clearly indicate the market parameters that influence the changes through time.

Finally the real judgement arrives: the hedging contest. The game is simple: buy a ATM swaption each day, delta hedge it with each model and sell the swaption and its hedge the next day. This is done over several years. The best hedger is the model producing the smallest standard deviation of profit. And the winner of the contest is ... only announced at the end of the paper.

The difference between the models is substantial. The choice of the model dynamic is an important decision that a risk manager has to take.

## 2. Different models: Black, Normal, Hull-White, and SABR

The sensitivity dependency with regards to choosing the model is analysed in this section. The models described in the introduction are calibrated to ATM swaption on 25 June 2004. For models with more freedom (SABR) the smile is calibrated at best to the smile as observed on that date.

The first (and main) risk measure analysed is the out-of-the-model delta to market rates. The numerical procedures are explained in the appendix.

For the Black, normal, and Hull-White models the sensitivity of the *in-the-model delta equivalent* is also analysed. The sensitivity of the underlying swap is computed and multiplied by that figure. The results are detailed in Table 1.

The *base* model is the SABR 0. The correlation is around 11% and the vol of vol around 30%. Using that model as base does not imply that it is the best model. However, to compare figures, a measuring stick is required and the SABR0 model is used for this purpose.

For the Black model, the sensitivity when the Black volatility is changed[1] is added. The volatility is increased by 5%. With the change of volatility, the prices also changes. This delta is not included in the title of the note; it is a eleventh delta but with a second price. It is interesting to note that the change of sensitivity coming from the (very) wrong volatility is a lot smaller than the change due to different models. In the first row of Table 1 showing the different model sensitivities, it is (almost) impossible to distinguish the Black model with the correct volatility from the one with the (very) wrong one but easy to distinguish between a model with normal and a model with log-normal assumption.

| Model | SABR 0 | Normal | Δ norm | Hull-White | Δ HW | Black | Black+5% |
|---|---|---|---|---|---|---|---|
| Sensi | -2,260 | -2,231 | -2,158 | -2,227 | -2,227 | -2,029 | -2,056 |
| Diff. | – | 29 | 103 | 34 | 33 | 231 | 203 |
| Perc. | – | -0.67 | -2.38 | -0.78 | -0.77 | -5.36 | -4.73 |
| **Model** | **Δ Black** | **Black fwd** | **SABR 1** | **SABR nc** | | | |
| Sensi | -1,955 | -1,955 | -1,836 | -2,205 | | | |
| Diff. | 305 | 305 | 424 | 55 | | | |
| Perc. | -7.07 | -7.07 | -9.83 | -1.28 | | | |

Table 1. Dependence of the total first order sensitivity on the model (1yx5y ATM receiver swaption, 25 June 04)

It is interesting to note the following fact[2]. Suppose the *correct* model is the SABR 0 one. The Black model gives a completely different dynamic. Correcting the *symptom* of the *incorrect* model by introducing a smile through the SABR 1 model worsen the hedging results. The SABR 1 sensitivity (1,836) is further apart from the SABR 0 (2,260) than the simple Black (2,029).

For the risk management of swaptions, the model fundamentals (dynamic) are more important than the appearance (smile). This will be detailed more in the next section. The choice of model

[1] This was suggested by M. Delzio when discussing a previous version of the note.

[2] This fact was pointed to the author by J.M. Malher when discussing a previous version of the note.

fundamentals (dynamics) is the most important decision that a hedger (or trader, or risk manager) has to take.

The models can be put in two different groups. The *normal-like* models comprising normal Black, SABR 0 and Hull-White and the *log-normal* ones comprising Black and SABR 1. Note that the normal hypothesis in the Hull-White model is not on the same rate than the one in the other models. The rate modelled is not the swap rate but the continuously compounded zero-coupon rates. Nevertheless the dynamic is in the same family.

The SABR nc is a little bit between the two groups. Depending on the shape of the smile, the elasticity coefficient $\beta$ can be different and give quite different sensitivities. In the data analysed, the calibration gives a $\beta$ of 0.13 and a dynamic close to a normal one. The results with the calibration done on 9 December 2003 are presented in Table 2. On that date the elasticity coefficient was 0.59. The sensitivity is somewhere between the normal and log-normal groups.

| Model | SABR 0 | Normal | $\Delta$ norm | H-W | $\Delta$ HW | Black |
|---|---|---|---|---|---|---|
| Sensi | -2,496 | -2,324 | -2,230 | -2,318 | -2,318 | -2,035 |
| Diff. | – | 172 | 266 | 178 | 178 | 460 |
| Perc. | – | -3.87 | -5.96 | -4.00 | -3.99 | -10.33 |

| Model | $\Delta$ Black | Black fwd | SABR 1 | SABR nc |
|---|---|---|---|---|
| Sensi | -1,942 | -1,942 | -1,911 | -2,152 |
| Diff. | 554 | 554 | 585 | 344 |
| Perc. | -12.42 | -12.42 | -13.12 | -7.71 |

TABLE 2. Dependence of the total first order sensitivity on the model (1yx5y ATM receiver swaption, 9 December 2003)

An heuristic explanation for those two groups can be proposed. The (Black) price of the (receiver) swaption for a given yield curve $C$ and an (implied Black) volatility $\sigma$ is denoted

$$S(C, \sigma).$$

If a model different from the Black is used and its parameters are kept constant when the curve is changed, its Black volatility will implicitly change with the curve. This is indicated by writing $\sigma = \sigma(C)$.

