On the Term Structure of Interest Rates with Basis Spreads, Collateral and Multiple Currencies *

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Outlines

This talk is based on the following three works:

- **M.Fujii, Y.Shimada and A.Takahashi, "A Note on Construction of Multiple Swap Curves with and without Collateral" (July, 2009). CARF Working Paper Series No. CARF-F-154. Available at SSRN: http://ssrn.com/abstract=1440633**
- **M.Fujii, Y.Shimada and A.Takahashi, "A Survey on Modeling and Analysis of Basis Spreads" (December, 2009). CARF Working Paper Series No. CARF-F-195. Available at SSRN: http://ssrn.com/abstract=1520619**
- **M.Fujii, Y.Shimada and A.Takahashi, "A Market Model of Interest Rates with Dynamic Basis Spreads in the Presence of Collateral and Multiple Currencies" (December, 2009). CARF Working Paper Series No. CARF-F-196. Available at SSRN: http://ssrn.com/abstract=1520618**

General Needs

- **Single Currency products**
  - **Swap**
  - **Swaption, Cancellable Swap, Cap/Floor.**
  - **TARN/Callable/Range Accrual of CMS, CMS-spread, Inverse Floater, etc.**
- **Multi-Currency products**
  - **Short term Vanilla FX products**
  - **Cross Currency Swap**
  - **TARN/Callable/Knockout of PRDC, PRDC with chooser option**
  - **Quantoed CMS, CMS-spread, etc.**

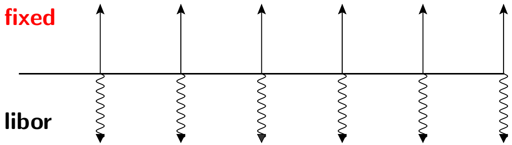
- **We need a term structure model in multi-currency environment.**

Textbook-style Implementation

- Basic Setup(an Example)
  - Complete probability space $(\Omega, \mathcal{F}, P)$ on which d-dimensional Brownian motion W is defined. $\{\mathcal{F}_t\}_{0 \le t \le \bar{T}}$: Augmented Brownian filtration
  - $r = \{r(t); 0 \le t \le \bar{T}\}$: The instantaneous risk-free short rate following an Itô process
  - $\beta(t)$: Price of the money market account at time t, where $\beta(t) = \exp\left(\int_0^t r(s)ds\right)$.
  - $P_{t,T}$: Price of the risk-free zero-coupon bond with maturity T at time t
  - $P_{t,T} = E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\right]$ $(E_t^Q[\cdot] \equiv E^Q[\cdot|\mathcal{F}_t])$
  - Q: The Equivalent Martingale Measure(EMM) or the risk-neutral measure, where the numeraire is the money market account.
  - Asset prices as well as all the factors and indexes affecting the asset prices are assumed to follow Itô processes.

Textbook-style Implementation

- **Curve Construction**
  - **Most Standard IRS : Fixed-vs-Libor**



- **Exchange Fixed Rate (Swap Rate) with Libor**
- **Market Quotes are the Swap Rates for various terms**

$$\mathrm{IRS}_M(t)\sum_{m=1}^{M}\Delta_m P_{t,T_m}=\sum_{m=1}^{M}\delta_m P_{t,T_m}E_t^{\mathcal{T}_m}[L(T_{m-1},T_m)]$$

Here, $P_{t,T}$ is the price of risk-free zero-coupon bond, and $E_t^{\mathcal{T}_m}[\cdot]$ denotes the forward expectation with P_{t,T_m} as the numeraire.

Textbook-style Implementation

- **Treat Libor as risk-free rate**

$$E_t^{\mathcal{T}_m}[L(T_{m-1},T_m)] = \frac{1}{\delta_m}\left(\frac{P_{t,T_{m-1}}}{P_{t,T_m}} - 1\right)$$

- **Determine $\{P_{t,T}\}$ by Bootstrap**

$$P_{t,T_M} = \frac{P_{t,T_0} - IRS_M(t)\sum_{m=1}^{M-1}\Delta_m P_{t,T_m}}{1 + IRS_M(t)\Delta_M}$$

- **Repeat the bootstrap for each currency $\Rightarrow$ Discounting rates (and hence, forward Libors) for each currency**

Textbook-style Implementation

- **Implementation in Multi-Currency Environment**
  - **Initial conditions**
    - **spot FXs**
    - **discounting rate for each currency**
  - **Simulation based on**
    - **Arbitrage-free dynamics of risk-free rate**
    - **Arbitrage-free dynamics of spot FX**
  - **Derive reference rates (such as Libor, Swap rate, etc) from simulated discounting rate.**
  - **Calibration on volatilities and correlations**

$\Rightarrow$ **Basically, we are done..**

Problems in Textbook-style Implementation

- **Is there any problem for the textbook-style implementation?**
  $\Rightarrow$ Yes,...., and in Very Critical way...
- **The potential loss can be a few percentage points of notional outstanding for IR/FX related trades...**

Problems in Textbook-style Implementation

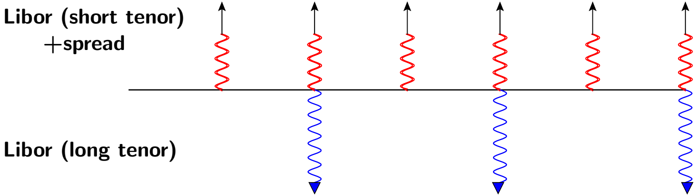
We will see...

List of problems:

- **wrong discounting** for secured (collateralized) trades
- **mispricing** of various important swaps
  - Tenor Swap (TS)
  - Cross Currency Swap (CCS)
    $\Rightarrow$ FX Forward
  - Overnight Index Swap (OIS)

Problems in Textbook-style Implementation

- Tenor Swap (TS)[1]



- Textbook-style Implementation ⇒ Zero spread.
- Market: Spread is quite significant and volatile since late 2007.

[1] It is also common that payment of short-tenor Leg is compounded and paid at the same time with the other Leg.

