A Market Model of Interest Rates With Dynamic Basis Spreads in the Presence of Collateral and Multiple Currencies

Masaaki Fujii
Graduate School of Economics, University of Tokyo

Yasufumi Shimada
Shinsei Bank Ltd

Akihiko Takahashi
Graduate School of Economics, University of Tokyo,
e-mail: akihikot@e.u-tokyo.ac.jp

Abstract
The recent financial crisis caused dramatic widening and elevated volatilities among basis spreads in cross-currency as well as domestic interest rate markets. Furthermore, the widespread use of cash collateral, especially in fixed-income contracts, has made the effective funding cost of financial institutions for the trades significantly different from the Libor of the corresponding payment currency. Because of these market developments, the textbook-style application of a market model of interest rates has now become inappropriate for financial firms; it cannot even reflect the exposures to these basis spreads in pricing, to say nothing of proper delta and vega (or kappa) hedges against their movements. This paper presents a new framework of the market model to address all these issues.

Keywords
market model, HJM model, Libor, tenor, swap, curve, OIS, cross-currency, basis spread, interest rate model, derivatives, multi-currency

1 Introduction
The recent financial crisis and the following liquidity and credit squeeze have caused significant widening and elevated volatilities among various types of basis spread.[1] In particular, we have witnessed dramatic moves in cross-currency swap (CCS), Libor-OIS, and tenor swap[2] (TS) basis spreads. On some occasions, the size of spreads has exceeded several tens of basis points, which is far wider than the general size of bid/offer spreads. Furthermore, there has been a dramatic increase of collateralization in financial contracts in recent years, and it has become almost a market standard at least in the fixed-income world [11]. As will be seen later, the existence of collateral agreement reduces the discounting rate significantly relative to the Libor of a given currency through frequent mark-to-market and collateral postings that follow. Although the Libor market model has been widely used among market participants since its invention, its textbook-style application does not provide an appropriate tool to handle these new realities; it can only treat one type of Libor, and is unable to reflect the movement of spreads among Libors with different tenors. The discounting of a future cash flow is done by the same Libor, which does not reflect the existence of collaterals and the funding cost differentials among multiple currencies in CCS markets.[3]

As a response to these market developments, the invention of a more sophisticated financial model which is able to reflect all the relevant swap prices and their behavior has risen as an urgent task among academics and market participants. Surprisingly, it is not at all a trivial task even constructing a set of yield curves explaining the various swap prices in the market consistently while keeping no-arbitrage conditions intact. Ametrano and Bianchetti (2009) proposed a simple scheme that is able to recover the level of each swap rate in the market, but gives rise to arbitrage possibilities due to the existence of multiple discounting rates within a single currency. The model proposed by Bianchetti (2008) using a multi-currency analogy does not seem to be a practical solution, although it is at least free from arbitrage. The main problem of the model is that the curve calibration cannot be separated from the option calibration due to the entanglement of volatility specifications, since it treats the usual Libor payment as a quanto of different currencies with a pegged FX rate. It also makes the daily hedge against the move of basis spreads quite complicated. In addition, neither Bianchetti (2008) nor Ametrano and Bianchetti (2009) have discussed how to make the model consistent with the collateralization and cross-currency swap markets.

Our recent work, "A note on construction of multiple swap curves with and without collateral" (Fujii, Shimada, and Takahashi, 2009) has developed a method of swap-curve construction which allows us to treat overnight index swaps (OIS), interest rate swaps (IRS), tenor swaps (TS), and cross-currency swaps (CCS) consistently with explicit considerations of the effects from collateralization. The current paper presents a framework of stochastic interest rate models with dynamic basis spreads addressing all the above-mentioned issues, whereas the output of curve calibrations in our recent work (Fujii et al.) can be used directly as a starting point of simulation. In the most generic setup, there remained a difficulty in calibrating all the parameters due to the lack of separate quotes of foreign currency collateralized swaps in the current market. This new work presents a simplified but practical way of implementation which allows exact fits to the domestic currency collateralized OIS, IRS, and TS, together with FX forward and mark-to-market CCS (MtMCCS) without referring to the quotes of foreign collateralized products. Also, this paper adopts an HJM (Heath–Jarrow–Morton)-type framework just for clarity of presentation: of course, it is quite straightforward to write the model using discretized interest rates, which becomes an extension of the Libor and swap market models; see Brace, Gataek, and Musiela, (1997) and Jamshidian (1997). Since our motivation is to explain the generic modeling framework, the details of volatility processes are not specified. Analytic expressions of vanilla options and implications for the risk management of various types of exotics will be presented elsewhere in the future, adopting a fully specified model.