The delta computed is the (total) derivative with respect to the yield curve change. Using the chain rule we obtain

$$\frac{dS}{dC} = D_1 S(C, \sigma(C)) + D_2 S(C, \sigma(C)) \frac{d\sigma}{dC} = \Delta_{\text{Black}} + \mathcal{V}_{\text{Black}} \frac{d\sigma}{dC}.$$

For a receiver, the Black price derivative with respect to the curve when the rates are increased ($D_1 S$) is negative. By convention the curve change corresponds to a rate increase. The derivative with respect to the second variable is the Black Vega and is positive when one is long the option. The model dependent part is the last factor: how does the Black volatility implied by another model evolves when the curve changes.

Normal-like models on the rate are analyzed. In those models the absolute volatility is constant. Consequently when rates increase the relative volatility is lower. In mathematical terms, the derivative of the implied volatility with respect to the curve is negative.

The summary for a receiver swaption is

$$\Delta_{\text{Model}} = \frac{dS}{dC} = \Delta_{\text{Black}} + (\text{positive}).(\text{negative}) < \Delta_{\text{Black}}.$$

The models studied are not purely normal so the absolute volatility of the swap rate is not constant. It is nevertheless constant enough to ensure that the indicated qualitative analysis holds.

The swaption behavior at several moneyness levels is also investigated. Results for three different moneyness (ATM, ATM$\pm$100bps) are presented in Table 3. The delta discrepancy is moneyness dependent. The same model grouping mentioned previously is still valid. For out-of-the-money options, the accuracy of the volatility is more crucial.

| Model | SABR 0 | Normal | Del norm | H-W | Del HW | Black |
|---|---|---|---|---|---|---|
| ATM-100bps | – | 0.35 | -0.89 | 0.20 | 0.21 | -5.36 |
| ATM | – | -0.67 | -2.38 | -0.78 | -0.77 | -5.36 |
| ATM+100bps | – | -0.59 | -1.75 | -0.72 | -0.72 | -6.11 |
| Model | Black+5% | Del Bl | Bl fwd | SABR 1 | SABR nc | |
| ATM-100bps | -1.58 | -6.29 | -5.74 | -6.61 | -0.92 | |
| ATM | -5.99 | -7.07 | -7.07 | -9.83 | -1.28 | |
| ATM+100bps | -11.03 | -7.59 | -10.16 | -6.35 | -0.79 | |

Table 3. Dependence of the total first order sensitivity on the model and moneyness (1yx5y ATM receiver swaption, 25 June 2004). Figures reported as percentage of ATM swap sensitivity.

In the case of the *gamma*, the grouping of models is not so clear any more (see Table 4). There the incorrect volatility has a larger impact on the results. Increasing the volatility decreases the gamma of the ATM swaption significantly (by 20%).

| Model | SABR 0 | Normal | H-W | Black |
|---|---|---|---|---|
| Gamma | 16.65 | 16.18 | 16.18 | 15.90 |
| Diff. | – | -0.47 | -0.47 | -0.76 |
| Perc. | – | -2.82 | -2.84 | -4.55 |
| Model | Black + 5% | Bl fwd | SABR 1 | SABR nc |
| Gamma | 13.31 | 14.45 | 16.24 | 16.58 |
| Diff. | -3.34 | -2.21 | -0.41 | -0.07 |
| Perc. | -20.06 | -13.25 | -2.47 | -0.44 |

Table 4. Dependence of the total gamma on the model (1yx5y ATM receiver swaption, 25 June 2004). Percentage difference relative to SABR0 gamma.

The aim of this paragraph is to explain the last part of this note's title. Only seven models were used to compute the gamma. We have not computed the in-the-model delta equivalent gamma has this would be quite meaningless. The only theoretically justified in-the-model sensitivity is the delta. For the Hull-White model the standard implementation is through a trinominal tree. It is known that trees produce very unstable sensitivities (and even meaningless gamma). This was documented for the CRR model by Pelseer and Vorst [7] and for the Hull-White model in [3]. Being unmindful in the implementation used can lead to not having gamma at all. The 1/2 in the title refers to the Hull-White gamma that is not always available in standard tree implementations.

## 3. Model and market impacts

3.1. **Market dynamic.** Different models have different probability distributions of the underlying, generating different price changes when the market moves (different sensitivities). This is often linked to the *smile* effect. This section shows that large sensitivity differences can appear even in absence of smile differences. Several versions of a model with different underlying dynamics are calibrated (very closely) to the same smile and still produce substantially different deltas. The models used is the SABR models with elasticity parameter ($\beta$) between 0 and 1.

As an example, the figures are computed with a 10 million 1yx5y receiver at-the-money (ATM) swaption in USD on 25 June 2004. The prices are calibrated to ATM Black volatility of 23.5%, for the SABR model with 0 elasticity ($\beta = 0$, normal-like model), the (arbitrarily chosen) correlation ($\rho$) is 10% and the volatility of volatility ($\nu$) is 30%.