Problems in Textbook-style Implementation

Historical data for JPY 3m/6m Tenor Swap Spread

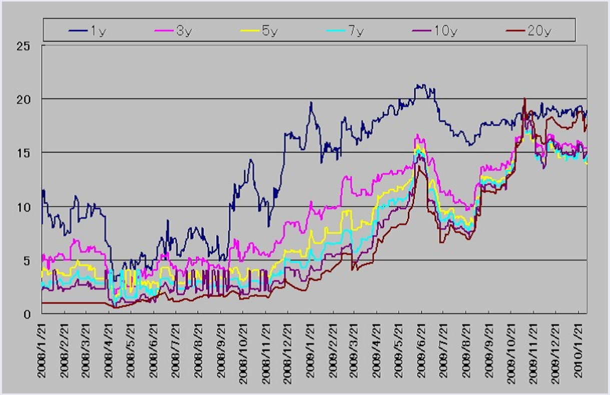


Figure: Source:Bloomberg

Problems in Textbook-style Implementation

Historical data for USD 3m/6m Tenor Swap Spread

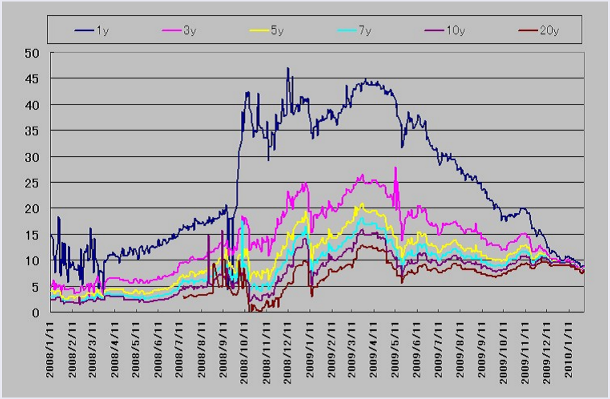
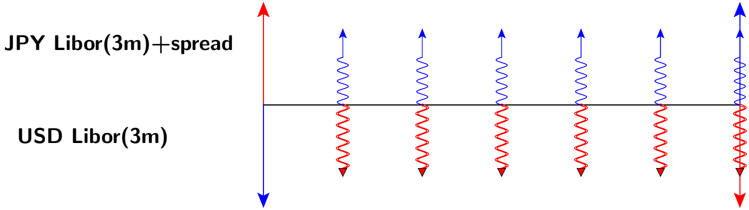


Figure: Source:Bloomberg

Problems in Textbook-style Implementation

- Cross Currency Swap (CCS)



- Textbook-style Implementation ⇒ Zero spread.
- Market:
  - Spread is quite significant and volatile for long time.
  - Drastic/Rapid change in recent years.

Problems in Textbook-style Implementation

Historical data for USDJPY CCS spread

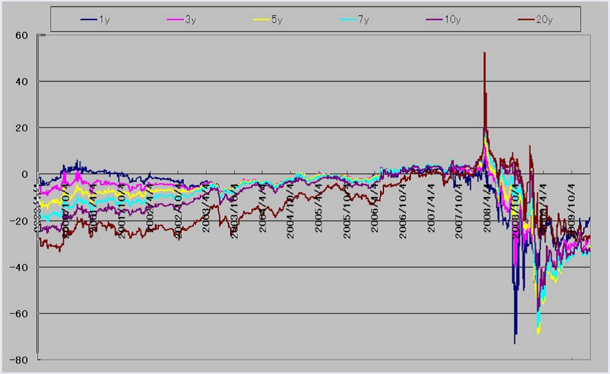
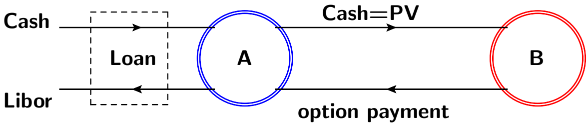


Figure: Source:Bloomberg

Problems in Textbook-style Implementation

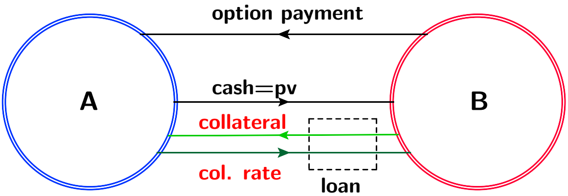
- **Unsecured Funding/Contract (old picture)**



- **Libor is unsecured offer rate in the interbank market.**
- **Libor discounting is appropriate for unsecured trades between financial firms with Libor credit quality.**
- **Libor discounting makes the present value of Loan zero.**

Problems in Textbook-style Implementation

- **Collateralized (Secured) Contract (current picture)**



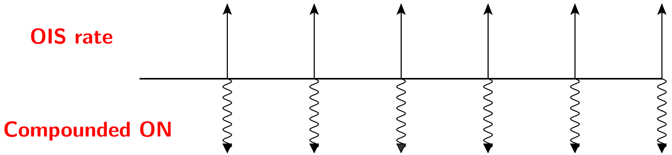
  - **No outright cash flow (collateral=PV)**
  - **No external funding is needed.**
  - **Funding is provided by collateral agreement**
    ⇒ Libor discounting is inappropriate.

Problems in Textbook-style Implementation

- Collateralized OTC Derivatives Market
  - 30%(2003) → 65% (2009) in terms of trade volume (ISDA)
  - Fixed income trade among financial firms are mostly collateralized
  - 84% of collateral(received) is Cash (USD=49.4%, EUR=29.5%)
  - Collateral rate on cash is the overnight rate controlled by the central bank: (Fed-Fund Rate, EONIA, MUTAN...)
  - Remainings are mostly government securities

Problems in Textbook-style Implementation

- **Overnight Index Swap (OIS)**



- **Floating side: Daily compounded ON rate**
- **Usually, there is only one payment for $< 1yr$.**
- **Market Quote : fixed rate, called OIS rate.**

Problems in Textbook-style Implementation

Historical data for USD&JPY Libor-OIS spread

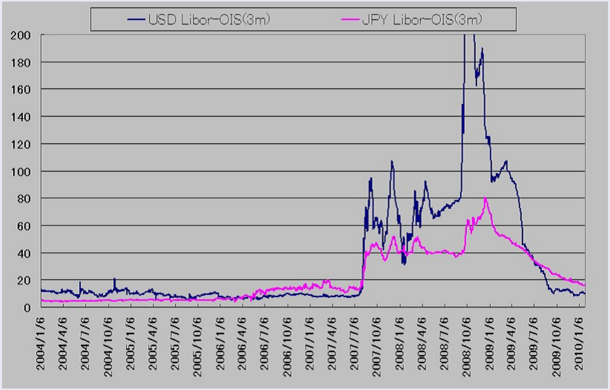


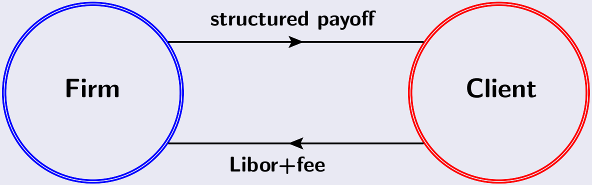
Figure: Source:Bloomberg

Implications for Financial Firms

- Wrong discounting for secured trades
  - Mispricing of future cash flow (over/under-estimate)
  - Significant impact on multi-currency trades where there usually exist final notional exchanges.
    Change$\sim$Notional$\times$Duration$\times$(Diff. of discounting rate)
- Inconsistency with CCS
  - Implied FX forwards are off the market
  - Mispricing of foreign Libors
  - Long-dated FX products are most severely affected.
- Wrong forward Libors
  - Overestimation of forward Libors with short tenors
  - Accumulation of receiver position of Libors with short tenor
    $\Rightarrow$ See next page example.