The organization of the paper is as follows: the next section firstly reminds readers of the pricing formula under the collateral agreement. Then, after reviewing the fundamental interest rate products, it presents the modeling framework with stochastic basis spreads in a single-currency environment, which enables us to explain these instruments consistently. Section 3 extends the model into the multi-currency environment and explains how to make the model consistent with the FX forward and MtMCCS. Finally, after Section 4 briefly comments on inflation modeling, Section 5 concludes.

2 Single-Currency Market

This section develops a HJM-type framework of an interest rate model in a single-currency market. Our goal is to construct a framework which is able to explain all the OIS, IRS, and TS markets consistently in a unified way. Here, it is assumed that every trade has a collateral agreement using a domestic currency as collateral.[4]

2.1 Collateralization

Firstly, let us briefly explain the effects of collateralization. Under the collateral agreement, the firm receives the collateral from the counterparty when the present value of the net position is positive and needs to pay the margin called "collateral rate" on the outstanding collateral in exchange. On the other hand, if the present value of the net position is negative, the firm is asked to post the collateral to the counterparty and receives the collateral rate in return. Although the details can possibly differ trade by trade due to the OTC nature of the fixed-income market, the most commonly used collateral is a currency of developed countries, such as USD, EUR, or JPY [11]. In this case, the collateral rate is usually fixed by the overnight rate of the collateral currency: for example, Fed-Fund rate, EONIA, Mutan for USD, EUR, JPY, respectively.

In the general setup, pricing of collateralized products is very hard due to the non-linearity arising from the residual credit risk. Due to the netting procedures, the pricing of each product becomes dependent on the whole contracts with the counterparty, which makes the use of a model impractical for daily pricing and hedging. In order to make the problem tractable, we will assume perfect and continuous collateralization with zero threshold by cash, which means that the mark-to-market and collateral posting is to be made continuously, and the posted amount of cash is 100 percent of the contract's present value. Actually, the daily mark-to-market and adjustment of collateral amount is the market best practice, and the approximation should not be too far from the reality. Under the above simplification, we can think that there remains no counterparty default risk and recover the linearity among different payments. This means that a generic derivative is treated as a portfolio of the independently collateralized strips of payments.

We would like to ask readers to consult section 3 of Fujii et al. (2009) for details, but the present value of a collateralized derivative with payment $h(T)$ at time T is given by:[5]

$$h(t) = E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right], \tag{2.1}$$

where $E_t^Q[\cdot]$ denotes the expectation under the money market (MM) measure Q conditioned on the time-t filtration, and $c(s)$ is the time-s value of the collateral rate. Note that $c(s)$ is not necessarily equal to the risk-free interest rate $r(s)$ of a given currency.

For our later purpose, let us define the collateralized zero-coupon bond D as

$$D(t,T) = E_t^Q\left[e^{-\int_t^T c(s)ds}\right], \tag{2.2}$$

which is the present value of the unit amount of payment under the contract of continuous collateralization with the same currency. In later sections, we will frequently use the expectation $E^{\tau^c}[\cdot]$ under the collateralized forward measure τ^c defined as

$$E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] = D(t,T)E_t^{T^c}\left[h(T)\right], \tag{2.3}$$

where the collateralized zero-coupon bond $D(\cdot,T)$ is used as numeraire.

2.2 Market Instruments

Before going on to discuss the modeling framework, this subsection briefly summarizes the important swaps in a domestic market as well as the conditions that par swap rates have to satisfy. They are the most important calibration instruments to fix the starting points of simulation.

2.2.1 Overnight Index Swap

An OIS is a fixed vs floating swap whose floating rate is given by the daily compounded overnight rate. Since the overnight rate is the same as the collateral rate of the corresponding currency, the following relation holds:[6]

$$\mathrm{OIS}_N(t)\sum_{n=1}^{N}\Delta_n E_t^Q\left[e^{-\int_t^{T_n}c(s)ds}\right]=\sum_{n=1}^{N}E_t^Q\left[e^{-\int_t^{T_n}c(s)ds}\left(e^{\int_{T_{n-1}}^{T_n}c(s)ds}-1\right)\right], \tag{2.4}$$

or equivalently,

$$\mathrm{OIS}_N(t)\sum_{n=1}^{N}\Delta_n D(t,T_n)=D(t,T_0)-D(t,T_N), \tag{2.5}$$

where $OIS_N(t) = OIS(t,T_0,T_N)$ is the market quote at time t of the T_0-start T_N-maturing OIS rate, and T_0 is the effective date in the case of spot-start OIS. Also, Δ_n denotes the fixed-leg day count fraction for the period (T_{n-1},T_n).