The ATM price and the smile are calibrated using the same SABR model but this time with an elasticity of 0.25, 0.50, 0.75 and 1.

| | $\beta$ | 0 | 0.25 | 0.50 | 0.75 | 1 |
|---|---|---|---|---|---|---|
| Delta | Sensitivity | -2,257 | -2,154 | -2,052 | -1,950 | -1,848 |
| | Difference | – | 103 | 205 | 307 | 410 |
| | Relative difference (%) | – | 2.38 | 4.75 | 7.12 | 9.49 |
| Gamma | Sensitivity | 16.64 | 16.54 | 16.75 | 16.51 | 16.28 |
| | Difference | – | -0.09 | 0.11 | -0.13 | -0.36 |
| | Relative difference (%) | – | -0.57 | 0.64 | -0.77 | -2.14 |

TABLE 5. Total sensitivity of a 1y x 5y receiver ATM swaption in SABR model with different elasticity parameters. Relative difference for delta is with respect to the one of the underlying swap and for gamma with respect to the one of the swaption with $\beta = 0$.

The out-of-the-model yield curve deltas are reported in Table 5 for different tenors. The relative difference is computed with respect to the sensitivity of the underlying ATM forward swap. As can be seen from the numbers, the difference is close to 5% for a parameter of 0.5 and close to 10% for 1. The version of the model with elasticity 0 has the largest delta in absolute value.

The yield curve gamma for ATM swaptions is also reported in Table 5.

The impact of the dynamic depends on the swaption moneyness. The sensitivities of receiver out-of-the-money (OTM) swaptions are computed with different strikes (1, 2, and 3% below and above forward rate). The yield curve sensitivities (delta and gamma) are reported in Table 6.

| | | ATM | Moneyness: ATM | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | swap | -3% | -2% | -1% | ATM | +1% | +2% | +3% |
| Delta | $\beta = 0$ | -4,315 | -31 | -202 | -872 | -2,257 | -3,651 | -4,412[4] | - 4,739 |
| | $\beta = 0.5$ | -4,315 | -19 | -150 | -717 | -2,052 | -3,530 | -4,357 | - 4,716 |
| | Diff. | – | -11 | -52 | -155 | -205 | -121 | -55 | -24 |
| | Rel. diff. (%) | – | 0.26 | 1.20 | 3.59 | 4.75 | 2.81 | 1.26 | 0.55 |
| Gamma | $\beta = 0$ | 3.55 | 0.65 | 3.50 | 10.98 | 16.64 | 12.50 | 6.63 | 4.26 |
| | $\beta = 0.5$ | 3.55 | 0.29 | 2.13 | 8.69 | 16.75 | 14.27 | 7.69 | 4.70 |
| | Diff. | | -0.36 | -1.38 | -2.28 | 0.11 | 1.77 | 1.05 | 0.44 |
| | Rel. diff. (%) | – | -2.15 | -8.26 | -13.73 | 0.64 | 10.63 | 6.32 | 2.64 |

TABLE 6. Sensitivity to market rates for different moneyness and two different elasticity parameters. Relative difference for delta is with respect to the one of the swap and for gamma with respect to the one of the ATM swaption with $\beta = 0$.

We have the same pattern with the 0 elasticity model giving a larger delta. The difference being between 0 and 5% of the risk of the underlying. The discrepancy is larger for the at-the-money than for out-the-money options.

It may seem paradoxal that all prices are equal for all strikes and the deltas are different. When the rate changes the models parameters are kept constant. After the movement the prices will not be equal anymore. The smile is a picture of the starting prices, the delta is an estimation of how the prices will change with the rates. The models describe how the rate changes through time. As an analogy we can say the the smile is a picture of a group (of prices) and the deltas are (short) movies of part of the live of each of them. Two board scripts (called models) of those movies are presented. Even if the characters are the same the stories are different.

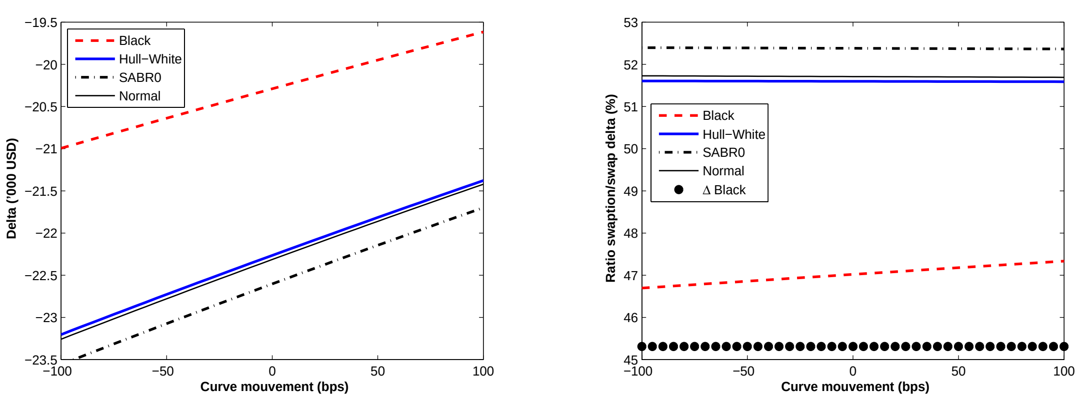
| | | | Volatility of volatility ($\nu$) | | | |
|---|---|---|---|---|---|---|
| Delta | Correlation ($\rho$) | | 0.20 | 0.30 | 0.40 | 0.50 |
| | -0.10 | Sensitivity | -2,214 | -2,205 | -2,197 | -2,187 |
| | | Rel. diff. (%) | -1.00 | -1.20 | -1.41 | -1.62 |
| | 0.10 | Sensitivity | -2,249 | -2,257 | -2,266 | -2,275 |
| | | Rel. diff. (%) | -0.20 | 0 | 0.20 | 0.41 |
| | 0.30 | Sensitivity | -2,283 | -2,309 | -2,336 | -2,362 |
| | | Rel. diff. (%) | -0.60 | 1.20 | 1.81 | 2.44 |
| | 0.50 | Sensitivity | -2,317 | -2,361 | -2,405 | -2,449 |
| | | Rel. diff. (%) | 1.39 | 2.40 | 3.41 | 4.44 |
| | 0.70 | Sensitivity | -2.352 | -2,412 | -2,473 | -2,534 |
| | | Rel. diff. (%) | 2.19 | 3.59 | 4.99 | 6.40 |