Implications for Financial Firms

Example: Funding legs of structured products



- **Tenor of Libor is usually 3m, or 6m.**
- **Simulated discounting rate is based on 6m Libor. (Standard IRS convention for JPY)**
- **Overestimation of value of receipt of 3m Libor**
- **It is easy to make deals with 3m Libors.**

Loss∼Notional(3m Funding)×PV01(or Annuity)×TS spread

Review of Recent Works and Un-addressed Issues

- **M.Johannes and S.Sundaresan (2007)**
  - **Point out the importance of collateralization on swap rates.**
  - **Provide a pricing formula for a collateralized contract.**
  - **Introducing an unobservable "convenience yield" and put more emphasis on the empirical study for the dynamics of the swap rate and the convenience yield in the US market.**
- **V.Piterbarg (2010)**
  - **Pricing formula for the collateralized stock options similar to the one given in M.Johannes et.al. (2007).**
  - **Treating partially collateralized case, but the counter-party default risk is neglected.**

Review of Recent Works and Un-addressed Issues

- **Kijima et.al. (2009)**
  - Consistent CCS pricing by separating JPY discounting and Libor curves, while assuming USD Libor as risk-free rate, and a numerical demonstration using a simple short-rate based model.
  - No discussion for collateralization and tenor spreads.
  - As for curve construction, it is a conventional method being used by US financial firms for many years where USD Libor is their funding cost.

$$IRS_N \sum_{n=1}^{N} \Delta_n P_{t,T_n} = \sum_{n=1}^{N} \delta_n E_t^{\mathcal{T}_n}[L(T_{n-1},T_n;\tau)]P_{t,T_n}$$

$$N_{JPY}\left\{-P_{t,T_0} + \sum_{n=1}^{N} \delta_n \left(E_t^{\mathcal{T}_n}[L(T_{n-1},T_n;\tau)] + b_N\right) P_{t,T_n} + P_{t,T_N}\right\}$$

$$= f_x(t)\left\{-P^{\$}_{t,T_0} + \sum_{n=1}^{N} \delta^{\$}_n E_t^{\mathcal{T}_n,\$}[L^{\$}(T_{n-1},T_n;\tau)]P^{\$}_{t,T_n} + P^{\$}_{t,T_N}\right\} \ (= 0)$$

$$\Rightarrow \sum_{n=1}^{N} \left(\Delta_n IRS_N + \delta_n b_N\right) P_{t,T_n} = P_{t,T_0} - P_{t,T_N}$$

Review of Recent Works and Un-addressed Issues

- **Ametrano and Bianchetti (2009)**
  - **Bootstrapping the swap quotes within each tenor separately, assuming "segmentation" of the market.**
  - **Arbitrage possibility due to multiple discounting curves within single currency.**
- **Bianchetti (2008)**
  - **Using FX analogy to remove arbitrage possibility in multi-curve setup.**
  - **Calibration of basis spreads needs to be done by quanto correction.**
  - **Curve construction cannot be separated from option calibration.**
  - **No guarantee that one can recover the observed basis spreads with reasonable size of volatility and correlation.**

Review of Recent Works and Un-addressed Issues

- **F.Mercurio (2008)**
  - **Introducing an efficient simulation scheme with multiple curves in Libor Market Model in single currency environment.**
  - **Referring to the work of Ametrano et.al. (2009) for details of curve construction and assuming the existence of constructed yield curves.**
- **F.Mercurio (March 2010)**
  - **Adopting the OIS-based curve construction in single currency, which is equivalent to a result given in one of our works ("A note on construction of multiple swap curves...(Jul. 2009)").**
  - **Assuming the independence between OIS and Libor-OIS spreads to get analytical tractability.**
  - **Proposing the specific form of volatility functions for the OIS process to retain the consistency among simple rates with different tenors in OIS.**

Un-addressed Issues in Existing Works

- **Un-addressed issues in these works**
  - **No discussion for the term structure construction under the collateralization.**
    - **It is crucial for the model to handle the situation where the payment currency and the collateral currency are different.**
    - **Example: CCS. Moreover, US financial firms to prefer USD cash collateral even for the JPY single currency products.**
  - **In addition to the collateralization issues, existing models can work only in single currency environment.**
    - **If financial firms really adopt these models, they are forced to have a set of curves for each currency desk, but they are inconsistent with cross currency markets.**
    - **It makes impossible to carry out the consistent risk-management for all currencies across the desks, which is crucial for most of the financial institutions.**

Criteria for the New Model

Criteria

- **Consistent discounting/forward curve construction**
  - **Price all types of IR swaps correctly: (OIS, IRS, TS)**
  - **Take collateralization into account.**
  - **Maintain consistency in multi-currency environment (FX forward, CCS, MtMCCS)**
- **Stochastic Modeling of Basis spreads**
  - **Systematic calibration procedures**
  - **Allow volatility risk management**

Contributions of Our Works

- "A note on construction of multiple swap curves...(Jul. 2009)"
    - Extend the formula in M.Johannes et.al. (2007) to the contract whose payment and collateral currencies are different.
    - Provide consistent term structure construction in the presence of basis spreads and collateral in a multi-currency environment for the first time.
- "A market model of interest rates...(Dec. 2009)"
    - Making the multi-currency curve construction more tractable by a simplifying assumption while maintaining the flexibility of calibration to the observable market quotes.
    - General multi-currency HJM framework in the presence of collateral and stochastic basis spreads.

First framework ready to be implemented for consistent pricing and risk-management in global derivatives business.

Pricing under the Collateralization

- **Assumption**
  - Continuous adjustment of collateral amount
  - Perfect collateralization by Cash
  - No threshold

- **Comments**
  - Daily margin call is the market best practice.
  - By making use of Repo / Reverse-Repo, other collateral assets can be converted into the equivalent amount of cash collateral.
  - General Collateral (GC) repo rate closely tracks overnight rate.

Pricing under the Collateralization

Proposition

T-maturing European option under the collateralization is given by

$$
\begin{aligned}
h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right] \\
\text{where,} & \\
y^{(j)}(s) &= r^{(j)}(s) - c^{(j)}(s)
\end{aligned}
$$

- $h^{(i)}(T)$: option payoff at time T in currency i
- collateral is posted in currency j
- $c^{(j)}(s)$: instantaneous collateral rate of currency j at time s
- $r^{(j)}(s)$: instantaneous risk-free rate of currency j at time s
- Q_i: Money-Market measure of currency i

Pricing under the Collateralization

- Collateral amount in currency j at time s is given by $\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}$, which is invested at the rate of $y^{(j)}(s)$:

$$
\begin{aligned}
h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T)\right] \\
&\quad + f_x^{(i,j)}(t) E_t^{Q_j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s)\left(\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}\right) ds\right] \\
&= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds\right].
\end{aligned}
$$

  - $f_x^{(i,j)}(t)$: Foreign exchange rate at time t representing the price of the unit amount of currency "j" in terms of currency "i".

Note that $X(t) = e^{-\int_0^t r^{(i)}(s)ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds$ is a Q_i-martingale. Then, the process of the option value is written by

$$
dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right) h^{(i)}(t) dt + dM(t)
$$

with some Q_i-martingale M. This establishes the proposition.

Pricing under the Collateralization

Remark : Derivation using the collateral account

- Dynamics of a collateral account in currency j, "$V^{(j)}$", is given by

$$dV^{(j)}(s) = y^{(j)}(s)V^{(j)}(s)ds + a(s)d\left[h^{(i)}(s)/f_x^{(i,j)}(s)\right] ,$$

where $a(s)$ is the number of positions of the derivative. We have

$$V^{(j)}(T) = e^{\int_t^T y^{(j)}(s)ds}V^{(j)}(t)+\int_t^T e^{\int_s^T y^{(j)}(u)du}a(s)d\left[h^{(i)}(s)/f_x^{(i,j)}(s)\right].$$

- Adopt a trading strategy:
  - $V^{(j)}(t) = h^{(i)}(t)/f_x^{(i,j)}(t)$, $a(s) = \exp\left(\int_t^s y^{(j)}(u)du\right)$,

which yields $V^{(j)}(T) = e^{\int_t^T y^{(j)}(s)ds}\left(h^{(i)}(T)/f_x^{(i,j)}(T)\right)$.

