A Note on Construction of Multiple Swap Curves with and without Collateral*

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Abstract

There are now available wide variety of swap products which exchange Libors with different currencies and tenors. Furthermore, the collateralization is becoming more popular due to the increased attention to the counter party credit risk. These developments require clear distinction among different type of Libors and the discounting rates. This note explains the method to construct the multiple swap curves consistently with all the relevant swaps with and without a collateral agreement.

Keywords : Libor, swap, tenor, yield curve, collateral, overnight index swap, cross currency, basis spread

1 Introduction

Among the market participants, Libor (London Inter Bank Offer Rate) has been widely used as a discounting rate of future cashflows. However, the basis spread observed in Cross Currency Swap (CCS) market has been far from negligible in recent years. Even in the single currency market, the tenor swap (TS), which exchanges the two Libors with different tenors, requires non-zero basis spread to be added in either side. From these facts, it is clear that we cannot treat all Libors equally as discounting rates in order to price the financial products consistently with existing swap markets. Furthermore, we are witnessing an increasing number of financial contracts are being made with collateral agreements. Due to the recent financial crisis and increasing attention to the counterparty credit risk, we can expect this tendency will accelerate in coming years. As we will see, the existence of the collateral agreement inevitably changes the funding cost of financial institutions, which makes the use of "Libor discounting" inappropriate for the

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proper pricing and hedging of collateralized contracts. In this brief note, we explain the method to construct the term structures of yield curves consistently with all the existing swap markets with and without the collateralization [1].

2 Swap curve construction without collateral

In this section, we develop the method to construct the term structures of yield curves consistently with the interest rate swaps (IRS), cross currency swaps (CCS) and tenor swaps (TS) without a collateral agreement. Here, we will concentrate on the traditional CCS, which keeps notional constant throughout the contract. The implication of the new type of CCS (mark-to-market CCS), which resets notional periodically using the spot exchange rate, will be discussed in Sec.3.5 under the context of collateralized swaps. We choose a single Libor as a discounting rate, and derive multiple index[2] curves in addition to the discounting curve to make the whole system consistent with the observable swap markets. As we will see, choosing a proper Libor as a discounting rate is important in order to reflect the difference of funding cost of each financial institution to the mark-to-market of its portfolio. We will also discuss the implications of the existence of multiple curves for the required hedges of interest rate products.

2.1 Case of Single IRS

As a preparation for later discussions, we first consider the situation where we have a single IRS market of a single currency only. For simplicity, let us assume that the payment dates of the fixed and floating rates of the IRS are the same. Then, the condition that the present value of the two legs are equal when we use the market swap rate as the coupon of the fixed leg is given by

$$C_N \sum_{n=1}^{N} \Delta_n P_{t,T_n} = \sum_{n=1}^{N} \delta_n E_t[L(T_{n-1}, T_n)] P_{t,T_n} \ . \tag{2.1}$$

Here, C_N is the swap rate of the length-N IRS at time t, Δ_n and δ_n are the daycount fractions of the fixed and floating legs, respectively. $L(T_{n-1}, T_n)$ is the Libor which is going to be reset at time T_{n-1} and maturing T_n. P_{t,T_n} denotes the time-t value of the risk-free zero coupon bond maturing at T_n. In the remainder of the paper, the expectation $E_t[\ ]$ is assumed to be taken under the appropriate forward measure unless it is specially mentioned.

In order to determine the set of $\{E_t[L(T_{n-1}, T_n)]\}$ and $\{P_{t,T_n}\}$ uniquely, we need to impose some relationship between these two type of variables since there is only one constraint of Eq.(2.1). Therefore, as we have mentioned, let us assume that the Libor is in fact the discounting rate. Then, the no-arbitrage condition between the zero coupon

[1] After the completion of the first version of this note, we were able to develop a fully dynamic term structure model of interest rates where all the basis spreads are stochastic. Please consult "A market model of interest rates with dynamic basis spreads in the presence of collateral and multiple currencies" of Ref. [1].

[2] We call the market rates (such as Libors) that are not equivalent to discounting rates as "index"-rates.

bond and the Libor floating payment gives

$$E_t[L(T_{n-1}, T_n)] = \frac{1}{\delta_n}\left(\frac{P_{t,T_{n-1}}}{P_{t,T_n}} - 1\right) . \tag{2.2}$$

Using this relation, we can write Eq.(2.1) as

$$C_N \sum_{n=1}^{N} \Delta_n P_{t,T_n} = P_{t,T_0} - P_{t,T_N} . \tag{2.3}$$

Now we can determine the set of discounting factor (and hence the forward Libors) sequentially, by transforming the above equation in the following form:

$$P_{t,T_N} = \frac{P_{t,T_0} - C_N\left(\sum_{n=1}^{N-1} \Delta_n P_{t,T_n}\right)}{1 + C_N \Delta_N}. \tag{2.4}$$

In the above formula, P_{t,T_0} is the discounting factor to the effective date, and can be determined by the overnight rate. Although, we need to carry out delicate splining to get a continuous set of discounting factor and forward Libor, which is important for practical application to the generic pricing, we will not step into the technical details, and concentrate on the conceptual understanding of the curve construction.

2.2 Case of IRS and CCS (USD Libor base)

In this section, we discuss the simple situation where there exist IRS and CCS markets and take the existing cross currency basis spread into account. To make the story concrete, we adopt USD and JPY as the relevant currencies and assume that the USD 3m-Libor as the discounting rate. This assumption is useful for the high rated firms whose funding currency is USD. For further simplification, we also assume that the payment frequency of the floating leg is quarterly both in the JPY IRS and USDJPY CCS[3].

In this setup, the curve construction for USD can be done in exactly the same way as explained in the previous section, since the Eq.(2.2) holds for the USD discounting factor and Libor. For JPY, this is not the case. The consistency conditions required from the JPY IRS and USDJPY CCS are given in the following forms, respectively.

$$C_N \sum_{n=1}^{N} \Delta_n P_{t,T_n} = \sum_{n=1}^{N} \delta_n E_t[L(T_{n-1}, T_n)] P_{t,T_n} , \tag{2.5}$$

$$N_{JPY}\left\{-P_{t,T_0} + \sum_{n=1}^{N} \delta_n \left(E_t[L(T_{n-1}, T_n)] + b_N\right) P_{t,T_n} + P_{t,T_N}\right\}$$

$$= f_x(t)\left\{-P^{\$}_{t,T_0} + \sum_{n=1}^{N} \delta^{\$}_n E^{\$}_t[L^{\$}(T_{n-1}, T_n)] P^{\$}_{t,T_n} + P^{\$}_{t,T_N}\right\} \tag{2.6}$$

[3] In reality, JPY IRS has semiannual payments and 6m tenor of Libor. On the other hand, the standard USDJPY CCS exchanges USD 3m-Libor flat against JPY 3m-Libor plus spread. The implications from the difference of tenor will be discussed in the next section

Here, the \$-index denotes that the variable is relevant for USD, b_N is the basis spread for length-N CCS, N_{JPY} is the JPY notional per USD and $f_x(t)$ is the USDJPY exchange rate at time t[4].