TABLE 7. Sensitivity to market rates for different correlation and vol of vol parameters and relative difference in percentage with respect to the base case. The percentage is relative to the sensitivity of the ATM forward swap.

3.2. **Smile.** In this section the fundamental dynamic of the market is kept unchanged (normal model on the swap rate) but the smile is modified. The SABR model with elasticity parameter $\beta = 0$ described in the previous section is used but the smile parameters (correlation $\rho$ and volatility of volatility $\nu$) are changed. The chosen parameters are in line with market figures. The correlation is between -10% and 70% and vol of vol between 20% and 50%. The results are given in Table 7. The base case for comparisons is the model used in the previous section with a correlation at 10% and a volatility of volatility at 30%.

One has to introduce significant changes of the smile to obtain effects of the same order of magnitude that the change of the dynamic. With these changes, the current picture of the prices (smile) would be incorrect and one could notice the error immediately.

3.3. **Rate level.** For all models, the ATM swaption delta is higher (in absolute value) when rates are lower. This is not too surprising. When rates are lower the discounting is less important and the values and sensitivities are higher (Figure 1(a)).




(a) Swaption sensitivity in USD

(b) Swaption relative sensitivity (in % of swap sensitivity)

FIGURE 1. Delta dependency on the rate level for three models.

If the relative sensitivity, computed as the ratio between the swap sensitivity and the swaption sensitivity, is analysed the picture is different. For the normal-like models (Hull-White and SABR), the ratio is almost constant while the (log-normal) Black model, the ratio is higher for higher rates (Figure 1(b)). The figure also contains the theoretical in-the-model delta. For ATM swaptions this delta is not affected by the curve level. The delta is given by $N(\sigma\sqrt{T/2})$ and depends only on the volatility level.

3.4. **Volatility level.** In the case of the delta change created by the volatility change, the picture is mixed. Figure 2 displays the delta as function of the volatility parameters. The parameters are changed between 70 and 130% of their initial value. For the Black model, the delta decreases (in absolute value) when the volatility increase. For the normal-like models, it is the opposite. This can be related to the ratio of deltas. When the ratio between the swap and the swaption delta is less than 50%, it decreases with the volatility decrease. When the ratio is above 50% the effect is the opposite. The volatility decrease concentrates the changes into a smaller place and the ratios get more extreme (away from the middle).

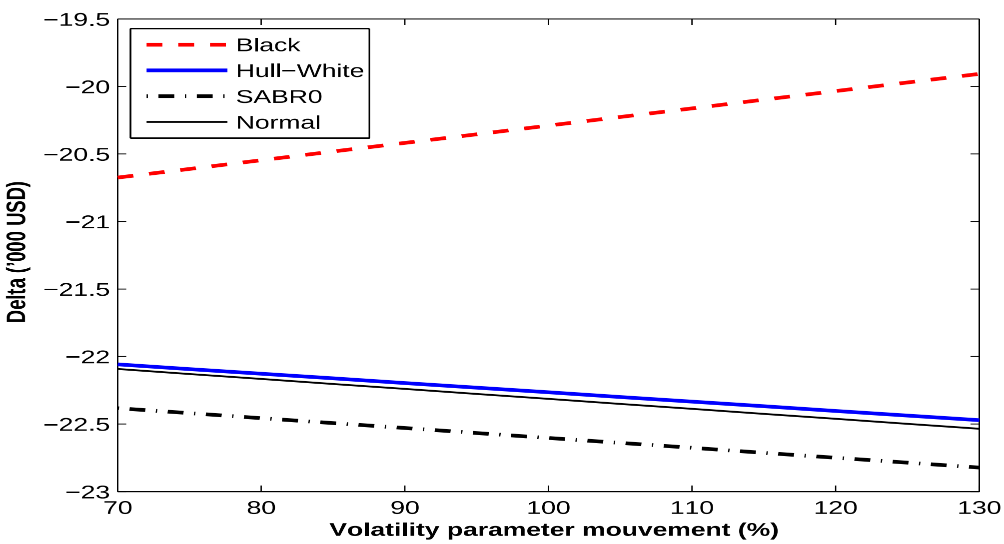
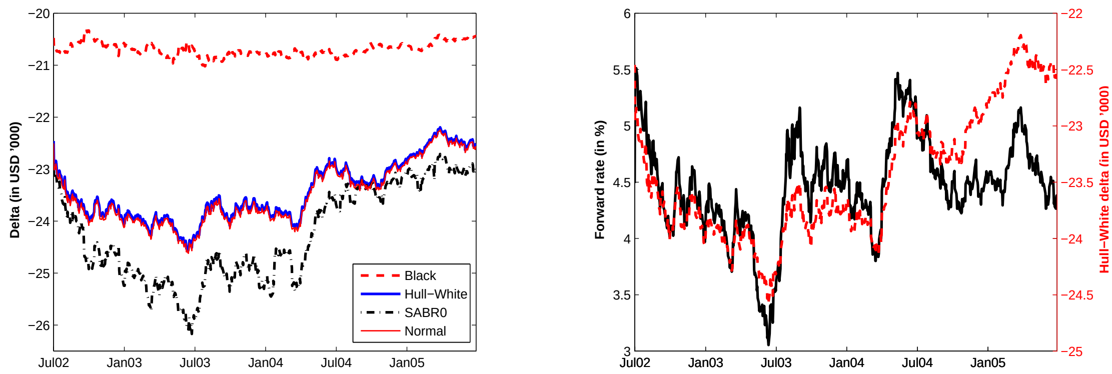