- Hence, we obtain

$$\begin{aligned} h^{(i)}(t) &= V^{(j)}(t)f_x^{(i,j)}(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}V^{(j)}(T)f_x^{(i,j)}(T)\right] \\ &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right] . \end{aligned}$$

Pricing under the Collateralization

Corollary

If payment and collateral currencies are the same, the option value is given by

$$\begin{aligned} h(t) &= E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] \\ &= D(t,T)E_t^{\mathcal{T}^c}\left[h(T)\right] \ . \end{aligned}$$

In the 2nd line, we have defined collateralized forward measure $\mathcal{T}^c$, where the collateralized zero coupon bond

$$D(t,T) = E_t^Q\left[e^{-\int_t^T c(s)ds}\right]$$

is used as the numeraire.

Curve Construction in Single Currency

- **Collateralized Overnight Index Swap**
  - **payment and collateral currencies are the same**
  - **collateral rate is given by the overnight rate**
- **Condition for the length-N OIS rate:**

$$\mathrm{OIS}_N(t)\sum_{n=1}^{N}\Delta_n E_t^Q\left[e^{-\int_t^{T_n}c(s)ds}\right]$$
$$=\sum_{n=1}^{N}E_t^Q\left[e^{-\int_t^{T_n}c(s)ds}\left(e^{\int_{T_{n-1}}^{T_n}c(s)ds}-1\right)\right]$$

or, equivalently,

$$\mathrm{OIS}_N(t)\sum_{n=1}^{N}\Delta_n D(t,T_n)=D(t,T_0)-D(t,T_N)\ .$$

Then, the collateralized zero coupon bond price can be bootstrapped as

$$D(t,T_N)=\frac{D(t,T_0)-\mathrm{OIS}_N(t)\sum_{n=1}^{N-1}\Delta_n D(t,T_n)}{1+\mathrm{OIS}_N(t)\Delta_N}.$$

Curve Construction in Single Currency

- **Collateralized IRS**

$$\mathrm{IRS}_M(t)\sum_{m=1}^{M}\Delta_m D(t,T_m)=\sum_{m=1}^{M}\delta_m D(t,T_m)E_t^{\mathcal{T}_m^c}[L(T_{m-1},T_m;\tau)]$$

- **Collateralized TS**[2]

$$\sum_{n=1}^{N}\delta_n D(t,T_n)\left(E_t^{\mathcal{T}_n^c}\left[L(T_{n-1},T_n;\tau_S)\right]+TS_N(t)\right)$$
$$=\sum_{m=1}^{M}\delta_m D(t,T_m)E_t^{\mathcal{T}_m^c}\left[L(T_{m-1},T_m;\tau_L)\right]$$

Market quotes of collateralized OIS, IRS, TS, and proper spline method allow us to determine

$$\{D(t,T)\},\quad \{E_t^{\mathcal{T}_m^c}\left[L(T_{m-1},T_m,\tau)\right]\}$$

for all the relevant T, T_m and tenor τ of Libor.

[2] The impact from the possible compounding of the short-tenor Leg is negligible.

Curve Construction in Multiple Currencies

- **OIS, IRS, TS**
  - Repeat the same procedures in the previous section.
  - Obtain $\{D^{(i)}(t,T)\}$, $\{E_t^{\mathcal{T}^c_{m},(i)}[L^{(i)}(T_{m-1},T_m,\tau)]\}$ for each currency "i"
- **CCS and FX forwards**
  - Various practical issues under the most generic setup.
    - forward expectation involves covariance between $y(t)=r(t)-c(t)$ and other variables.
    - No separate market quote is available for each collateral currency.
    - ...

Curve Construction in Multiple Currencies

- **Collateralized FX Forward: $f_x^{(i,j)}(t,T)$**
  - **Amount of currency i to be exchanged by the unit amount of currency j, assuming collateralization is done by currency k:**

$$\begin{aligned} 0 &= f_x^{(i,j)}(t,T)E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}e^{\int_t^T y^{(k)}(s)ds}\right] \\ &\quad -f_x^{(i,j)}(t)E_t^{Q_j}\left[e^{-\int_t^T r^{(j)}(s)ds}e^{\int_t^T y^{(k)}(s)ds}\right] \end{aligned}$$

and then,

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)}\left(\frac{E_t^{\mathcal{T}(j)}\left[e^{\int_t^T y^{(k)}(s)ds}\right]}{E_t^{\mathcal{T}(i)}\left[e^{\int_t^T y^{(k)}(s)ds}\right]}\right).$$

  - **If the spread y is stochastic, the currency triangle, such as**

$$\text{USD/JPY} \times \text{EUR/USD} = \text{EUR/JPY},$$

holds only among those with the same collateral currency.

Curve Construction in Multiple Currencies

Assumption

Spread between the risk-free and collateral rate of each currency

$$y^{(i)}(t) = r^{(i)}(t) - c^{(i)}(t)$$

is a deterministic function of time.

We will achieve:

- **Enough flexibility to fit available market quotes.**
- **Currency triangle relation holds among FX forwards.**

Curve Construction in Multiple Currencies

- **Remark:**
  - **Option in currency i collateralized with currency j under the assumption of deterministic spread:**

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right] \\ &= P^{(i)}(t,T)e^{\int_t^T y^{(j)}(s)ds}E_t^{\mathcal{T}(i)}\left[h^{(i)}(T)\right] \\ &= D^{(i)}(t,T)e^{\int_t^T y^{(j,i)}(s)ds}E_t^{\mathcal{T}(i)}\left[h^{(i)}(T)\right] \ , \end{aligned}$$

where $y^{(j,i)}(s) = y^{(j)}(s) - y^{(i)}(s)$. On the other hand,

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T c^{(i)}(s)ds}e^{\int_t^T y^{(j,i)}(s)ds}h^{(i)}(T)\right] \\ &= D^{(i)}(t,T)e^{\int_t^T y^{(j,i)}(s)ds}E_t^{\mathcal{T}^c_{(i)}}\left[h^{(i)}(T)\right] \ , \end{aligned}$$

and thus,

$$E_t^{\mathcal{T}^c_{(i)}}\left[h^{(i)}(T)\right] = E_t^{\mathcal{T}(i)}\left[h^{(i)}(T)\right] \ .$$

Curve Construction in Multiple Currencies

- When spread y is deterministic, the previous forward FX becomes

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)} = f_x^{(i,j)}(t)\frac{D^{(j)}(t,T)}{D^{(i)}(t,T)}e^{\int_t^T y^{(i,j)}(s)ds},$$

which is independent from the choice of collateral currency.

- Fitting to FX forward
  - Bootstrap the spread $\{y^{(i,j)}(s)\}$ using the relation:

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{D^{(j)}(t,T)}{D^{(i)}(t,T)}e^{\int_t^T y^{(i,j)}(s)ds}$$

  - Except the $y^{(i,j)}$, all the variables are already fixed or observable in the market.
  - It can be used only for relatively short maturities due to liquidity reason.