2.2.2 Interest Rate Swap

In an IRS, two parties exchange a fixed coupon and Libor for a certain period with a given frequency. The tenor τ of Libor is determined by the frequency of floating payments (6m tenor for semi-annual payments, for example). For a T_0-start T_M-maturing IRS with Libor of tenor τ, we have

$$\mathrm{IRS}_M(t)\sum_{m=1}^{M}\Delta_m D(t,T_m)=\sum_{m=1}^{M}\delta_m D(t,T_m)E_t^{T_m^c}[L(T_{m-1},T_m;\tau)] \tag{2.6}$$

as consistency condition. Here, $\mathrm{IRS}_M(t) = \mathrm{IRS}(t, T_0,T_M; \tau)$ is the time-t value of the corresponding IRS quote, $L(T_{m-1},T_m;\tau)$ is the Libor rate with tenor τ for a period (T_{m-1}, T_m), and δ_m is its day count fraction. In the remainder of this paper, we distinguish the difference in day count conventions between the fixed and floating legs by Δ and δ, respectively.

Here, it is assumed that the frequencies of both legs are equal just for simplicity, and it does not affect our later arguments even if this is not the case. Usually, IRS with a specific choice of τ has dominant liquidity in a given currency market, such as 6m for JPY IRS and 3m for USD IRS. Information on forward Libors with other tenors is provided by tenor swaps, which will be explained next.

2.2.3 Tenor Swap

A tenor swap is a floating vs floating swap where the parties exchange Libors with different tenors with a fixed spread on one side, which we call the TS basis spread in this paper. Usually, the spread is added on top of the Libor with shorter tenor. For example, in a 3m/6m tenor swap, quarterly payments with 3m Libor plus spread are exchanged by semi-annual payments of 6m Libor flat. The condition that the tenor spread should satisfy is given by

$$\sum_{n=1}^{N}\delta_n D(t,T_n)\left(E_t^{T_n^c}[L(T_{n-1},T_n;\tau_S)]+TS(t)\right)=\sum_{m=1}^{M}\delta_m D(t,T_m)E_t^{T_m^c}[L(T_{m-1},T_m;\tau_L)], \tag{2.7}$$

where $T_N = T_M$, m, and n distinguish the difference in payment frequency. $TS(t) = TS(t, T_0, T_N; \tau_S, \tau_L)$ denotes the time-t value of the TS basis spread for the T_0-start T_N-maturing tenor swap. The spread is added on to the Libor with shorter tenor τ_s in exchange for the Libor with longer tenor τ_L.

Here, we have explained the use of slightly simplified terms of the contract. In the actual market, terms of the contract in which coupons of the leg with the shorter tenor are compounded by Libor flat and paid with the same frequency as the other leg are more popular. However, the size of correction from the above simplified result can be shown to be negligibly small. See Appendix A for details.

2.2.4 Underlying Factors in the Model

Using the above instruments and the method explained in Fujii et al. (2009), we can extract

$$\{D(t,T)\},\ \{E_t^{T^c}[L(T-\tau,T;\tau)]\} \tag{2.8}$$

for continuous time $T \in [0, T_H]$, where T_H is the time horizon of relevant pricing,[7] with each relevant tenor τ (1m, 3m, 6m, 12m, for example).[8] The next section will explain how to make these underlying factors consistent with no-arbitrage conditions in an HJM-type framework.

2.3 Model with Dynamic Basis Spreads in a Single Currency

As seen in Section 2.1, the collateral rate plays a critical role as the effective discounting rate, which leads us to consider its dynamics first. Let us define the continuous forward collateral rate as

$$c(t,T)=-\frac{\partial}{\partial T}lnD(t,T) \tag{2.9}$$

or, equivalently,

$$D(t,T)=e^{-\int_t^T c(t,s)ds}, \tag{2.10}$$

where it is related to the spot rate as $c(t,t) = c(t)$. Then, assume that the dynamics of the forward collateral rate under the MM measure Q is given by

$$dc(t,s)=\alpha(t,s)dt+\sigma_c(t,s)\cdot dW^Q(t), \tag{2.11}$$

where $\alpha(t,s)$ is a scalar function for its drift, and $W^Q(t)$ is a d-dimensional Brownian motion under the Q-measure. $\sigma_c(t,s)$ is a d-dimensional vector and the following abbreviation has been used:

$$\sigma_c(t,s)\cdot dW^Q(t)=\sum_{j=1}^{d}[\sigma_c(t,s)]_j\,dW_j^Q(t). \tag{2.12}$$

As mentioned in the Introduction, the details of the volatility process will not be specified; it can depend on the collateral rate itself, or any other state variables.