Since we are assuming that the USD Libor as the discounting rate, the right-hand side of Eq.(2.6) is actually zero. Therefore, eliminating the floating parts in Eqs.(2.5) and (2.6) gives us the simple formula:

$$\sum_{n=1}^{N} (\Delta_n C_N + \delta_n b_N) P_{t,T_n} = P_{t,T_0} - P_{t,T_N} \ . \tag{2.7}$$

From the above equation, just as we did in the previous section, we can determine the set of $\{P_{t,T_n}\}$ sequentially and make it continuous with the help of appropriate spline method. Once this is done, we get the set of forward Libors by substituting the derived $P_{t,T}$ into the Eq.(2.5).

As a result, we are forced to have two different curves, one for discounting and the other for the forward Libor index of JPY rates. One can see that the effective rate determining the JPY discounting factor is approximately given by

$$C_N^{\text{eff}} \simeq C_N + \frac{\delta}{\Delta} b_N \ , \tag{2.8}$$

and it is clear that the popular relation given in Eq.(2.2) does not hold for JPY Libor as long as there exists non-zero basis spread. Eq.(2.8) tells us that the JPY discounting curve lies below the index curve by the size of basis spread, which is usually negative $b_N < 0$ in the current market.

2.3 Case of IRS and CCS with TS basis into account (USD Libor base)

In the previous section, we have assumed the common payment frequency and tenor of JPY floating rates both in the IRS and CCS. However, in reality, the JPY Libor used in CCS has 3m tenor and quarterly payments, but it has 6m tenor and semiannual payments in JPY IRS. In addition, there exists 3m/6m tenor swap, in which one party pays 3m-Libor plus spread quarterly in exchange for receiving 6m-Libor semiannually, where the observed spread is often non-negligible, say more than 10bps. In this section, we continue to treat USD 3m-Libor as the discounting rate, but extend the previous method to take the observed TS basis spread into account consistently with IRS and CCS. In the remainder of the paper, we distinguish semiannual and quarterly payments and corresponding Libors by the ind of "m" and "n", respectively.

[4]At the inception of the CCS, N_{JPY} is determined by the spot exchange rate, which is the "forward" exchange rate maturing at the $T+2$ effective date. Due to this fact, the current $f_x(t)$ and N_{JPY} are slightly different in reality. However, we will neglect this small difference throughout this paper since it does not affect the main discussion.

The required conditions for the JPY rates are given as follows:

$$C_M \sum_{m=1}^{M} \Delta_m P_{t,T_m} = \sum_{m=1}^{M} \delta_m E_t[L(T_{m-1}, T_m)] P_{t,T_m} \ , \tag{2.9}$$

$$\sum_{n=1}^{N} \delta_n \left(E_t[L(T_{n-1}, T_n)] + \tau_N \right) P_{t,T_n} = \sum_{m=1}^{M} \delta_m E_t[L(T_{m-1}, T_m)] P_{t,T_m} \ , \tag{2.10}$$

$$\begin{aligned} N_{JPY} & \left\{ -P_{t,T_0} + \sum_{n=1}^{N} \delta_n \left(E_t[L(T_{n-1}, T_n)] + b_N \right) P_{t,T_n} + P_{t,T_N} \right\} \\ & = f_x(t) \left\{ -P^{\$}_{t,T_0} + \sum_{n=1}^{N} \delta^{\$}_n E^{\$}_t [L^{\$}(T_{n-1}, T_n)] P^{\$}_{t,T_n} + P^{\$}_{t,T_N} \right\} \ , \end{aligned} \tag{2.11}$$

where, we have assumed $N = 2M$, and τ_N denotes the time-t market spread of the length-N 3m/6m tenor swap. Since we are treating USD 3m-Libor as the discounting rate, the right hand side of Eq.(2.11) is zero as before. Eliminating the floating parts from these relations, one can easily show the equation

$$C_M \sum_{m=1}^{M} \Delta_m P_{t,T_m} + \sum_{n=1}^{N} \delta_n (b_N - \tau_N) P_{t,T_n} = P_{t,T_0} - P_{t,T_N} \tag{2.12}$$

holds among the JPY discounting factors. From this formula, it is straightforward to derive the (continuous) set of discounting factor by appropriate spline method as before. Then, using the determined discounting factors, we can derive $\{E_t[L(T, T + 3m)]\}$ and $\{E_t[L(T, T + 6m)]\}$, the set of 3m and 6m forward Libors, from Eqs.(2.11) and (2.9), respectively[5].

Now that, under the assumption of USD 3m-Libor being the discounting rate, we have derived the set of JPY discounting and two index curves, which make it possible to carry out JPY mark-to-market consistently with IRS, CCS and TS at the same time. If there exists a different type of TS market, one can easily extend the method to derive forward Libors with different tenors, such as 1m and 12m. We can also use the same method to derive JPY Tibor since there exists a swap exchanging Libor with Tibor plus spread. As for USD rates, we can use the method in sec.2.1 to derive the discounting factors and forward 3m-Libors, and then use USD TS information to derive the Libors with different tenors.

In order to understand the relation among the JPY discounting and index curves, it is convenient to use the following approximation:

$$\Delta_m P_{t,T_m} \simeq \frac{\Delta_m}{2} \left(P_{t,T_m - 3m} + P_{t,T_m} \right) \ . \tag{2.13}$$

By putting $\Delta_n = \Delta_m/2$, we can simplify the Eq.(2.12) as

$$\sum_{n=1}^{N} \left\{ \Delta_n C_M + \delta_n (b_N - \tau_N) \right\} P_{t,T_n} \simeq P_{t,T_0} - P_{t,T_N} \tag{2.14}$$

[5]We have not included the information available from USDJPY foreign exchange (FX) market. Since the FX forward contracts can be replicated by CCSs, the implied forward FX from the resultant discounting factors is mostly consistent with the market. Due to the liquidity issues, it is also common to use forward FX contracts instead of CCSs in the short end of the curve.

and then we see the effective swap rate implying the discounting factor is given by

$$C_M^{\rm eff} \simeq C_M + \frac{\delta}{\Delta}(b_N - \tau_N) \ . \tag{2.15}$$

It is clear from the above relation that JPY discounting factor depends not only on swap rates $\{C_M\}$ but also on CCS and TS spreads, $\{b_N, \tau_N\}$. Therefore, even if we have a position only in the standard JPY IRS, we need to hedge the exposures to the sensitivities of these spreads. It is also instructive to understand the relation among JPY discounting and two index curves. If the market quotes of IRS, TS and CCS are all flat, one can easily understand the relation

$$L^{3m} = R_{\rm discount} - b \ , \tag{2.16}$$
$$L^{6m} = L^{3m} + \tau = R_{\rm discount} - (b - \tau) \tag{2.17}$$

holds among the corresponding forward rates. Here, b and τ denote the flat CCS and TS basis spreads, respectively. We have also neglected the difference in the daycount fractions.

2.4 Case of IRS and CCS with TS basis into account (JPY Libor base)

In the previous sections, we have assumed that the USD 3m-Libor is the discounting rate. However, for the financial institutions which funding bases are located in Japan, it would be more appropriate to consider JPY Libor as the discounting rate. In this sections, we carry out the same exercise under the assumption that JPY 3m-Libor is the discounting rate[6].