Curve Construction in Multiple Currencies

- **Fitting to CCS with Constant Notional**

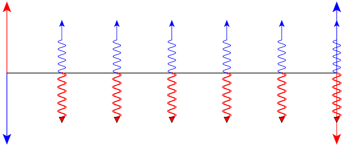
"j" Libor+spread

$N_j = 1$

"i" Libor

N_i(set by spot fx)

col. currency



$$PV_i(t) = -E_t^{Q_i}\left[e^{-\int_t^{T_0} c^{(i)}(s)ds}\right] + E_t^{Q_i}\left[e^{-\int_t^{T_N} c^{(i)}(s)ds}\right]$$

$$+\sum_{n=1}^{N}\delta_n^{(i)} E_t^{Q_i}\left[e^{-\int_t^{T_n} c^{(i)}(s)ds} L^{(i)}(T_{n-1}, T_n; \tau)\right]$$

$$PV_j(t) = -E_t^{Q_j}\left[e^{-\int_t^{T_0}\left(r^{(j)}(s)-y^{(i)}(s)\right)ds}\right] + E_t^{Q_j}\left[e^{-\int_t^{T_N}\left(r^{(j)}(s)-y^{(i)}(s)\right)ds}\right]$$

$$+\sum_{n=1}^{N}\delta_n^{(j)} E_t^{Q_j}\left[e^{-\int_t^{T_n}\left(r^{(j)}(s)-y^{(i)}(s)\right)ds}\left(L^{(j)}(T_{n-1}, T_n; \tau) + B_N^{\mathrm{CCS}}(t)\right)\right]$$

Curve Construction in Multiple Currencies

After simplification,

$$PV_i(t) = -D^{(i)}(t,T_0) + D^{(i)}(t,T_N) + \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t,T_n) E_t^{\mathcal{T}^c_{n,(i)}}\left[L^{(i)}(T_{n-1},T_n;\tau)\right]$$

$$PV_j(t) = -D^{(j)}(t,T_0)e^{\int_t^{T_0} y^{(i,j)}(s)ds} + D^{(j)}(t,T_N)e^{\int_t^{T_N} y^{(i,j)}(s)ds} + \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n)e^{\int_t^{T_n} y^{(i,j)}(s)ds}\left(E_t^{\mathcal{T}^c_{n,(j)}}\left[L^{(j)}(T_{n-1},T_n;\tau)\right] + B_N^{\mathrm{CCS}}(t)\right) ,$$

where $y^{(i,j)}$ is the only unknown.

Bootstrap $\{y^{(i,j)}(s)\}$ using $N_i PV_i(t) = f_x^{(i,j)}(t) PV_j(t)$.

Curve Construction in Multiple Currencies

Interpretation of the spread between the risk-free and collateral rates

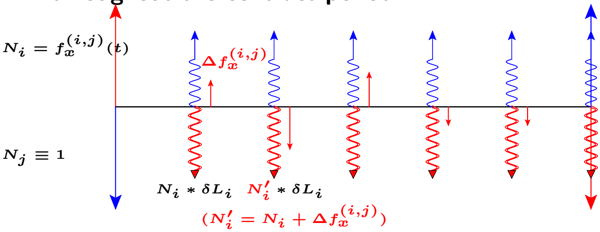
- **The ON rate controlled by the central bank is not necessarily equal to the risk-free rate.**
- **Imposing $r \equiv c$ leaves us no freedom to calibrate FX forwards and CCS.**
- **$y^{(i,j)}$ may be reflecting the difference of the stance between the two central banks of currency "i" and "j".**

Two different types of Cross Currency Swap

- **There are two different types of Cross Currency Swap.**
  - **Constant Notional Cross Currency Swap (CNCCS)**
    CCS which we have already explained
  - **Mark-to-Market Cross Currency Swap (MtMCCS)**
- **Both of the instruments play critical roles in long-dated FX market.**
- **We will see quite different risk characteristics between the twos.**

Two different types of Cross Currency Swap

- Mark-to-Market Cross Currency Swap (MtMCCS)
  - One of the most liquid long-dated cross currency product
  - Smaller FX exposure than CCS with constant notional
  - The notional of the Leg which pays Libor flat (usually USD) is reset at the every start of the Libor calculation period based on the spot FX at that time.
    $(\rightarrow \Delta f_x^{(i,j)} = f_x^{(i,j)}(t+\tau) - f_x^{(i,j)}(t))$
  - The notional and spread of the other leg is kept constant throughout the contract period.



Two different types of Cross Currency Swap

Definition: Libor-OIS (with single payment) spread

$$B(t,T_k;\tau) = L^c(t,T_{k-1},T_k;\tau) - L^{\mathrm{OIS}}(t,T_{k-1},T_k)$$

where

$$\begin{aligned} L^c(t,T_{k-1},T_k;\tau) &= E_t^{\mathcal{T}_k^c}[L(T_{k-1},T_k;\tau)] \\ L^{\mathrm{OIS}}(t,T_{k-1},T_k) &= E_t^{\mathcal{T}_k^c}\left[\frac{1}{\delta_k}\left(\frac{1}{D(T_{k-1},T_k)} - 1\right)\right] \\ &= \frac{1}{\delta_k}\left(\frac{D(t,T_{k-1})}{D(t,T_k)} - 1\right) \end{aligned}$$

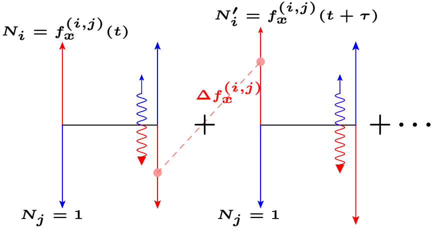
Two different types of Cross Currency Swap

- **j-Leg of T_0-start T_N-maturing MtMCCS collateralized by currency i**
  The notional is fixed, and hence exactly the same with the j-Leg of CNCCS.

$$
\begin{aligned}
PV_j(t) &= -E_t^{Q_j}\left[e^{-\int_t^{T_0}\left(r^{(j)}(s)-y^{(i)}(s)\right)ds}\right] + E_t^{Q_j}\left[e^{-\int_t^{T_N}\left(r^{(j)}(s)-y^{(i)}(s)\right)ds}\right] \\
&+ \sum_{n=1}^{N}\delta_n^{(j)}E_t^{Q_j}\left[e^{-\int_t^{T_n}\left(r^{(j)}(s)-y^{(i)}(s)\right)ds}\left(L^{(j)}(T_{n-1},T_n;\tau)+B_N^{\mathrm{MtM}}(t)\right)\right] \\
&= \sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}\left(B^{(j)}(t,T_n;\tau)+B_N^{\mathrm{MtM}}(t)\right) \\
&+ \sum_{n=1}^{N}D^{(j)}(t,T_{n-1})e^{\int_t^{T_{n-1}}y^{(i,j)}(s)ds}\left(e^{\int_{T_{n-1}}^{T_n}y^{(i,j)}(s)ds}-1\right)
\end{aligned}
$$

Two different types of Cross Currency Swap

- i-Leg of T_0-start T_N-maturing MtMCCS collateralized by currency i



$$PV_i(t) = -\sum_{n=1}^{N} E_t^{Q_i}\left[e^{-\int_t^{T_{n-1}} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\right]$$

$$+\sum_{n=1}^{N} E_t^{Q_i}\left[e^{-\int_t^{T_n} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\left(1+\delta_n^{(i)} L^{(i)}(T_{n-1},T_n;\tau)\right)\right]$$

$$=\sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t,T_n) E_t^{\mathcal{T}^c_{n,(i)}}\left[f_x^{(i,j)}(T_{n-1}) B^{(i)}(T_{n-1},T_n;\tau)\right]$$