FIGURE 2. Dependency of delta on the volatility level for three models.

Note also that in practice their is a strong dependency between the rate level and the Black volatility level. When the rates are higher, the Black volatility tends to be lower. Models with this characteristic have enhanced chances to improve Black's delta hedging. In the historical data we used (see next section) the relation is around a decrease of 7.5% of Black volatility for an 1% forward rate increase. The rate (Figure 1(a)) and volatility (Figure 2) movements used have approximately that ratio. When rates increase, the delta decreases. But simultaneously the volatility decreases which imply a delta increase. The two movements partially compensate each other.

## 4. Historical time-series

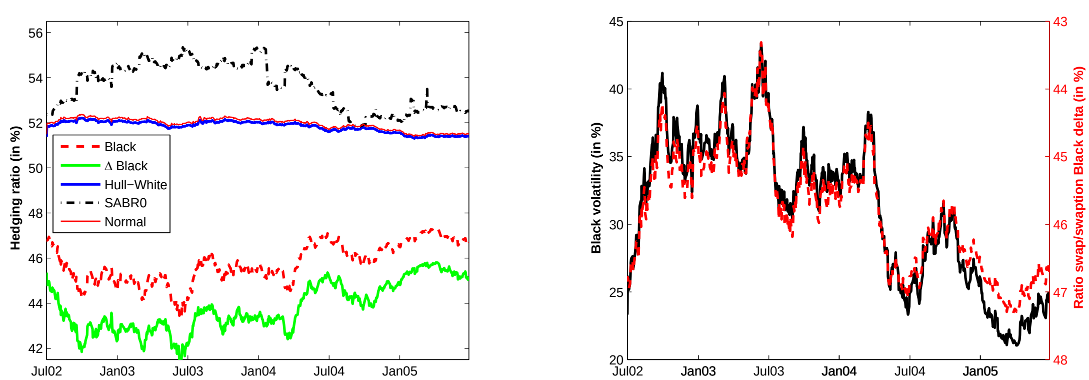
In Figure 3(a) the ATM sensitivity time series are graphed for the models studied in the previous section (Black, Hull-White, SABR0, normal). The Black sensitivity is quite stable in dollar terms. The Hull-White delta is more volatile. As described in the previous section, the different rate levels create variability. The forward and Hull-White delta time series are graphed in Figure 3(b). The two series use very different scales but it is clear that their is a very strong relationship between them.




(a) Sensitivity (b) Forward rate vs Hull-White sensitivity

FIGURE 3. Time series for different models. July 2002 to June 2005.

The ratio between swap and swaption delta is graphed in Figure 4(a) for the same models. The theoretical in-the-model Black delta ($\Delta$) is also graphed.




(a) Deltas (b) Black volatility vs ratio swap/swaption delta

FIGURE 4. Time series for different models (July 2002 to June 2005).

Note that the theoretical Hull-White delta is almost equal to the ratio swap to swaption delta. This can be explained in the following way. The theoretical delta ([2, Theorem 5.1]) is

$$\frac{\sum_{i=0}^{n} c_i P(0,t_i)\nu(0,t_i)N(\kappa+\alpha_i)}{\sum c_i P(0,t_i)\nu(0,t_i)}$$

where $\nu(0,t) = (1-\exp(-at_i))\sigma/a$. If the exponential in $\nu$ is (Taylor) expanded to the first order ($\alpha$ is around 1% and $t_i$ between 1 and 5), the formula simplifies to

$$\frac{\sum_{i=0}^{n} c_i P(0,t_i)t_i N(\kappa+\alpha_i)}{\sum c_i P(0,t_i)t_i}.$$

On the other side the ratio of swap sensitivity is

$$\frac{\sum_{i=0}^{n} c_i N(\kappa+\alpha_i) dP(0,t_i)/dC}{\sum c_i dP(0,t_i)/dC}$$

where $dP(0,t_i)/dC$ is the change of discount factor implied by the parallel (market) rate movement. If we write the price as $\exp(-r_i t_i)$ and approximate the market movement by the zero-coupon movement we obtain

$$\frac{\sum_{i=0}^{n} c_i P(0,t_i) t_i N(\kappa+\alpha_i)}{\sum c_i P(0,t_i) t_i}.$$

This is the same approximation as previously. The two numbers are almost indistinguishable.