In this setup, Eq.(2.2) holds between the JPY 3m-Libor and the discounting factor, which allows us to rewrite Eq.(2.10) as

$$P_{t,T_0} - P_{t,T_N} + \sum_{n=1}^{N} \delta_n \tau_N P_{t,T_n} = \sum_{m=1}^{M} \delta_m E_t[L(T_{m-1}, T_m)] P_{t,T_m} \ . \tag{2.18}$$

And then, eliminating the floating parts from the above equation using Eq.(2.9) yields the following formula:

$$C_M \sum_{m=1}^{M} \Delta_m P_{t,T_m} - \sum_{n=1}^{N} \delta_n \tau_N P_{t,T_n} = P_{t,T_0} - P_{t,T_N} \ . \tag{2.19}$$

Once we calculate the set of $\{P_{t,T}\}$ with proper splining from the above equation, we can easily recover the set of forward 3m-Libors from the relation given in Eq.(2.2), and that of forward 6m-Libors from Eq.(2.18). As was explained in the previous section, it is easy to obtain the forward Libors with different tenors if there exist additional TS markets.

Now, let us construct the USD curves consistently with the assumption of JPY 3m-Libor discounting. Note that Eq.(2.2) now holds for JPY 3m-Libor, Eq.(2.11), which is the condition from the CCS, is rewritten as

$$N_{\$} \left(b_N \sum_{n=1}^{N} \delta_n P_{t,T_n} \right) = -P_{t,T_0}^{\$} + \sum_{n=1}^{N} \delta_n^{\$} E_t^{\$}[L^{\$}(T_{n-1}, T_n)] P_{t,T_n}^{\$} + P_{t,T_N}^{\$}, \tag{2.20}$$

[6]It is straightforward to apply the same methodology for JPY Libor with different tenors, or even Tibor as the discounting rate.

where

$$N_{\$} = \frac{N_{JPY}}{f_x(t)} \ , \tag{2.21}$$

and it is almost "1" and we treat it as a constant[7]. We also have

$$C_K^{\$} \sum_{k=1}^{K} \Delta_k^{\$} P_{t,T_k}^{\$} = \sum_{n=1}^{N} \delta_n^{\$} E_t^{\$}[L^{\$}(T_{n-1}, T_n)] P_{t,T_n}^{\$} \tag{2.22}$$

as the constraint from USD IRS. Here, $N = 4K$ and we have distinguished the annual payment of fixed coupon by the index of "k" from the quarterly payment in the floating side in the standard USD IRS. As before, by eliminating the floating parts from Eqs.(2.20) and (2.22), we get the following equation among the USD discounting factors:

$$-P_{t,T_0}^{\$} + P_{t,T_N}^{\$} + C_K^{\$} \sum_{k=1}^{K} \Delta_k^{\$} P_{t,T_k}^{\$} = N_{\$} \left(b_N \sum_{n=1}^{N} \delta_n P_{t,T_n} \right) \ . \tag{2.23}$$

Since the right hand side is already known, we can repeat the same spline method to get the set of the discounting factors, $\{P_{t,T}^{\$}\}$. Then, forward 3m-Libors can be obtained from Eq.(2.22) by substituting the derived discount factors, and forward Libors with different tenors if there exist corresponding USD TS markets.

Under the assumption of JPY 3m-Libor discounting, the interdependence among discounting and index curves are quite different from that of the last section. It is clear from Eq.(2.19) that the basis spread in CCS does not affect the JPY discouting factors but that the USD discounting factors depend not only on the USD IRS quotes, but also on the basis spreads in CCS and JPY TS. It is important to notice that we need an aggregate risk management system to deal with the interdependence among USD and JPY interest rates both in the last and current cases.

2.5 Implications from different choice of discounting curve

Let us consider the implication from the different choice of Libor as a discounting rate. As is clear from the previous two sections, different choice leads to the different discounting curves, which inevitably leads to different present values even for the same cashflow. Although it does allow the arbitrage if the two methods coexist, we will now see that it is precisely reflecting the asymmetry of funding cost of financial firms.

For concreteness, let us first take a look at a high-rated financial firm located in the United States, which can borrow USD loan with 3m-Libor flat. In this case, the present value of the initial receipt of USD notional followed by 3m-Libor and the final notional repayments should be zero in total, which will make it convenient for this firm to use USD 3m-Libor as the discounting rate. Now, we want to know how much it costs to borrow JPY loan for the same firm. The firm can first borrow USD loan in US market, and then swap it into JPY loan by entering USDJPY CCS. The implied JPY funding cost is then given by JPY 3m-Libor + basis spread. Since in the USDJPY basis spread is usually

[7] Precisely speaking, it depends on the overnight rates of USD and JPY, since N_{JPY} is usually determined by the spot FX rate, which is actually "T+2" forward rate as we have mentioned before. We will neglect its rate dependency for simplicity throughout the paper.

negative, it can borrow JPY cash at cheaper cost than the Japanese domestic market. Therefore, the firm can make profit when it accesses the domestic market to provide JPY loan with JPY 3m-Libor flat. One can see that our curve construction based on USD Libor can explain this fact by making the JPY discounting rate displaced from the Libor by the CCS basis spread.

On the other hand, we have a quite different story for a high-rated Japanese financial firm. Since its funding cost of JPY loan is JPY Libor, it cannot raise any profit by lending a loan with JPY Libor flat in the domestic market. Now, let us consider the case where the firm wants to provide USD loan with USD Libor flat to its client. Since it does not have ample pool of USD cash, it needs to swap the JPY cash to USD by entering CCS market. The firm pays the USD Libor to the CCS counter party by passing the repayments from the client in return for receiving JPY Libor + basis spread. This essentially means that the firm provided a loan at lower yield than its funding cost because of the negative basis spread. Thus the firm has to recognize the loss from this contract. If we use the JPY Libor as the discounting rate and follows the construction explained in the last section, we can take this fact into account for the pricing of financial products.

As is now clear from the above examples, each financial firm needs to choose the appropriate reference as its discounting rate when constructing the set of curves[8]. It should be emphasized that the coexistence of different assumptions within the single firm needs to be avoided. It would allow the arbitrage within the system, and make it impossible to carry out consistent hedges against the exposures to the various spreads in the market[9].

3 Swap curve construction with collateral

Up to now, we have assumed that the swap contract is made without a collateral agreement and explained the curve construction based on the specific Libor treated as a discounting (or funding) rate. However, in recent years, more and more financial products have been made with collateral agreements due to the increased attention to the counter party credit risk. It is especially the case for major fixed income products such as swaps [2]. It seems that the tendency will accelerate further and will be applied to wider variety of products as a fallout of the current financial turmoil. As we will see later, the existence of collateral not only reduces the credit risk but also changes the funding cost significantly and hence affects the valuation of financial products in an important fashion. In the remainder of the paper, we will discuss the implication of the existence of collateral for the swap curve construction.

[8]The different Libor choice among market participants is difficult to be recognized at the inception of swaps, since it is common to enter the swap with the outstanding par rate which results in zero present value. However, we in fact experience some difficulty in the price agreement when we close the position.