Two different types of Cross Currency Swap

- **i-Leg of T_0-start T_N-maturing CNCCS collateralized by currency i (Revisited)**

$$\begin{aligned} PV_i(t) &= N_i\Big\{-E_t^{Q_i}\left[e^{-\int_t^{T_0} c^{(i)}(s)ds}\right] + E_t^{Q_i}\left[e^{-\int_t^{T_N} c^{(i)}(s)ds}\right] \\ &\quad + \sum_{n=1}^{N} \delta_n^{(i)} E_t^{Q_i}\left[e^{-\int_t^{T_n} c^{(i)}(s)ds} L^{(i)}(T_{n-1}, T_n; \tau)\right]\Big\} \\ &= N_i \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t, T_n) B^{(i)}(t, T_n; \tau) \end{aligned}$$

- **N_i is set by the spot FX at the trade inception, and kept constant.**

Par CCS basis spread is obtained by

$$PV_i(t)/f_x^{(i,j)}(t) = PV_j(t)$$

Two different types of Cross Currency Swap

T_0-start T_N-maturing (i,j)-MtMCCS par spread (collateralized by currency i)

$$
\begin{aligned}
&B_N^{\mathrm{MtM}}(t,T_0,T_N;\tau) = \\
&\Bigg[\sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}\times \\
&\left\{\frac{\delta_n^{(i)}}{\delta_n^{(j)}}E_t^{T^c_{n,(i)}}\left[\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(t,T_n)}B^{(i)}(T_{n-1},T_n;\tau)\right]-B^{(j)}(t,T_n;\tau)\right\} \\
&-\sum_{n=1}^{N}D^{(j)}(t,T_{n-1})e^{\int_t^{T_{n-1}}y^{(i,j)}(s)ds}\left(e^{\int_{T_{n-1}}^{T_n}y^{(i,j)}(s)ds}-1\right)\Bigg] \\
&/\sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}
\end{aligned}
$$

- **MtMCCS spread observable in the market can be calibrated by adjusting correlation between $f_x^{(i,j)}$ and Libor-OIS spread $B^{(i)}$.**

Two different types of Cross Currency Swap

T_0-start T_N-maturing (i,j)-CNCCS par spread (collateralized by currency i)

$$
\begin{aligned}
&B_N^{\mathrm{CCS}}(t,T_0,T_N;\tau) = \\
&\quad\Bigg[\sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \times \\
&\quad\left\{ \frac{\delta_n^{(i)}}{\delta_n^{(j)}} \frac{N^{(i)}}{f_x^{(i,j)}(t,T_n)} B^{(i)}(t,T_n;\tau) - B^{(j)}(t,T_n;\tau) \right\} \\
&\quad - \sum_{n=1}^{N} D^{(j)}(t,T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1 \right)\Bigg] \\
&\quad / \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}
\end{aligned}
$$

- **If Libor-OIS spread of i-Leg (USD) is zero, par basis spreads of the two CCSs are exactly the same.**

Two different types of Cross Currency Swap

- Difference of FX exposure between the two CCSs
  - Suppose that we are now at time T after the inception of trade at time t
  - Label the next closest payment time as T_S.
  - j-Leg (eg. JPY) has the same value both for the CNCCS and MtMCCS

i-Leg value of CNCCS at time T:

$$PV_i(T) = N_i \left\{ D^{(i)}_{T,T_S} \delta^{(i)}_S L(T_{S-1}, T_S; \tau) + D^{(i)}_{T,T_N} \right\}$$

$$+ N_i \sum_{n=S+1}^{N} \delta^{(i)}_n E^{Q_i}_T \left[e^{-\int_T^{T_n} c^{(i)}(s)ds} L^{(i)}(T_{n-1}, T_n; \tau) \right]$$

$$= N_i \left\{ D^{(i)}_{T,T_S} \left(1 + \delta^{(i)}_S L^{(i)}(T_{S-1}, T_S; \tau) \right) + \sum_{n=S+1}^{N} D^{(i)}_{T,T_n} \delta^{(i)}_n B^{(i)}(T, T_n; \tau) \right\}$$

Two different types of Cross Currency Swap

i-Leg value of MtMCCS at time T:

$$
\begin{aligned}
PV_i(T) &= f_x^{(i,j)}(T_{S-1})D^{(i)}_{T,T_S}\left(1+\delta_S^{(i)}L^{(i)}(T_{S-1},T_S;\tau)\right) \\
&- \sum_{n=S+1}^{N} E_T^{Q_i}\left[e^{-\int_T^{T_{n-1}} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\right] \\
&+ \sum_{n=S+1}^{N} E_T^{Q_i}\left[e^{-\int_T^{T_n} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\left(1+\delta_n^{(i)}L^{(i)}(T_{n-1},T_n;\tau)\right)\right] \\
&= f_x^{(i,j)}(T_{S-1})D^{(i)}_{T,T_S}\left(1+\delta_S^{(i)}L^{(i)}(T_{S-1},T_S;\tau)\right) \\
&\quad + \sum_{n=S+1}^{N} D^{(i)}_{T,T_n}\delta_n^{(i)}E_T^{\mathcal{T}^c_n,(i)}\left[f_x^{(i,j)}(T_{n-1})B^{(i)}(T_{n-1},T_n;\tau)\right]
\end{aligned}
$$

Two different types of Cross Currency Swap

- **The value of j-Legs are the same between the twos, and remains "~ 1", in terms of currency j.**

i-Leg value in terms of currency j

- CNCCS

$$\frac{PV_i(T)}{f_x^{(i,j)}(T)} = \frac{N_i}{f_x^{(i,j)}(T)}\Big\{D^{(i)}_{T,T_S}\left(1+\delta^{(i)}_S L^{(i)}(T_{S-1},T_S;\tau)\right) + \sum_{n=S+1}^{N} D^{(i)}_{T,T_n}\delta^{(i)}_n B^{(i)}(T,T_n;\tau)\Big\}$$

- MtMCCS

$$\frac{PV_i(T)}{f_x^{(i,j)}(T)} = \frac{f_x^{(i,j)}(T_{S-1})}{f_x^{(i,j)}(T)} D^{(i)}_{T,T_S}\left(1+\delta^{(i)}_S L^{(i)}(T_{S-1},T_S;\tau)\right) + \sum_{n=S+1}^{N} D^{(i)}_{T,T_n}\delta^{(i)}_n E_T^{T^c_n,(i)}\left[\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(T)} B^{(i)}(T_{n-1},T_n;\tau)\right]$$

Two different types of Cross Currency Swap

Summary of CNCCS and MtMCCS

- **Both CCSs have the same par basis spread if $B^{(i)}$ (or, USD Libor-OIS spread) is zero.**
- **Potentially significant mis-pricing.**
- **CNCCS has significant FX exposure.**
- **MtMCCS has only a limited size of FX exposure.**

Term Structure Model with Single Currency

Make the multiple reference rates stochastic consistently with no-arbitrage conditions in an HJM-type framework.