In Figure 4(b) the time series of Black volatility and the sensitivity ratio are displayed. The scales are not identical and the ratio is on an inverse scale but the strong relationship between them is conspicuous. The variability of Black implied volatility through time has an impact on the Black hedging ratio.

## 5. Hedging contest

The section is devoted to the analysis of the models delta hedging performance.

The exact procedure, repeated on a daily basis for more than three years and for each model in the contest is the following. An at-the-money 1y x 5y receiver swaption with a 100m notional is purchased. Its delta is computed. The delta is the out-of-the-model first order sensitivity computed by moving the market rates up and down by one basis point. At the same time a 1y x 5y forward payer swap with the same sensitivity (and with a notional around 50% of the swaption one) is entered into. This procedure creates portfolios that are delta hedged with its corresponding models in the best possible way. The four models used for this part of the study are the (log-normal) Black, the normal, the Hull-White and the SABR models. We also add the hedging using the theoretical Black delta (in-the-model) and the *naive* hedging with a swap notional equal to exactly 50% of the swaption one (which correspond also to the in-the-model normal delta).

The clock is moved one day forward and the swaption and swap price are recomputed. Also to be precise we pay interest on the premium amount we need to borrow overnight. For reevaluation the historical rates and smile data of the next day are obviously used.

This produce a profit (or loss) figure for each day and each model. The data set is analysed in several ways.

First the total profit is computed by simply adding all the profits. On one extreme the not hedged swaption created a profit of 6.6m and on the other extreme the Hull-White portfolio created a loss of less than 50,000. This figure is not the most interesting as one can arrive close to 0 through large positive and negative jumps.

| | Swaption | Naive | Black | Hull-White | SABR0 | Normal |
|---|---|---|---|---|---|---|
| Total profit | 6,761,605 | 65,099 | 454,409 | -180,515 | -266,372 | -197,936 |
| Maximum profit | 7,641,803 | 1,100,569 | 1,419,573 | 918,911 | 841,282 | 906,230 |
| Minimum profit | -105,017 | -32,825 | 0 | -269,337 | -348,096 | -286,036 |

Table 8. Cumulative profit over the analysis period (July 2002 – June 2005)

The most interesting analysis, and the yard stick for the contest, is the standard deviation of the profit. If the hedging was perfect it would be 0. Obviously none of the models can claim that fame. The unhedged swaption position created a daily standard deviation of around 190,000 where all the hedged portfolios are below 40,000. The first (reassuring) conclusion is that hedging is generally efficient; it reduces significantly the risk.

Among the four real competitors (Black, normal, Hull-White and SABR0), the contest winner is the *Hull-White model* (27,000) and the looser is the Black model (35,000). The in-the-model Black delta is performing worst that all the out-of-the-model hedging. The normal-like models (Hull-White, SABR0, normal) all have similar results. Surprisingly[7]the naive approach is performing almost as well as the best model.

| | Swaption | Naive | Black | Black $\Delta$ | Hull-White | SABR0 | Normal |
|---|---|---|---|---|---|---|---|
| Standard deviation | 190,875 | 27,636 | 34,828 | 40,054 | 26,935 | 28,716 | 26,952 |

TABLE 9. Standard deviation of the profit over the analysis period (July 2002 – June 2005)

It is interesting to extend the naive approach by computing the hedging efficiency for several fixed hedging ratios. Figure 5 reports the experiment. The parabola-like curve is the profit standard deviation for hedging ratios between 40 and 60%. The minimum is at the ratio of 51.75% and has a value of 26,900. The figure also represents the extend of the hedging ratio for different models and the level of profit standard deviation. The Hull-White and normal models have their hedging ratios close to the minimum ($[51.40, 52.37]$ and $[51.30, 52.21]$ respectively). The SABR0 model is slightly above ($[51.84, 55.37]$) and the Black model significantly below ($[43.31, 47.32]$). This emphasize again that the delta hedging performance depends on the delta level and for a given model this level is quite stable through time. For ATM option a hedging performance at the level of the best model used (even marginally better) can be obtained by using a fixed ratio. This is bad news for modellers (and perhaps calibrators) as a naive and fixed approach is competing with them. On the good news side the best model is similar in performance to the best naive approach with the immense advantage to be usable with off-the-money swaptions and other instruments.