Applying Itô's formula to (2.10), we have

$$\begin{aligned}\frac{dD(t,T)}{D(t,T)}&=\left\{c(t)-\int_t^T\alpha(t,s)ds+\frac{1}{2}\left\|\int_t^T\sigma_c(t,s)ds\right\|^2\right\}dt\\&\quad-\left(\int_t^T\sigma_c(t,s)ds\right)\cdot dW_t^Q.\end{aligned} \tag{2.13}$$

On the other hand, from the definition of (2.2), the drift rate of $D(t,T)$ should be $c(t)$. Therefore, it is necessary that

$$\alpha(t,s) = \sum_{j=1}^{d} [\sigma_c(t,s)]_j \left(\int_t^s \sigma_c(t,u)du \right)_j \quad (2.14)$$

$$= \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right), \quad (2.15)$$

and as a result, the process of $c(t,s)$ under the Q-measure is obtained by

$$dc(t,s) = \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right) dt + \sigma_c(t,s) \cdot dW^Q(t). \quad (2.16)$$

Now, let us consider the dynamics of Libors with various tenors. Mercurio [14] has proposed an interesting simulation scheme.[9] He follows the original idea of the Libor market model, and has modeled the market observables or forward expectations of Libors directly, instead of considering the corresponding spot process as in Bianchetti (2008). We will adopt Mercurio's scheme, but separating the spread processes explicitly.

Firstly, define the collateralized forward Libor, and the OIS forward as

$$L^c(t,T_{k-1},T_k;\tau) = E_t^{\tau_k^c}\left[L(T_{k-1},T_k;\tau)\right], \quad (2.17)$$

$$L^{\mathrm{OIS}}(t,T_{k-1},T_k) = E_t^{\tau_k^c}\left[\frac{1}{\delta_k}\left(\frac{1}{D(T_{k-1},T_k)} - 1\right)\right] \quad (2.18)$$

$$= \frac{1}{\delta_k}\left(\frac{D(t,T_{k-1})}{D(t,T_k)} - 1\right), \quad (2.19)$$

and also define the Libor–OIS spread process:

$$B(t,T_k;\tau) = L^c(t,T_{k-1},T_k;\tau) - L^{\mathrm{OIS}}(t,T_{k-1},T_k). \quad (2.20)$$

By construction, $B(t,T;\tau)$ is a martingale under the collateralized forward measure τ^c, and its stochastic differential equation can be written

$$dB(t,T;\tau) = B(t,T;\tau)\sigma_B(t,T;\tau) \cdot dW^{\mathcal{T}_c}(t), \quad (2.21)$$

where the d-dimensional volatility function σ_B can depend on B or other state variables as before. Using Maruyama–Girsanov's theorem, one can see that the Brownian motion under the τ^c-measure, $W^{\tau^c}(t)$, is related to $W^Q(t)$ by the following relation:

$$dW^{\mathcal{T}^c}(t) = \left(\int_t^T \sigma_c(t,s)ds \right) dt + dW^Q(t). \quad (2.22)$$

As a result, the process of $B(t,T;\tau)$ under the Q-measure is obtained by

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t,T;\tau) \cdot \left(\int_t^T \sigma_c(t,s)ds \right) dt + \sigma_B(t,T;\tau) \cdot dW^Q(t). \quad (2.23)$$

We need to specify the B-processes for all the relevant tenors in the market (1m, 3m, 6m, and 12m, for example). If one wants to guarantee the positivity for $B(\cdot,T;\tau_S)$ where $\tau_L > \tau_S$, it is possible to model this spread as (2.23) directly.

The list of what we need only consists of these two types of underlying. As one can see, there is no explicit need to simulate the risk-free interest rate in a single-currency environment if all the interested trades are collateralized with the same domestic currency. Let us summarize the relevant equations:

$$dc(t,s) = \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right) dt + \sigma_c(t,s) \cdot dW^Q(t), \quad (2.24)$$

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t,T;\tau) \cdot \left(\int_t^T \sigma_c(t,s)ds \right) dt + \sigma_B(t,T;\tau) \cdot dW^Q(t). \quad (2.25)$$