- **Definition: Instantaneous Forward Collateral Rate**

$$c(t,T) = -\frac{\partial}{\partial T} \ln D(t,T)$$

or

$$D(t,T) = \exp\left(-\int_t^T c(t,s)ds \right)$$

Proposition

The SDE of the forward collateral rate under the Money-Market measure Q is given by

$$dc(t,s) = \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right) dt + \sigma_c(t,s) \cdot dW^Q(t) \ ,$$

where W^Q is the d-dimensional Brownian motion under the measure Q.

Term Structure Model with Single Currency

Write the dynamics of $c(t,s)$ as

$$dc(t,s) = \alpha(t,s)dt + \sigma_c(t,s) \cdot dW^Q(t) \ .$$

Applying Itô's formula,

$$\begin{aligned}\frac{dD(t,T)}{D(t,T)} &= \left\{ c(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2} \left\| \int_t^T \sigma_c(t,s)ds \right\|^2 \right\} dt \\ &\quad - \left(\int_t^T \sigma_c(t,s)ds \right) \cdot dW_t^Q \ .\end{aligned}$$

Imposing the fact that the drift rate of $D(t,T)$ is $c(t)$:

$$\begin{aligned}\alpha(t,s) &= \sum_{j=1}^d [\sigma_c(t,s)]_j \left(\int_t^s \sigma_c(t,u)du \right)_j \\ &= \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right) \ .\end{aligned}$$

Term Structure Model with Single Currency

- Instantaneous forward collateral rate $\{c(t,T)\}$ and Libor-OIS spread $\{B(t,T;\tau)\}$ for each tenor fully determine the IR model in Single Currency.

Proposition

The SDE of Libor-OIS spread in Money-Market measure is given by

$$dB(t,T;\tau)/B(t,T;\tau)$$
$$= \sigma_B(t,T;\tau)\cdot\left(\int_t^T \sigma_c(t,s)ds\right)dt + \sigma_B(t,T;\tau)\cdot dW^Q(t)\ .$$

$B(\cdot,T;\tau)$ is a martingale under the collateralized forward measure $\mathcal{T}^c$:

$$dB(t,T;\tau) = B(t,T;\tau)\sigma_B(t,T;\tau)\cdot dW^{\mathcal{T}^c}(t)$$

Maruyama-Girsanov's theorem indicates

$$dW^{\mathcal{T}^c}(t) = \left(\int_t^T \sigma_c(t,s)ds\right)dt + dW^Q(t)\ .$$

Term Structure Model with Single Currency

Summary in Single Currency Environment

- Bootstrap $\{D(t,T)\}$ and $\{E_t^{\mathcal{T}_m^c}[L(T_{m-1},T_m;\tau)]\}$ from OIS, IRS and TS.
- Construct continuous curves for the forward collateral rate and Libor-OIS spread of each tenor.
  $\Rightarrow$ Initial conditions: $\{c(t,s)\}$, $\{B(t,T;\tau)\}$.
- Simulation based on SDEs:

$$dc(t,s) = \sigma_c(t,s)\cdot\left(\int_t^s \sigma_c(t,u)du\right)dt + \sigma_c(t,s)\cdot dW^Q(t)$$

$$dB(t,T;\tau)/B(t,T;\tau)$$

$$= \sigma_B(t,T;\tau)\cdot\left(\int_t^T \sigma_c(t,s)ds\right)dt + \sigma_B(t,T;\tau)\cdot dW^Q(t)$$

- Calibration to the Option Market.

Term Structure Model with Multiple Currencies

SDE for the spot FX process

$$\begin{aligned} df_x^{(i,j)}(t)/f_x^{(i,j)}(t) & \\ &= \left(r^{(i)}(t) - r^{(j)}(t)\right) dt + \sigma_X^{(i,j)}(t) \cdot dW^{Q_i}(t) \\ &= \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(s)\right) dt + \sigma_X^{(i,j)}(t) \cdot dW^{Q_i}(t) \end{aligned}$$

The Maruyama-Girsanov's theorem indicates

$$dW^{Q_i}(t) = \sigma_X^{(i,j)}(t)dt + dW^{Q_j}(t) \ ,$$

which determines the SDEs of the foreign interest rates.

Term Structure Model with Multiple Currencies

Set of SDEs in Multi-Currency Environment

$$\frac{df_x^{(i,j)}(t)}{f_x^{(i,j)}(t)} = \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(s)\right) dt + \sigma_X^{(i,j)}(t) \cdot dW^{Q_i}(t)$$

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s) \cdot \left(\int_t^s \sigma_c^{(i)}(t,u) du\right) dt + \sigma_c^{(i)}(t,s) \cdot dW^{Q_i}(t)$$

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t,T;\tau) \cdot \left(\int_t^T \sigma_c^{(i)}(t,s) ds\right) dt + \sigma_B^{(i)}(t,T;\tau) \cdot dW^{Q_i}(t)$$

$$dc^{(j)}(t,s) = \sigma_c^{(j)}(t,s) \cdot \left[\left(\int_t^s \sigma_c^{(j)}(t,u) du\right) - \sigma_X^{(i,j)}(t)\right] dt + \sigma_c^{(j)}(t,s) \cdot dW^{Q_i}(t)$$

$$\frac{dB^{(j)}(t,T;\tau)}{B^{(j)}(t,T;\tau)} = \sigma_B^{(j)}(t,T;\tau) \cdot \left[\left(\int_t^T \sigma_c^{(j)}(t,s) ds\right) - \sigma_X^{(i,j)}(t)\right] dt + \sigma_B^{(j)}(t,T;\tau) \cdot dW^{Q_i}(t)$$

Term Structure Model with Multiple Currencies

Summary in Multi-Currency Environment

- **Curve Construction**
  - **Bootstrap $\{D^{(i)}(t,T)\}$ and $\{E_t^{\mathcal{T}^c_m,(i)}[L^{(i)}(T_{m-1},T_m;\tau)]\}$ from OIS, IRS and TS of each currency "i".**
  - **Bootstrap $\{y^{(i,j)}(s)\}$ for all the relevant currency pairs from FX forwards and CCS.**
  - **Construct continuous curves using appropriate spline techniques. $\Rightarrow$ Initial conditions: $\{c^{(i)}(t,s)\}$ and $\{B^{(i)}(t,T;\tau)\}$ for each currency, and $\{y^{(i,j)}(s)\}$ for all the relevant currency pairs.**
- **Simulation based on the SDEs in the previous page.**
- **Calibration to the Option Market and MtMCCS.**

Pricing of Single Currency Products

- **Collateralized Swaption on OIS**
  - **T_0-start T_N-maturing forward OIS rate:**

$$\mathrm{OIS}(t,T_0,T_N) = \frac{D(t,T_0) - D(t,T_N)}{A(t,T_0,T_N)}$$

$$A(t,T_0,T_N) = \sum_{n=1}^{N} \Delta_n D(t,T_n)$$

  - **Define annuity measure $\mathcal{A}$, where $A(\cdot,T_0,T_N)$ is the numeraire.**

Collateralized payer swaption on T_0-start T_N-maturing OIS with strike K

$$PV(t) = A(t,T_0,T_N) E_t^{\mathcal{A}}\left[\left(\mathrm{OIS}(T_0,T_0,T_N) - K \right)^+ \right]$$