[9]It does allow the arbitrage among market participants if they have different funding currencies. The situation is even more striking in some emerging markets where the implied basis spreads are astonishingly large (and negative). Although some of the foreign financial firms are actually taking advantage of the asymmetry of funding cost among different currencies to make profit, it seems that the activities are not enough to make spreads disappear. Some possible reasons are various regulations on foreign firms, their limited penetration in domestic markets, accounting rules making the recognition of profit from these activities difficult, and large USD demand to fulfill the hedge needs from domestic exporting companies and financial institutions with big foreign asset exposures. It would be important to study the economic reasons that lead to the existence of significant size of the currency basis spread.

3.1 Pricing of collateralized products

In this section, before going to the details of curve construction, we will discuss the generic pricing of collateralized trades. Under the collateral agreement, the firm receives the collateral from the counter party when the present value of the contract is positive, and needs to pay the margin called "collateral rate" on the outstanding collateral to the payer. Although the details can differ trade by trade, the most commonly used collateral is a currency of developed countries, such as USD, EUR and JPY, and the mark-to-market of the contracts is to be made quite frequently. In the case of cash collateral, the overnight rate for the collateral currency, such as Fed-fund rate for USD, is usually used as the collateral rate.

In general setup, carrying out the pricing of collateralized products is quite hard due to the non-linearity arising from the credit risk. In the remainder of the paper, in order to make the problem tractable, we will assume the perfect and continuous collateralization with zero threshold by cash, which means that mark-to-market and collateral posting is to be made continuously, and the posted amount of cash is 100% of the contract's present value. Actually, the daily adjustment of the collateral should be the best practice in the market and seems becoming popular, and hence the approximation should not be too far from the reality. Under the above simplification, we can neglect the counter party default risk and recover the linearity among different payments. Therefore, we can decompose the cashflow of a collateralized swap and treat them as a portfolio of the independently collateralized strips of payments.

Let us consider the stochastic process of the collateral account $V(t)$ with an appropriate self-financing trading strategy under the risk-neutral measure, following the method sometime used in the pricing of futures. Since one can invest the posted collateral with the risk-free interest rate but need to pay the collateral rate, the process of the collateral account is given by

$$dV(s) = y(s)V(s)ds + a(s)dh(s) \ , \tag{3.1}$$

where, $y(s) = r(s) - c(s)$ is the difference of the risk-free rate $r(s)$ and the collateral rate $c(s)$ at time s, $h(s)$ denotes the time-s value of the derivative which matures at T with the cashflow $h(T)$, and $a(s)$ is the number of positions of the derivative. We get

$$V(T) = e^{\int_t^T y(u)du}V(t) + \int_t^T e^{\int_s^T y(u)du} a(s)dh(s) \tag{3.2}$$

by integrating Eq.(3.1). Adopting the trading strategy specified by

$$\begin{aligned} V(t) &= h(t) \\ a(s) &= \exp\left(\int_t^s y(u)du\right) \end{aligned} \tag{3.3}$$

allows us to rewrite Eq.(3.2) as

$$V(T) = e^{\int_t^T y(s)ds}h(T) \ . \tag{3.4}$$

Then, we see the present value of the underlying derivative is given by

$$h(t) = E_t^Q\left[e^{-\int_t^T (r(s)-y(s))ds}h(T)\right] = E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] \ . \tag{3.5}$$

Here, $E^Q[\,]$ denotes the expectation where the money-market account is being used as the numeraire[10].

Next, let us consider the case where the collateral is posted by a foreign currency. In this case, the process of the collateral account V^f is

$$dV^f(s) = y^f(s)V^f(s)ds + a(s)d[h(s)/f_x(s)] \ , \tag{3.9}$$

where $f_x(s)$ is the foreign exchange rate at time s, and $y^f(s) = r^f(s) - c^f(s)$ denotes the difference of the risk-free and collateral rate of the foreign currency. Integrating it, we obtain

$$V^f(T) = e^{\int_t^T y^f(s)ds}V^f(t) + \int_t^T e^{\int_s^T y^f(u)du}a(s)d[h(s)/f_x(s)] \ . \tag{3.10}$$

This time, we adopt the trading strategy

$$\begin{aligned} V^f(t) &= h(t)/f_x(t) \\ a(s) &= \exp\left(\int_t^s y^f(u)du\right) \ , \end{aligned} \tag{3.11}$$

which yields

$$V^f(T) = e^{\int_t^T y^f(s)ds}h(T)/f_x(T) \ . \tag{3.12}$$

Then, we see the price of the derivative in terms of the domestic currency is given by

$$\begin{aligned} h(t) &= V^f(t)f_x(t) = E_t^Q\left[e^{-\int_t^T r(s)ds}V^f(T)f_x(T)\right] \\ &= E_t^Q\left[e^{-\int_t^T r(s)ds}\left(e^{\int_t^T (r^f(s)-c^f(s))ds}\right)h(T)\right] \ . \end{aligned} \tag{3.13}$$

From the above discussion, it is now clear that "Libor discounting" is not appropriate for the pricing of collateralized trades. As we can see from Eq.(3.5), we have to discount the future cashflow by the collateral rate, which can be significantly lower than the Libor for the corresponding currency, especially under the distressed market conditions. It is also useful to interpret the results in terms of the funding cost for the possessed positions. First, let us consider the case where there is a receipt of cash at a future time (hence, positive present value) from the underlying contract. In this case, we are immediately posted an equivalent amount of cash as its collateral, on which we need to pay the collateral rate and return its whole amount in the end. We consider it as a loan where we fund the position

[10] Considering the continuous and perfect collateralization and its investment with rate $y(t)$, we see

$$h(t) = E_t^Q\left[e^{-\int_t^T r(s)ds}h(T) + \int_t^T e^{-\int_t^s r(u)du}y(s)h(s)ds\right] \tag{3.6}$$

should hold. From this equation, one can show that

$$X(t) = e^{-\int_0^t r(s)ds}h(t) + \int_0^t e^{-\int_0^s r(u)du}y(s)h(s)ds \tag{3.7}$$

is a martingale process, which then implies that the price process of $h(t)$ is expressed with a certain martingale process $M(t)$ as

$$dh(t) = c(t)h(t)dt + dM(t) \ . \tag{3.8}$$

This would also leads to the formula given in Eq.(3.5).

at the expense of the collateral rate. On the other hand, if there is a payment of cash at future time (negative present value), the required collateral posting can be interpreted as a loan provided to the counter party with the same rate. Therefore, compared to the non-collateralized trade (and hence, Libor funding), we get more in the case of positive present value since we can fund the loan cheaply, but lose more in the case of negative value due to the lower return from the loan lent to the client.

3.2 Overnight Index Swap

As we have seen in the previous section, it is critical to determine the forward curve of overnight rate for the pricing of collateralized swaps. Fortunately, there is a product called "overnight index swap" (OIS), which exchanges the fixed coupon and the daily-compounded overnight rate.