Since we already have $\{c(t,s)\}_{s\geq t}$, and $\{B(t,T;\tau)\}_{T\geq t}$ teach for the relevant tenor, after curve construction as explained in Fujii et al. (2009), we can use them directly as starting points for simulation. If one needs an equity process $S(t)$ with an effective dividend yield given by $q(t)$ with the same collateral agreement, we can model it as

$$dS(t)/S(t) = (c(t) - q(t))\,dt + \sigma_S(t) \cdot dW^Q(t), \quad (2.26)$$

and σ_S and q can be state dependent. Note that the effective dividend yield q is not equal to the dividend yield in the non-collateralized trade, but should be adjusted by the difference between the collateral rate and the risk-free rate.[10] In practice, it is likely not a big problem to use the same value or process as the usual definition of dividend yield. Here, we are not trying to reflect the details of repo cost for an individual stock, but rather trying to model a stock index, such as S&P500, for IR–equity hybrid trades.

2.4 Simple Options in a Single Currency

This subsection explains the procedures for simple option pricing in a single-currency environment. In the following, suppose that all the forward and option contracts themselves are collateralized with the same domestic currency.

2.4.1 Collateralized Overnight Index Swaption

As was seen in Section 2.2.1, a T_0-start T_N-maturing forward OIS rate at time t is given by

$$\mathrm{OIS}(t,T_0,T_N) = \frac{D(t,T_0) - D(t,T_N)}{\sum_{n=1}^{N} \Delta_n D(t,T_n)}. \quad (2.27)$$

When the length of OIS is very short and there is only one final payment, we can get the correct expression by simply replacing the annuity in the denominator by $\Delta_N D(t,T_N)$, a collateralized zero-coupon bond times a day count fraction for the fixed payment.

Under the annuity measure $\mathcal{A}$, where the annuity $A(t,T_0,T_N) = \Sigma_{n=1}^{N} \Delta_n D(t,T_n)$ is being used as numeraire, the above OIS rate becomes a martingale. Therefore, the present value of a collateralized payer option on the OIS with strike K is given by

$$PV(t) = A(t,T_0,T_N) E_t^{\mathcal{A}}\left[(\mathrm{OIS}(T_0,T_0,T_N) - K)^+\right], \quad (2.28)$$

where one can show that the stochastic differential equation for the forward OIS is given as follows under the $\mathcal{A}$-measure:

$$dOIS(t,T_0,T_N) = OIS(t,T_0,T_N)\left\{\frac{D(t,T_N)}{D(t,T_0)-D(t,T_N)}\left(\int_{T_0}^{T_N}\sigma_c(t,s)ds\right)\right.$$
$$\left.+\frac{1}{A(t,T_0,T_N)}\sum_{n=1}^{N}\Delta_n D(t,T_n)\left(\int_{T_0}^{T_n}\sigma_c(t,s)ds\right)\right\}\cdot dW^{\mathcal{A}}(t), \tag{2.29}$$

where $W^{\mathcal{A}}(t)$ is the Brownian motion under the $\mathcal{A}$-measure, and is related to $W^{Q}(t)$ as

$$dW^{\mathcal{A}}(t) = dW^{Q}(t) + \frac{1}{A(t,T_0,T_N)}\sum_{n=1}^{N}\Delta_n D(t,T_n)\left(\int_{t}^{T_n}\sigma_c(t,s)ds\right)dt. \tag{2.30}$$

We can derive an accurate approximation of (2.28) by applying an asymptotic expansion technique (e.g., Takashashi, 1995; Takashashi, 1999; Takashashi, Takehara, and Toda, 2009), or ad hoc but simpler methods given, for example, by Brigo and Mercurio (2006).

2.4.2 Collateralized Interest Rate Swaption

Next, let us consider the usual swaption with the collateral agreement. As we have seen in Section 2.2.2, a T_0-start T_N-maturing collateralized forward swap rate is given by

$$IRS(t,T_0,T_N;\tau) = \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)L^c(t,T_{n-1},T_n;\tau)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)} \tag{2.31}$$

$$= \frac{D(t,T_0)-D(t,T_N)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)} + \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)} \tag{2.32}$$

$$= OIS(t,T_0,T_N) + Sp^{OIS}(t,T_0,T_N;t), \tag{2.33}$$

where we have defined the IRS–OIS spread Sp^{OIS} as

$$Sp^{OIS}(t,T_0,T_N;\tau) = \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)}{\sum_{n=1}^{N}\Delta_n D(t,T_n)}. \tag{2.34}$$

Note that we have slightly abused the notation of OIS(t). In reality, there is no guarantee that the day count conventions and frequencies are the same between IRS and OIS, which may require appropriate adjustments.