Pricing of Single Currency Products

Maruyma-Girsanov's theorem indicates

$$dW^{\mathcal{A}}(t) = dW^{Q}(t) + \frac{1}{A(t,T_0,T_N)} \sum_{n=1}^{N} \Delta_n D(t,T_n) \left(\int_t^{T_n} \sigma_c(t,s)ds \right) dt$$

and the SDE of the forward OIS rate is given by

$$\begin{aligned} d\mathrm{OIS}(t,T_0,T_N) \quad &= \quad \mathrm{OIS}(t,T_0,T_N) \left\{ \frac{D(t,T_N)}{D(t,T_0) - D(t,T_N)} \left(\int_{T_0}^{T_N} \sigma_c(t,s)ds \right) \right. \\ &\left. + \frac{1}{A(t,T_0,T_N)} \sum_{n=1}^{N} \Delta_n D(t,T_n) \left(\int_{T_0}^{T_n} \sigma_c(t,s)ds \right) \right\} \cdot dW^{\mathcal{A}}(t) \end{aligned}$$

Pricing of Single Currency Products

- **Collateralized Swaption on IRS**
  - **T_0-start T_N-maturing forward IRS rate:**

$$
\begin{aligned}
\mathrm{IRS}(t,T_0,T_N;\tau) &= \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)L^c(t,T_{n-1},T_n;\tau)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)} \\
&= \frac{D(t,T_0)-D(t,T_N)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)} + \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)} \\
&= \mathrm{OIS}(t,T_0,T_N) + Sp^{\mathrm{OIS}}(t,T_0,T_N;\tau)
\end{aligned}
$$

Collateralized payer swaption on T_0-start T_N-maturing IRS with strike K

$$
\begin{aligned}
&PV(t) \\
&= A(t,T_0,T_N)E_t^{\mathcal{A}}\left[\left(\mathrm{OIS}(T_0,T_0,T_N)+Sp^{\mathrm{OIS}}(T_0,T_0,T_N;\tau)-K\right)^+\right]
\end{aligned}
$$

Pricing of Single Currency Products

$$Sp^{\mathrm{OIS}}(t,T_0,T_N;\tau)=\frac{\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)}$$

SDE for Sp under the $\mathcal{A}$-measure is given by

$$dSp^{\mathrm{OIS}}(t,T_0,T_N;\tau)=Sp^{\mathrm{OIS}}(t)\left\{\frac{1}{A(t)}\sum_{j=1}^{N}\Delta_j D(t,T_j)\left(\int_{T_0}^{T_j}\sigma_c(t,s)ds\right)\right.$$

$$\left.+\frac{1}{A_{sp}(t)}\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)\left(\sigma_B(t,T_n;\tau)-\int_{T_0}^{T_n}\sigma_c(t,s)ds\right)\right\}\cdot dW^{\mathcal{A}}(t)$$

where

$$A_{sp}(t)=\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)$$

Pricing of Multi-Currency Products

FX-(i/j) call option collateralized with currency "k"

$$
\begin{aligned}
PV(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(k)}(s)ds}\left(f_x^{(i,j)}(T)-K\right)^+\right] \\
&= D^{(i)}(t,T)e^{\int_t^T y^{(k,i)}(s)ds} E_t^{\mathcal{T}_{(i)}^c}\left[\left(f_x^{(i,j)}(T,T)-K\right)^+\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{df_x^{(i,j)}(t,T)}{f_x^{(i,j)}(t,T)} &= \sigma_{FX}^{(i,j)}(t,T)\cdot dW^{\mathcal{T}_{(i)}^c}(t) \\
&= \left\{\sigma_X^{(i,j)}(t)+\int_t^T \sigma_c^{(i)}(t,s)ds-\int_t^T \sigma_c^{(j)}(t,s)ds\right\}\cdot dW^{\mathcal{T}_{(i)}^c}(t)
\end{aligned}
$$

Risk Management

Risk Management

There are three important points :

- **Hedges**
- **Monitoring**
- **Risk Reserve**

Risk Management

Hedges

- **Delta Hedge**
  - The most important risk factor for all the books.
  - Blipping each input of market quotes (1y,2y,...) separately and perform mark-to-market.
  - Take the difference between the original scenario to calculate the exposure.
  - Entering the relevant swap to reduce the exposure within a certain limit.
  - Accurate modeling of the curve-level dependence on ATMF volatilities is important for the efficiency of the delta hedges.

Risk Management

Hedges

- **Kappa Hedge**
  - **A very important risk factor for all the derivative books.**
  - **Although the market of basis swaps is liquid enough, derivatives on spreads and OIS are still quite rare.**
  - **IRS and FX kappa hedges would be enough for the daily operation.**
  - **Sensitivities for the change of market implied volatilities rather than the model parameters are important in practice.**
  - **Recalibration for each blipped scenario of implied volatility would require too much time...**

Risk Management

- A practical method of Kappa Hedge (ATMF)
  - Choose N hedge instruments with high liquidity.
    Label their implied volatilities as $\{\sigma_i\}_{i=1}^{N}$.
    Use the delta-neutral form of option, such as Straddle.
  - Partitioning the volatility curves/surfaces into N regions.
  - Label the partitions as $\{V_i\}_{i=1}^{N}$.
  - Make sure that $N \times N$-matrix, $\left(\frac{\partial \sigma_i}{\partial V_j}\right)$, is invertible.
  - For each scenario of blipped V_i, calculate $\frac{\partial PV}{\partial V_i}$, $\left\{\frac{\partial \sigma_j}{\partial V_i}\right\}_{j=1}^{N}$.
  - Calculate the exposure to the j-th hedge instrument as

$$\frac{\partial PV}{\partial \sigma_j} = \sum_{i=1}^{N} \left(\frac{\partial V_i}{\partial \sigma_j}\right) \times \left(\frac{\partial PV}{\partial V_i}\right)$$

  - Hedge the exposure by using the j-th instrument.

Risk Management

Monitoring

Everyday PL decomposition is very important to check the reliability of hedges and find a signal of unexpected risk factor, or bugs of system.

- **Using the change of market data and the calculated Greeks such as (Deltas, Gammas, Kappas, Thetas,...) to derive the expected PL[a].**
- **Take the difference between the actual and the expected PLs, and check the size and dominant source of residuals.**
- **Understand the cause of residual if it is significant.**

[a] To make cross Gamma calculation feasible, it is convenient to use several principal components.

Risk Management

Risk Reserve

There are a lot of risk factors very difficult to hedge in practice. It requires to hold reasonable amount of risk reserve.

- **Model limitation.**
- **Illiquidity of basis spread options.**
- **Illiquidity of Far OTM options.**
- **Exposure to correlation change.**
- **Stochastic correlations and their dependence on yield curve level/slope.**
- **etc...**

Conclusions

Conclusions

- Textbook-style implementation of IR model is not appropriate in the current market conditions.
  - Existence of various basis spreads and their movements
  - Widespread use of collateral
  - Significant implication for profit/loss of financial firms and their risk management
- We have proposed a framework of
  - Curve Construction consistent with all the relevant swaps (and hence basis spreads)
  - IR model with stochastic basis spreads

  in the presence of multiple currencies and collateral agreement.