Here, let us assume that the OIS itself is continuously and perfectly collateralized with zero threshold, and approximate the daily compounding with continuous compounding [11]. In this case, using the Eq.(3.5), we get the condition from the OIS as

$$S_N \sum_{n=1}^{N} \Delta_n E_t^Q \left[e^{-\int_t^{T_n} c(s)ds} \right] = \sum_{n=1}^{N} E_t^Q \left[e^{-\int_t^{T_n} c(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} c(s)ds} - 1 \right) \right] \ . \tag{3.14}$$

Here, S_N is the time-t par rate for the length-N OIS, and $c(t)$ is the overnight (and hence collateral) rate at time t. We can simplify the above equation into the form

$$S_N \sum_{n=1}^{N} \Delta_n D_{t,T_n} = D_{t,T_0} - D_{t,T_N} \tag{3.15}$$

by defining the discounting factor of the collateral rate:

$$D_{t,T} = E_t^Q \left[e^{-\int_t^{T} c(s)ds} \right] \ . \tag{3.16}$$

Now, from Eq.(3.15), we can obtain the continuous set of $\{D_{t,T}\}$ by appropriate splining as before.

3.3 Case of collateralized swaps in single currency

In the case of single currency, calculation of the forward Libors is quite straightforward. The consistency conditions from the collateralized IRS and TS corresponding to Eqs.(2.9) and (2.10) are

$$C_M \sum_{m=1}^{M} \Delta_m D_{t,T_m} = \sum_{m=1}^{M} \delta_m D_{t,T_m} E_t^c[L(T_{m-1}, T_m)] \ , \tag{3.17}$$

$$\sum_{n=1}^{N} \delta_n \left(E_t^c[L(T_{n-1}, T_n)] + \tau_N \right) D_{t,T_n} = \sum_{m=1}^{M} \delta_m D_{t,T_m} E_t^c[L(T_{m-1}, T_m)] \ , \tag{3.18}$$

[11] Typically, there is only one payment at the very end for the swap with short maturity ($< 1yr$) case, and otherwise periodical payments, quarterly for example.

where $E_t^c[\ ]$ denotes the expectation taken under the measure where $D_{t,T}$ is used as the numeraire. Since all the relevant $\{D_{t,T}\}$ are already known from the OIS market, we can easily calculate the set of forward Libors from these conditions. Here, we have assumed that OIS swap market is available up to necessary range to determine the entire forward curve.

3.4 Case of collateralized swaps in multiple currencies (with Constant Notional CCS)

In this section, we consider the method to construct the term structures of collateralized swaps in the multi-currency setup, where we continue to use the constant notional CCS as a calibration instrument. We will discuss the implications of new type of CCS, "Mark-to-Market CCS", in the next section. In the single currency case, it is common to use the same currency as the collateral, and we can easily derive the relevant curves as we have seen in Secs.3.2 and 3.3. However, there inevitably appear the payments with different currency from that of the collateral in CCS, which makes the determination of the forward Libors complicated due to the involvement of the risk-free and collateral rate at the same time as indicated by Eq.(3.13). In the actual market, USD is being widely used as the collateral for the trades including multiple currencies.

As in the previous sections, let us use USD and JPY swaps to demonstrate the method. To make the problem simpler, we treat the Fed-Fund rate, which is the collateral rate for USD, to be the risk-free interest rate. Then, we have the relation

$$D_{t,T}^{\$} = E_t^{Q^{\$}}\left[e^{-\int_t^T c^{\$}(s)ds}\right] = E_t^{Q^{\$}}\left[e^{-\int_t^T r^{\$}(s)ds}\right] = P_{t,T}^{\$} \ . \tag{3.19}$$

The required conditions from JPY-collateralized JPY swaps are given by

$$S_N \sum_{n=1}^{N} \Delta_n D_{t,T_n} = D_{t,T_0} - D_{t,T_N} \ , \tag{3.20}$$

$$C_M \sum_{m=1}^{M} \Delta_m D_{t,T_m} = \sum_{m=1}^{M} \delta_m D_{t,T_m} E_t^c[L(T_{m-1},T_m)] \ , \tag{3.21}$$

$$\sum_{n=1}^{N} \delta_n \left(E_t^c[L(T_{n-1},T_n)] + \tau_N\right) D_{t,T_n} = \sum_{m=1}^{M} \delta_m D_{t,T_m} E_t^c[L(T_{m-1},T_m)], \tag{3.22}$$

and, those of USD-collateralized USD swaps are

$$S_N^{\$} \sum_{n=1}^{N} \Delta_n^{\$} P_{t,T_n}^{\$} = P_{t,T_0}^{\$} - P_{t,T_N}^{\$} \ , \tag{3.23}$$

$$C_K^{\$} \sum_{k=1}^{K} \Delta_k^{\$} P_{t,T_k}^{\$} = \sum_{n=1}^{N} \delta_n^{\$} P_{t,T_n}^{\$} E_t^{\$}[L^{\$}(T_{n-1},T_n)] \ , \tag{3.24}$$

$$\sum_{n=1}^{N} \delta_n^{\$} \left(E_t^{\$}[L^{\$}(T_{n-1},T_n)] + \tau_N^{\$}\right) P_{t,T_n}^{\$} = \sum_{m=1}^{M} \delta_m^{\$} P_{t,T_m}^{\$} E_t^{\$}[L^{\$}(T_{m-1},T_m)], \tag{3.25}$$

where, the conditions are from OIS, IRS and TS, respectively. As before, we can add additional TS condition if exists. One can now derive the discounting factors $\{D_{t,T}\}$ and $\{P^{\$}_{t,T}\}$ from the OIS conditions, and then the remaining forward Libors in turn.

Now, let us consider the determination of USD-collateralized JPY interest rates. If the USDJPY CCS is collateralized by USD cash, which is the common practice in the market, we get the following condition by applying the result in Eq.(3.13):

$$\sum_{n=1}^{N} \delta_n \left(E_t[L(T_{n-1}, T_n)] + b_N \right) P_{t,T_n} - P_{t,T_0} + P_{t,T_N} = V_N \ . \tag{3.26}$$

Here,

$$V_N = \left\{ \sum_{n=1}^{N} \delta^{\$}_n E^{\$}_t[L^{\$}(T_{n-1}, T_n)] P^{\$}_{t,T_n} - P^{\$}_{t,T_0} + P^{\$}_{t,T_n} \right\} / N_{\$} \tag{3.27}$$

and it is given by the result of previous calculations for USD swaps. As you can see, it is impossible to determine the JPY risk-free zero coupon bond price $\{P_{t,T_n}\}$ and the forward Libors $\{E_t[L(T_{n-1}, T_n)]\}$ uniquely, from these standard set of swaps only. However, if there exist USD-collateralized JPY IRS and TS markets[12], we get the additional information as

$$\tilde{C}_M \sum_{m=1}^{M} \Delta_m P_{t,T_m} = \sum_{m=1}^{M} \delta_m P_{t,T_m} E_t[L(T_{m-1}, T_m)] \ , \tag{3.28}$$

$$\sum_{n=1}^{N} \delta_n \left(E_t[L(T_{n-1}, T_n)] + \tilde{\tau}_N \right) P_{t,T_n} = \sum_{m=1}^{M} \delta_m P_{t,T_m} E_t[L(T_{m-1}, T_m)] \ . \tag{3.29}$$

Here, $\tilde{C}_M$ and $\tilde{\tau}_N$ denote the par rates of the USD-collateralized JPY swaps, which differ from C_M and τ_N, the par rates of JPY collateralized swaps in general. We can now eliminate the floating parts from Eqs.(3.26), (3.28) and (3.29), and obtain

$$\sum_{n=1}^{N} \delta_n (b_N - \tilde{\tau}_N) P_{t,T_n} + \tilde{C}_M \sum_{m=1}^{M} \Delta_m P_{t,T_m} - V_N = P_{t,T_0} - P_{t,T_N} \ . \tag{3.30}$$

Then, as we did in Sec.2.3, we can determine the set of $\{P_{t,T}\}$ and the forward Libors with the both tenors by applying an appropriate spline method.