Sp^{OIS} is a martingale under the $\mathcal{A}$-measure, and one can show that its stochastic differential equation is given by

$$dSp^{OIS}(t,T_0,T_N;\tau) = Sp^{OIS}(t)\left\{\frac{1}{A(t,T_0,T_N)}\sum_{j=1}^{N}\Delta_j D(t,T_j)\left(\int_{T_0}^{T_j}\sigma_c(t,s)ds\right)\right.$$
$$+\frac{1}{A_{sp}(t,T_0,T_N;\tau)}\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)$$
$$\left.\times\left(\sigma_B(t,T_n;\tau)-\int_{T_0}^{T_n}\sigma_c(t,s)ds\right)\right\}\cdot dW(t), \tag{2.35}$$

where we have defined

$$A_{sp}(t,T_0,T_N;\tau) = \sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau) \tag{2.36}$$

Since the IRS forward rate is a martingale under the annuity measure $\mathcal{A}$, the present value of a T_0 into T_N collateralized payer swaption is expressed as

$$PV(t) = A(t,T_0,T_N)E_t^{\mathcal{A}}\left[\left(OIS(T_0,T_0,T_N)+Sp^{OIS}(T_0,T_0,T_N;\tau)-K\right)^+\right]. \tag{2.37}$$

As in the previous OIS case, we can use the asymptotic expansion technique or other methods to derive an analytic approximation for this option.

2.4.3 Collateralized Tenor Swaption

Finally, consider an option on a tenor swap. From Section 2.2.3, the forward TS spread for a collateralized T_0-start $T_N(=T_M)$-maturing swap which exchanges Libors with tenors τ_S and τ_L is given by

$$TS(t,T_0,T_N;\tau_S,\tau_L)$$
$$= \frac{\sum_{m=1}^{M}\delta_m D(t,T_m)L^c(t,T_{m-1},T_m;\tau_L)-\sum_{n=1}^{N}\delta_n D(t,T_n)L^c(t,T_{n-1},T_n;\tau_S)}{\sum_{n=1}^{N}\delta_n D(t,T_n)}$$
$$= \frac{\sum_{m=1}^{M}\delta_m D(t,T_m)B(t,T_m;\tau_L)}{\sum_{n=1}^{N}\delta_n D(t,T_n)} - \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau_L)}{\sum_{n=1}^{N}\delta_n D(t,T_n)} \tag{2.38}$$

where we have distinguished the different payment frequencies by n and m. In the case of a 3m/6m tenor swap, for example, $N = 2M$, $\tau_S = 3\text{m}$, and $\tau_L = 6\text{m}$. Since the two terms in (2.38) are equal to Sp^{OIS} except for the difference in day count conventions, the tenor swaption is basically equivalent to a spread option between two different Sp^{OIS}s. The present value of a collateralized payer tenor swaption with strike K can be expressed as

$$PV(t) = \left(\sum_{n=1}^{N}\delta_n D(t,T_n)\right)E_t^{\tilde{\mathcal{A}}}\left[\left(TS(T_0,T_0,T_N;\tau_S,\tau_L)-K\right)^+\right]. \tag{2.39}$$

Here, $E_t^{\tilde{\mathcal{A}}}[\cdot]$ denotes the expectation under the annuity measure with day count fraction specified by that of the floating leg, δ.

These options, explained in Sections 2.4.1, 2.4.2, and 2.4.3, can allow us to extract volatility information for our model. Considering the current situation where there is no liquid market of options on the relevant basis spreads, we probably need to combine some historical estimation for the volatility calibration.

3 Multiple-Currency Market

This section extends the framework developed in the previous section into a multi-currency environment. For our later purpose, let us define several variables first. The T-maturing risk-free zero-coupon bond of currency k is denoted by $P^{(k)}(\cdot, T)$, and is calculated from the equation

$$P^{(k)}(t,T) = E_t^{Q_k}\left[e^{-\int_t^T r^{(k)}(s)ds} \right], \tag{3.1}$$

where Q_k and $r^{(k)}$ denote the MM measure and the risk-free interest rate for the k-currency. Also, define the instantaneous risk-free forward rate by

$$f^{(k)}(t,T) = -\frac{\partial}{\partial T} \ln P^{(k)}(t,T) \tag{3.2}$$

as usual, and $r^{(k)}(t) = f^{(k)}(t,t)$.