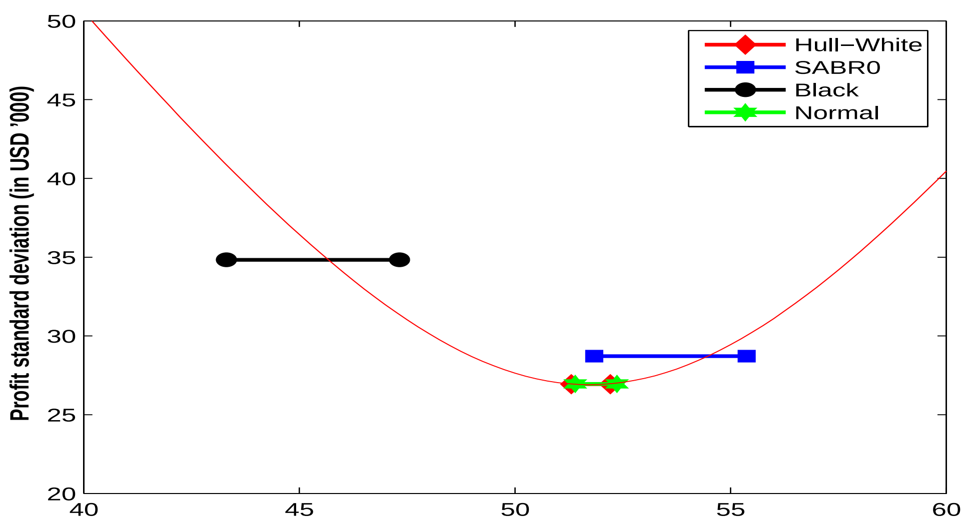



FIGURE 5. Delta hedging level and profit standard deviation for the naive fixed hedging ratio approach and different models (July 2002 – June 2005).

With only one number, it runs the risk to be dependent on the period on which it is computed. For that reason we also compute the same standard deviations for 50 consecutive days periods. We do this for every day. Obviously the figures obtained are not independent as the second period

[7]Maybe this is not really surprising as the naive 50% hedging is equivalent to the in-the-model delta for the normal model.

overlaps the first by 49 days. But it gives an overview of the results consistency over time (see Figure 6).

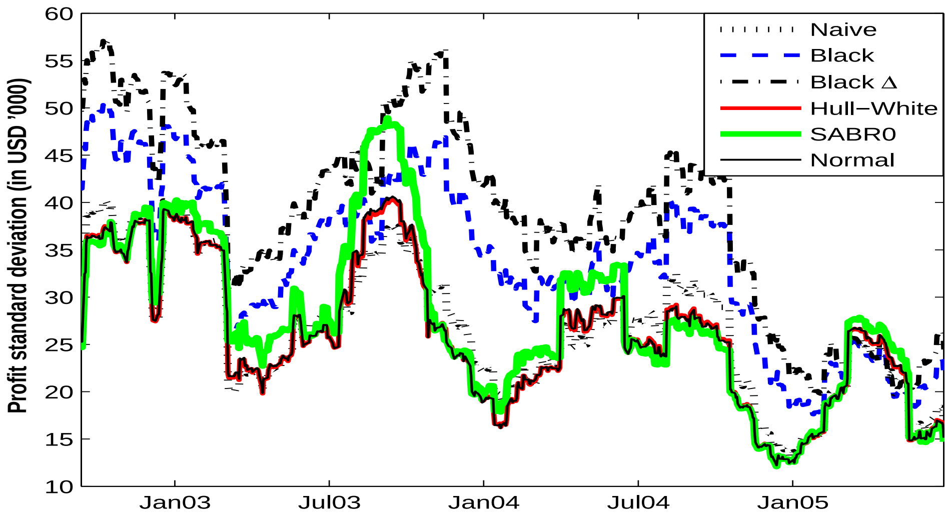



FIGURE 6. Time series of the standard deviation on 50-day periods (June 2002 – June 2005).

The Hull-White model performs better than the Black one on around 90% of the 50-day periods. SABR wins over the same Black around 80% of the periods. Maybe surprisingly, Hull-White wins over SABR about 70% of the time. The difference between Hull-White and SABR was the most significant in 2003. In the last year of data (June 04 – June 05), the SABR model was marginally better. One difference between the two models is the smile fitting for the stochastic volatility model. One possible explanation is that part of the smile reflects demand and supply and not purely the market modeling and expectation. Perhaps the instantaneous correlation between the volatility and the rate changes in the model does not reflect the real dynamic of the market (see next paragraph) but only flow idiosyncrasy.

Another interesting figure is the correlation between the daily profit and the change of forward rate. For the Hull-White hedged portfolio this correlation is almost 0 (2%) while for the Black hedged portfolio it is around 65%. In some sense it means that in the Black model the volatility and the forward rate parameters are not exactly what their names say they are. The volatility parameter contains some part of the rates. The delta hedging is not hedging this part. To be convinced that the volatility parameter contains some rate information, one can also note that the correlation between the change of Black implied volatility and change of forward is close to 90%.

## 6. At-the-money straddle

The delta of at-the-money (ATM) straddles is analysed for some models. The ATM straddles are the most liquid volatility instruments. The reason for this is that the combination of a call (payer) and a put (receiver) gives a double exposure to volatility and (almost) cancel the exposure to the underlying. An exposure to the volatility is created without (local) exposure to the underlying. This is a pure volatility trade.

The question answered here are: How small is the exposure to the underlying? How good is the cancellation process between the payer and the receiver?

The questions are analysed for two models from a theoretical view point (normal and Black) and from a practical one for all of them.