If it is difficult to obtain the separate quotes for USD-collateralized JPY swaps, we may not be able to use Eqs.(3.28) and (3.29) for the curve construction. If this is the case, one possible approach is to set

$$E_t[L(T_{n-1}, T_n)] = E^c_t[L(T_{n-1}, T_n)] \tag{3.31}$$

by neglecting the correction arising from the change of numeraire. This approximation would be reasonable if the dynamic properties of the JPY risk-free and the overnight interest rates are similar with each other. Once admitting the assumption, one can determine the set of discount factors from Eq.(3.26). If there exists enough liquidity in the FX forward market, then using the FX forward quotes and the USD discounting factor to derive $\{P_{t,T}\}$ is another possible way.

[12]In fact, it seems that the US banks tend to ask their counter parties to post USD collateral even for the JPY IRS and TS.

Finally, let us mention the case where we have JPY-collateralized USD swap markets. Since we have not assumed that the JPY overnight rate is risk-free, the difference between the risk-free and collateral rates appears in the expression of present value as given in Eq.(3.13). The conditions from the JPY-collateralized USD IRS is given by

$$\tilde{C}_K^{\$}\sum_{k=1}^{K}\Delta_k^{\$}P_{t,T_k}^{\$}E_t^{\$}\left[e^{\int_t^{T_k}y(s)ds}\right]=\sum_{n=1}^{N}\delta_n^{\$}P_{t,T_n}^{\$}E_t^{\$}\left[e^{\int_t^{T_n}y(s)ds}L^{\$}(T_{n-1},T_n)\right] \;, \tag{3.32}$$

where $y(s)=r(s)-c(s)$ is the difference between the JPY risk-free and collateral rates, and $\tilde{C}_K^{\$}$ is the par rate of the length-K IRS. In the same way, if there exists JPY-collateralized USDJPY CCS, we also have the following condition:

$$\begin{aligned}&\sum_{n=1}^{N}\delta_n^{\$}P_{t,T_n}^{\$}E_t^{\$}\left[e^{\int_t^{T_n}y(s)ds}L^{\$}(T_{n-1},T_n)\right]\\&\qquad=N_{\$}\left(\sum_{n=1}^{N}\delta_n(E_t^c[L(T_{n-1},T_n)]+\tilde{b}_N)D_{t,T_n}-D_{t,T_0}+D_{t,T_N}\right),\end{aligned} \tag{3.33}$$

where, $\tilde{b}_N$ is the par spread of the length-N CCS. Since the right hand side of Eq.(3.33) and USD discount factors are already known, we can determine the set of

$$E_t^{\$}\left[e^{\int_t^{T_n}y(s)ds}\right],\quad E_t^{\$}\left[e^{\int_t^{T_n}y(s)ds}L^{\$}(T_{n-1},T_n)\right] \;. \tag{3.34}$$

This completes the calculation of whole set of curves, which are USD-collateralized USD rates, JPY-collateralized JPY rates, USD-collateralized JPY rates, and JPY-collateralized USD rates.

3.5 Case of collateralized swaps in multiple currencies (with Mark-to-Market Cross Currency Swap)

In this section, we discuss a different type of swap called mark-to-market cross currency swap (MtMCCS) and its implication to the curve construction. Similarly to the traditional CCS, the participants exchange the Libor in one currency and the Libor plus spread in another currency with notional exchanges. The different feature of the MtMCCS is that the notional on the currency paying Libor flat is adjusted at the every start of the Libor calculation period based on the spot FX, and the difference between the notional used in the previous period and the next one is also paid or received at the reset time. Here, the notional for the other currency is kept constant throughout the contract. For pricing, we can consider it as a portfolio of the strips of the one-period traditional CCS with the common notional and the spread for the side paying Libor plus spread. Here, the net effect from the final notional exchange of the (i)-th CCS and the initial exchange of the (i+1)-th CCS is equivalent to the notional adjustment at the start of (i+1)-th period of MtMCCS. Usually, we need to adjust the notional of the USD side, since it is the market standard to exchange USD Libor flat against Libor plus spread in another currency.

For concreteness, let us consider the case of USDJPY MtMCCS with USD collateral, and continue to identify collateral rate (Fed-Fund rate) as the USD risk-free rate. It is simple to calculate the present value in JPY side, since the notional is kept constant.

Using the same notation, the present value from the view point of JPY Libor receiver is given by

$$\begin{aligned} PV_{JPY} &= -\sum_{n=1}^{N} P_{t,T_{n-1}} + \sum_{n=1}^{N} P_{t,T_n} \left(1 + \delta_n (b_N + E_t[L(T_{n-1}, T_n)])\right) \\ &= -P_{t,T_0} + P_{t,T_N} + \sum_{n=1}^{N} P_{t,T_n} \delta_n (b_N + E_t[L(T_{n-1}, T_n)]) \ , \end{aligned} \tag{3.35}$$

which is equivalent to the left hand side of Eq.(3.26).

On the other hand, the present value of USD side is expressed as

$$\begin{aligned} PV_{USD} &= -\sum_{n=1}^{N} E_t^{Q^\$}\left[\frac{e^{-\int_t^{T_{n-1}} r^\$(s)ds}}{f_x(T_{n-1})}\right] + \sum_{n=1}^{N} E_t^{Q^\$}\left[\frac{e^{-\int_t^{T_n} r^\$(s)ds}(1 + \delta_n^\$ L^\$(T_{n-1}, T_n))}{f_x(T_{n-1})}\right] \\ &= -\sum_{n=1}^{N} \frac{P^\$_{t,T_{n-1}}}{FX(t, T_{n-1})} + \sum_{n=1}^{N} E_t^{Q^\$}\left[\frac{e^{-\int_t^{T_n} r^\$(s)ds}(1 + \delta_n^\$ L^\$(T_{n-1}, T_n))}{f_x(T_{n-1})}\right] . \end{aligned} \tag{3.36}$$

Here, $FX(t, T)$ denotes the time-t forward exchange rate maturing at T. If we assume that the USD Libor is the risk-free rate, then the second term cancels the first one and turns out to be zero in total, $PV_{USD} = 0$ [13]. However, we are now making a distinction between USD Libor and the risk-free Fed-Fund rate, there inevitably appears a model dependent term. To understand it more clearly, let us decompose the market Libor into the risk-free part and the residual part:

$$L^\$(T_{n-1}, T_n) = \frac{1}{\delta_n}\left(\frac{1}{P^\$_{T_{n-1},T_n}} - 1\right) + S(T_{n-1}, T_n), \tag{3.37}$$

where the second term $S(T_{n-1}, T_n)$ denotes the residual part in the Libor $L^\$(T_{n-1}, T_n)$ at time T_{n-1}. Then, we get the USD side value as

$$PV_{USD} = \sum_{n=1}^{N} E_t^{Q^\$}\left[\frac{e^{-\int_t^{T_n} r^\$(s)ds}\delta_n S(T_{n-1}, T_n)}{f_x(T_{n-1})}\right] \ , \tag{3.38}$$

which depends on the covariance of the risk-free zero coupon bonds and the FX rate even if the spread is deterministic. The correction from the forward value arises when we change the numeraire into the risk-free zero coupon bond with maturity T_n. If we can evaluate this model dependent term, it is possible to repeat the same discussions following Eq.(3.26) after replacing V_N by $f_x(t) \times PV_{USD}$:

$$\sum_{n=1}^{N} \delta_n \left(E_t[L(T_{n-1}, T_n)] + b_N\right) P_{t,T_n} - P_{t,T_0} + P_{t,T_N} = f_x(t) PV_{USD} \ . \tag{3.39}$$

[13] Therefore, if we consider the non-collateralized swaps with USD Libor as the discounting rate, we can repeat exactly the same arguments in Secs.2.2 and 2.3.