As is well known, its stochastic differential equation under the domestic MM measure Q_k is given by

$$df^{(k)}(t,s) = \sigma^{(k)}(t,s)\cdot\left(\int_t^s \sigma^{(k)}(t,u)du\right)dt + \sigma^{(k)}(t,s)\cdot dW^{Q_k}(t), \tag{3.3}$$

where $W^{Q_k}(t)$ is the d-dimensional Brownian motion under the Q_k-measure. The volatility term $\sigma^{(k)}$ is a d-dimensional vector and possibly depends on $f^{(k)}$ or any other state variables. Here, we have shown that the risk-free interest rate to make the structure of the model easy to understand through our scheme does not simulate it directly, as will be seen later.

Let us also define the spot foreign exchange rate between currency i and j:

$$f_x^{(i,j)}(t). \tag{3.4}$$

It denotes the time-t value of a unit amount of currency j in terms of currency i. Then, define its dynamics under the Q_i-measure as

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = (r^{(i)}(\tilde{t}) - r^{(j)}(t))dt + \sigma_X^{(i,j)}(t)\cdot dW^{Q_i}(t). \tag{3.5}$$

The volatility term can depend on $fx^{(i,j)}$ or any other state variables. The Brownian motions of two different MM measures are connected to each other by the relation

$$dW^{Q_i}(t) = \sigma_x^{(i,j)}(t)dt + dW^{Q_j}(t), \tag{3.6}$$

as indicated by Maruyama–Girsanov's theorem.

3.1 Collateralization with Foreign Currencies

Until this point, the collateral currency has been assumed to be the same as the payment currency of the contract. However, this assumption cannot be maintained in a multi-currency environment, since multi-currency trades contain different currencies in their payments in general. In fact, this currency mismatch is inevitable in a CCS trade whose payments contain two different currencies, but only one collateral currency.

Our previous work (Fujii et al., 2009) has provided a pricing formula for a generic financial product whose collateral currency j is different from its payment currency k:

$$h^{(k)}(t) = E_t^{Q_k}\left[e^{-\int_t^T r^{(k)}(s)ds}\left(e^{\int_t^T (r^{(j)}(s)-c^{(j)}(s))ds}\right) h^{(k)}(T)\right] \tag{3.7}$$

$$= P^{(k)}(t,T)E_t^{(k)}\left[\left(e^{\int_t^T (r^{(j)}(s)-c^{(j)}(s))ds}\right) h^{(k)}(T)\right]. \tag{3.8}$$

Here, $h^{(k)}(t)$ is the present value of a financial derivative whose payment $h^{(k)}(T)$ is to be made at time T in the k-currency. The collateralization is assumed to be made continuously by cash of the j-currency with zero threshold, and $c^{(j)}$ is the corresponding collateral rate. $E_t^{\tau(k)}[\cdot]$ denotes the expectation under the risk-free forward measure of currency k, $\tau_{(k)}$, where the risk-free zero-coupon bond $P^{(k)}(\cdot,T)$ is used as numeraire.

As is clear from these arguments, the price of a financial product depends on the choice of collateral currency. Let us check this impact for the most fundamental instruments (i.e., FX forward contracts and Libor payments) in the next sections.

3.1.1 FX Forward and Currency Triangle

As is well known, the currency triangle relation should be satisfied among arbitrary combinations of currencies (j,k,l):

$$f_x^{(j,k)}(t) = f_x^{(j,l)}(t)\times f_x^{(l,k)}(t), \tag{3.9}$$

otherwise the difference will soon be arbitraged away in the current liquid foreign exchange market. In the default-free market without collateral agreement, this relation should also hold in the FX forward market. However, it is not a trivial issue in the presence of collateral, as will be seen below.[11]

Let us consider a k-currency collateralized FX forward contract between the currencies (i,j). The FX forward rate $f_x^{(i,j)}(t,T)$ is given by the amount of i-currency to be exchanged by the unit amount of j-currency at time T with zero present value:

$$f_x^{(i,j)}(t,T)P^{(i)}(t,T)E_t^{\tau_{(i)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right] = f_x^{(i,j)}(t)P^{(j)}(t,T)E_t^{\tau_{(j)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right], \tag{3.10}$$

and hence

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)}\left(\frac{E_t^{\tau_{(j)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right]}{E_t^{\tau_{(i)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right]}\right). \tag{3.11}$$

From the above equation, it is clear that the currency triangle relation only holds among the trades with common collateral currency, in general.