6.1. **Normal.** For the normal swaption model on forward rate, the in-the-model hedging for receiver is $N(-\kappa)$ of the underlying and for payer is $N(\kappa)$ (see Appendix for formulas). For ATM options ($F = K$), $\kappa = 0$ and we obtain a total nominal of receiver swap of

$$N(-\kappa) - N(\kappa) = N(0) - N(0) = 0.$$

As a conclusion, for the normal model, the delta is *perfectly* cancelled between ATM receiver and payer, both have a 50% delta. The straddle is a pure *volatility* trade.

6.2. **Black swaption.** For the Black swaption model, the in-the-model delta for receiver is $N(-d_1)$ and for payer of $N(d_1)$. For ATM swaptions, $F = K$ and so $d_1 = \sigma\sqrt{T}/2$. The total delta in term of receiver swap is

$$N(-d_1) - N(d_1) = 1 - 2N(d_1) = 1 - 2N(\sigma\sqrt{T}/2) < 0.$$

To have no exposure to the underlying, the strike of the straddle should be such that $d_1 = 0$ or

$$K = F\exp(\sigma^2 T/2).$$

For that strike, both the receiver and the payer swaptions have a delta of 50%.

6.3. **Numerical results.** We used the same models with the same data as in the previous section. The total straddle sensitivity with different models is given in Table 10.

| Model | SABR 0 | Normal | Δ norm | H-W | Δ HW | Black | Del Bl | SABR 1 | SABR nc |
|---|---|---|---|---|---|---|---|---|---|
| Sensi | -205 | -148 | 0 | -138 | -139 | 257 | 405 | 643 | -95 |
| Perc. | 4.76 | 3.42 | 0 | 3.19 | 3.22 | -5.96 | -9.38 | -14.89 | 2.20 |

TABLE 10. Sensitivity of ATM straddle in different models

As expected, for normal like model the residual is smaller. Using the in-the-model delta for the normal model gives a completely flat exposure. For the Black model, the in-the-model delta is slightly above 45% for the receiver and close to 55% for the payer. In term of receiver swap, the the difference is 45%-55%, or almost -10%, a number than can be recognized in the table. Note also that the normal-like model gives a positive percentage of the receiver swap while the Black-like model give a negative exposure. Not only the size of the exposure is different but its sign also. In one case the position is long the market and in the other case it is short.

## 7. CONCLUSION

The most important fact to note in this study on *out-of-the-model* delta is that the *model dynamic* is the most significant factor. Even if models are calibrated to the same prices the difference in the delta can reached 10% or more of the underlying. The models studied can be grouped depending on their dynamic. The models with rates following a geometric Brownian motion can be clearly distinguished from the ones with arithmetic Brownian motion. The smile also plays a role but to a lesser extend. Even by changing significantly the shape of the smile, the arithmetic Brownian motion risk figures can not be confused with geometrical ones. Similarly changing the volatility level does not affect the risk significantly. Replication of current prices (even all of them) is not enough to ensure a correct hedging of a book. The problem will only appear through time and with the movements of the market. Choosing the model dynamic is the most important decision the hedger (trader or risk manager) has to take.

Obviously the sensitivity difference creates a difference in the hedging efficiency. A delta hedging contest is run with three years of historical data. The hedging efficiency measured by the standard deviation of the profit differs by almost 30% between the best and the worst model. The normal-like dynamic models perform consistently better. The standard Black model hedging efficiency is the poorest in most of the periods. The extended Vasicek model is the best performing model, even performing better than a stochastic volatility model that reproduces the market smile.

## Appendix A. Numerical procedures

This appendix briefly describes the way the number of this note are computed. For the Black model the in-the-model is given by (for receiver and payer) (see [5, Section 20.4])

$$\Delta_R = N(-d_1) \quad \Delta_P = N(d_1) \quad \text{with} \quad d_1 = \frac{1}{\sigma\sqrt{t}}\left(\ln(F/K) + \frac{1}{2}\sigma^2 t\right)$$

for the normal model (see [6])

$$\Delta_R = N(-\kappa) \quad \Delta_P = N(\kappa) \quad \text{with} \quad \kappa = \frac{1}{\sigma\sqrt{t}}(F-K)$$

and with the Hull-White model (see [2])

$$\Delta_R = \frac{\sum_{i=0}^n c_i P(0,t_i) N(\kappa+\alpha_i)\nu(0,t_i)}{\sum_{i=0}^n c_i P(0,t_i)\nu(0,t_i)} \quad \Delta_P = -\frac{\sum_{i=0}^n c_i P(0,t_i) N(-\kappa-\alpha_i)\nu(0,t_i)}{\sum_{i=0}^n c_i P(0,t_i)\nu(0,t_i)}$$

where $\kappa$ is given implicitly by $\sum_{i=0}^n c_i P(0,t_i)\exp(-1/2\alpha_i^2 - \alpha_i\kappa) = 0$.

For the computation of the sensitivities, the following procedure is used. Firstly the yield curve is reconstructed once by moving all market rates up by one basis point and once by moving them down. For each of these shifted curves the value of the instrument is recomputed. The sensitivity is the *symmetrical difference* of the prices (price with rate up minus price with rate down divided by two). This procedure is used for discounting the swap cash-flows and all the models except the Hull-White model.

The Hull-White pricing of a swaption involves solving a non-linear equation and applying this procedure would require solving it several times. To avoid this we use a mixture of theoretical approach and numerical technique. The details of which can be found in [3].

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