For simplicity, let us assume the deterministic spread[14] and the geometric Brownian motion for both of the forward FX and the USD forward risk-free Bond:

$$FX(t,T_{n-1}) = f_x(t)\frac{P^{\$}_{t,T_{n-1}}}{P_{t,T_{n-1}}}, \qquad FB(t,T_{n-1},T_n) = \frac{P^{\$}_{t,T_{n-1}}}{P^{\$}_{t,T_n}} \ . \tag{3.40}$$

We denote their deterministic log-normal volatilities and the correlation between $FX(t,T_{n-1})$ and $FB(t,T_{n-1},T_n)$ as $(\sigma_{FX_{n-1}}(t), \sigma_{FB_{n-1,n}}(t), \rho_{n-1}(t))$, respectively. In this simplest case, the USD side present value can be evaluated as

$$PV_{USD} = \sum_{n=1}^{N} \frac{P^{\$}_{t,T_n}\delta_n S(T_{n-1},T_n)}{FX(t,T_{n-1})} \exp\left(\int_t^{T_{n-1}} \rho_{n-1}(s)\sigma_{FX_{n-1}}(s)\sigma_{FB_{n-1,n}}(s)ds\right) \ . \tag{3.41}$$

Therefore, in this simple setup, the curve calibration can be done in the following way. Firstly, construct the USD Fed-Fund rate curve and the collateralized USD Libor curve as discussed in the last section, and then extract the spread between them assuming that it is deterministic. Secondly, although the available maturity is limited, we can extract the Fed-Fund rate volatility from the OIS option market. The forward FX volatility can be directly read from the vanilla FX option market. As for the correlation between the USD risk-free bond and the forward FX, we need to use either the historical data, or possibly make use of the information in quanto products. Now the last remaining ingredient is the FX forward rate. Of course, we can directly read the quotes from the market if there is enough liquidity in the FX forward contracts. Even if this is not the case, there is a way around requiring only swap information. Since the maturity of FX forward is shorter than that of the MtMCCS by one period, if we have the JPY discounting factor up to $P_{t,T_{n-1}}$, then we can sequentially derive P_{t,T_n} by using Eq.(3.39) and the discussion following Eq.(3.26) in the last section. Therefore, although the procedure is more complicated, we can still construct the curves under the simplifying assumptions.

Finally, let us check the case where the MtMCCS is collateralized by JPY cash. The present value of the JPY side is

$$PV_{JPY} = -D_{t,T_0} + D_{t,T_N} + \sum_{n=1}^{N} D_{t,T_n}\delta_n(\tilde{b}_N + E^c_t[L(T_{n-1},T_n)]) \ , \tag{3.42}$$

where the $\tilde{b}_N$ denotes the JPY-collateralized MtMCCS spread. The USD side is now given by

$$\begin{aligned} PV_{USD} \ = \ & -\sum_{n=1}^{N} E^{Q^{\$}}_t\left[\frac{e^{-\int_t^{T_{n-1}} r^{\$}(s)ds}\, e^{\int_t^{T_{n-1}} y(s)ds}}{f_x(T_{n-1})}\right] \\ & + \sum_{n=1}^{N} E^{Q^{\$}}_t\left[\frac{e^{-\int_t^{T_n} r^{\$}(s)ds}\, e^{\int_t^{T_n} y(s)ds}(1+\delta^{\$}_n L^{\$}(T_{n-1},T_n))}{f_x(T_{n-1})}\right] \ , \end{aligned} \tag{3.43}$$

where $y(s) = r(s) - c(s)$ denotes the difference of JPY risk-free rate and the collateral rate. If we assume that $y(s)$ is deterministic, or independent from the other variables in addition

[14]Precisely speaking, the independence of the spread motion from the risk-free USD rate and FX is enough to apply the following discussion.

to the assumption on the residual spread of Libor, we can repeat the same calculation to derive the convexity correction. Following the similar discussion after Eq.(3.33) in the last section, we can obtain the correction to the forward USD Libor in the case of JPY collateralization, which is the factor of $\exp(\int_t^T y(s)ds)$ [15].

4 Importance of appropriate curve construction

Up to this point, we have explained how to construct multiple swap curves which can mark various swaps to the market consistently with and without collateral agreements. Some of the readers may wonder if this is totally unnecessary complication to explain anyway "very small" basis spreads by inferring that the spreads affect the profit/loss of the financial firms only through the proportion :

$$\frac{\text{spread size}}{\text{level of interest rate}} .$$

However, it is not at all the case since their profit and loss are made only through the "change" of interest rate instead of its level. Therefore, the potential impact would be disastrous if the system cannot recognize the existence of basis spreads and if it is unable to risk manage the exposure to their movements. Basically, the existence of basis spreads affects the mark-to-market of the trades through the following two routes:
(1) Change of the forward expectation of Libors;
(2) Change of the discounting rate.
In the following, let us explain each effect using simple examples so that the readers can easily recognize the importance of consistent curve construction.

Let us start from the first case. Suppose there is one firm which does not recognize the tenor swap spreads and working in structured product business; The firm pays the structured payoff to its clients and receives Libor (plus spread to cover the optionality premia) in return as its funding. Let us suppose the funding legs of the firm's portfolio contain the two frequencies with equal fractions, 3m and 6m JPY-Libor, reflecting the different demands among the clients. If the firm constructs the swap curve based on JPY IRS with semiannual frequency, and if it is not able to handle the 3m/6m tenor spread, both of the 3m and 6m forward Libors are derived from the common discounting curve based on the IRS. In this case, the model implied 3m/6m tenor spread is zero. As one can easily imagine, the firm is significantly overestimating the value of 3m-Libor funding legs. The easiest way to estimate its impact is to convert the stream of 3m-Libor payments into that of 6m-Libor by entering the 3m/6m JPY-Libor tenor swap as the payer side of 3m-Libor. Since the firm needs to pay the 3m/6m tenor spread on top of the 3m-Libor, the loss of the firm from the mis-pricing of the funding legs can be estimated as

$$\text{Loss} \simeq \text{Outstanding Notional} \times \text{PVO1(Average Duration)} \times (\text{ 3m/6m tenor spread }) ,$$

where the PV01 denotes the annuity of the corresponding swap, which is the sum of the discount factors times daycount fractions. If the average duration and the tenor spread is around 10yr and 10bp respectively, the loss would be about one percentage point of the

[15]Under the assumption that y is a deterministic function of time, we can make the curve construction more straightforward. Please see the related discussion in Ref. [1].

total notional outstanding, which would be far from negligible for the firm. Of course, if the structured payoffs are dependent on the 3m-Libor, there will be additional contributions. Furthermore, when the firm is an active participant of IRS market at the same time, the potential impact would be much worse. Since the system unable to recognize the spread gives the traders an incentive to enter the positions as 3m-Libor receivers, since they can offer very "competitive" prices relative to their competitors while making their profit positive within the firm's faulty system.

Now, let us discuss the impact from the second effect, or the change in the discounting factors. This is the dominant change when we properly take the collateralization into account. Although the impact will be smaller than the direct change of the forward Libors, there would be quite significant impact especially from the cross currency trades, where we usually have final notional exchanges. In the presence of 10bp Libor-OIS spread, the present value of the notional payment in 10yrs time would be different by around one percentage point of its notional. For the whole portfolio, the impact from the difference between the Libor and the collateral rate of each currency can be tremendous. In addition, there is another route through which the change of discounting factors affects the firm's profit in an important fashion. If, as a more preliminary level, the firm is not capable of treating the CCS basis spread correctly, the resultant discounting curves never reproduce the market level of FX forwards [16]. If they are participating in FX derivatives business without having developed the proper system, the effect through FX forward will be quite critical, if it is not fatal. On the other hand, even if the discounting curve of foreign currency is properly constructed to reproduce the FX forwards, if the system neglects the difference between the resultant discounting curve and the forward Libor of the corresponding currency, the value of future cash flow dependent on the foreign Libor will be totally wrong. This effect would be particularly important for the FX-IR hybrid products, such as PRDCs.

5 Use of multiple curves in a trading system

It is now clear that we need a large number of Libor index and discounting curves to price the financial products consistently with the observable swap markets. In the remaining part of the paper, we will discuss some important points related to the use of the multiple curves in an actual trading system.

5.1 Use of curves for non-collateralized products

Here, we will discuss the case of non-collateralized products. In this case, what we need to do first is to choose a single appropriate reference rate, which should reflect the funding cost of the relevant firm reasonably well, and also have good liquidity in the market, such as the Libor of the funding currency. It will be used as the base discounting rate when we construct the multiple curves. Although the complexity of hedge does depend on the choice, it should be unique throughout the firm to avoid the arbitrage within the system and to retain the consistency of hedges. After the choice of a single funding rate, we can

[16] Note that the combination of IRS and CCS effectively replicate FX forward contracts.

uniquely determine the discounting and forward Libor curves for each currency except the freedom associated with the details of spline method.

For the practical use, it would be convenient to create following quantities:

$$\{P_{0,T}\},\ \{P_{0,T}^{1m}\},\ \{P_{0,T}^{3m}\},\ \{P_{0,T}^{6m}\},\cdots \tag{5.1}$$

where the first one is the discounting factor, and the others are recursively defined by the relation

$$P_{0,0}^{\tau}=1,\quad \frac{1}{\tau}\left(\frac{P_{0,T-\tau}^{\tau}}{P_{0,T}^{\tau}}-1\right)=E[L(T-\tau,T)]\ . \tag{5.2}$$

The quantity, $P_{0,T}^{\tau}$, can be considered as the risky discounting factor reflecting the relative risk among the Libors with different tenors. Since it is natural to assume that the relative risk of the Libor changes smoothly in terms of its tenor, we can approximate $P_{0,T}^{\tau}$ with an arbitrary τ by interpolating the set of (5.1). This would be quite useful for the pricing of over-the-counter products, which sometimes require the Libor with a tenor which is not available in the liquid TS market.

The pricing of products without optionality is then carried out straightforwardly, by calculating the appropriate forward rate using the interpolation of P^{τ} if necessary, and then multiplying the discounting factor of the payment date. Delta (and hence gamma) sensitivities are calculable by using different set of curves after blipping the market quotes of the relevant swaps, $\{C_M, b_N, \tau_N\}$. As we have seen, it is important to notice that the movement of quotes even in different currencies can affect the hedges through the effect of CCS.

5.2 Use of curves for collateralized products

Now let us discuss the case of collateralized products. Firstly, we need to choose a "risk-free" interest rate to construct the curves. Considering the available length of the OIS, the Fed-Fund rate would be useful. Although the basic idea is the same, the operation under the collateralization is more complicated than the non-collateralized case. As we have seen, under the collateralization, the effective discounting factors and associated expectation of forward Libors depend on the collateral currencies, and hence, it would be convenient to setup separate books for each of them in the trading system.

Ideally, we would like to have all the types of swaps for each collateral currency, which then allows to determine the curves uniquely, and makes it possible to close the hedges within the swaps with the same collateral. However, it is not the case in general, and we are required to use the approximate relation, such as Eq.(3.31), to relate the exposure to the available swaps. Except these complications, dealing with the Libors with different tenors and the hedge operations are the same as those in the non-collateralized case.

5.3 Comments on Simulation Scheme

Finally, let us comment on the issue related to the simulation scheme in the multi-curve setup. Generally speaking, we need to make all the curves dynamic if we want to fully capture the optionality related to the spreads among different Libors. However, as one can easily imagine, it would be a quite demanding task to develop the system due to the complicated calibration mechanism even for the vanilla options, and the need of delicate noise reduction to recover the observed swap prices within a reasonable calculation time. On the other hand, despite the difficulties, we also know the importance to incorporate the multi-curve setup into the simulation system so that we can properly reflect the observed market swap prices in the structured derivatives, and appropriately manage the exposures to the various spreads in the market.

The simplest approach is to assume constant and time-homogeneous spreads among the discounting curve and the Libor index curves within each currency. Under the assumption, we can simply adopt the usual interest-rate term structure model to drive the discounting curve. For pricing, we check the relevant tenor of the reference rate, adding up the relevant spread to the simulated discounting rate to get the pathwise realization of the Libor index. We can look at the model with the above simplification as the minimum requirement for most of the financial firms so that they can properly manage the exposure to the existing spreads in various swaps.

Of course, however, there are a lot of potentially important problems arising from this simplification. Especially, the dynamics of the overnight rate set by the central bank and the Libor index in the market can be significantly different especially when the credit condition is tight, which suggests the need of independent modeling of these two underlyings. It is an important remaining research topic to develop the model which can handle multiple dynamic curves and its practical calibration scheme [17].

References

[1] Fujii, Masaaki, Shimada, Yasufumi and Takahashi, Akihiko, "A Market Model of Interest Rates with Dynamic Basis Spreads in the Presence of Collateral and Multiple Currencies" December 2009. Available at SSRN: http://ssrn.com/abstract=1520618. CARF Working Paper Series CARF-F-196.

[2] ISDA Margin Survey 2009
www.isda.org/c_and_a/pdf/ISDA-Margin-Survey-2009.pdf

[17] Recently, after the completion of first version of this note, we have written the paper proposing a new framework of interest rate model which allows fully stochastic basis spreads [1], where the resultant curves constructed in this note are directly used as initial conditions of the simulation. We also presented the more straightforward curve construction in multi-currency environment under the collateralization.