3.1.2 Libor Payment Collateralized with Foreign Currency

Next, let us consider the implications for a foreign currency collateralized Libor payment. Using the result of Section 3.1, the present value of a k-currency Libor payment with cash collateral of j-currency is given by

$$PV(t) = \delta_n P^{(k)}(t, T_n) E_t^{\tau_{n,(k)}} \left[e^{\int_t^{T_n} (r^{(j)}(s) - c^{(j)}(s)) ds} L^{(k)}(T_{n-1}, T_n; \tau) \right]. \tag{3.12}$$

Remember that if the Libor is collateralized by the same domestic currency k, the present value of the same payment is given by

$$PV(t) = \delta_n D^{(k)}(t, T_n) E_t^{\tau^c_{n,(k)}} \left[L^{(k)}(T_{n-1}, T_n; \tau) \right] \tag{3.13}$$

$$= \delta_n P^{(k)}(t, T_n) E_t^{\tau_{n,(k)}} \left[e^{\int_t^{T_n} (r^{(k)}(s) - c^{(k)}(s)) ds} L^{(k)}(T_{n-1}, T_n; \tau) \right]. \tag{3.14}$$

Here, the superscript c in $\tau^c_n,(k)$ of $E_t^{\tau^c_n,(k)}[\cdot]$ denotes that the expectation is taken under the collateralized forward measure instead of the risk-free forward measure. The above results suggest that the price of an interest rate product, such as IRS, does depend on the choice of its collateral currency.

3.1.3 Simplification for Practical Implementation

The findings of Sections 3.1.1 and 3.1.2 give rise to a big difficulty for practical implementation. If all the relevant vanilla products have separate quotes as well as sufficient liquidity for each collateral currency, it is possible to set up a separate multi-currency model for each choice of collateral currency. However, separate quotes for different collateral currencies are unobservable in the actual market. Furthermore, closing the hedges within each collateral currency is unrealistic. This is because one would like to use JPY domestic IR swaps to hedge the JPY Libor exposure in a complicated multi-currency derivative collateralized by EUR, for example. The setup of a separate model for each collateral currency will make these hedges too complicated.

In order to avoid these difficulties, let us adopt a very simple assumption that

$$\sigma^{(k)}(t, s) = \sigma_c^{(k)}(t, s). \tag{3.15}$$

or

$$y^{(k)}(t, s) = f^{(k)}(t, s) - c^{(k)}(t, s). \tag{3.16}$$

is a deterministic function of t for each s and for every currency k. Here, $\sigma_c^{(k)}$ is the volatility term defined for the forward collateral rate of the k-currency as in (2.16). Under this assumption, one can show that

$$r^{(k)}(t) - c^{(k)}(t) = f^{(k)}(s, t) - c^{(k)}(s, t) \tag{3.17}$$

for any $s \le t$. Hence, it follows that

$$y^{(k)}(t) = r^{(k)}(t) - c^{(k)}(t) \tag{3.18}$$

as a deterministic function of time.

Under this assumption, one can see that the FX forward rate in (3.11) becomes

$$f_x^{(i,j)}(t, T) = f_x^{(i,j)}(t) \frac{P^{(j)}(t, T)}{P^{(i)}(t, T)}, \tag{3.19}$$

and it is independent of the choice of collateral currency. Therefore, the cross-currency triangle relation holds among FX forwards even when they contain multiple collateral currencies.

In addition, the collateralized forward expectation and the risk-free forward expectation are equal for each currency k:

$$E_t^{\mathcal{T}^c_{(k)}}[\cdot] = E_t^{\mathcal{T}_{(k)}}[\cdot], \tag{3.20}$$

since the corresponding Radon–Nikodym derivative becomes constant:

$$e^{-\int_0^{\cdot} (r^{(k)}(s) - c^{(k)}(s)) ds} \frac{P^{(k)}(\cdot, T)}{D^{(k)}(\cdot, T)} \frac{D^{(k)}(0, T)}{P^{(k)}(0, T)} \equiv 1. \tag{3.21}$$

Now, (3.12) turns out to be

$$PV(t) = \delta_n P^{(k)}(t, T_n) e^{\int_t^{T_n} y^{(j)}(s) ds} E_t^{\tau_{n,(k)}} \left[L^{(k)}(T_{n-1}, T_n; \tau) \right] \tag{3.22}$$

$$= \delta_n D^{(k)}(t, T_n) e^{\int_t^{T_n} (y^{(j)}(s) - y^{(k)}(s)) ds} E_t^{\tau_{n,(k)}} \left[L^{(k)}(T_{n-1}, T_n; \tau) \right]. \tag{3.23}$$

Since it holds that $E_t^{\tau^c_{(k)}}[\cdot] = E_t^{\tau_{(k)}}[\cdot]$ under the current assumption, even if the Libor payment is collateralized by a foreign j-currency, it is straightforward to calculate the exposure in terms of the standard IRS collateralized by the domestic currency.

One can see that all the corrections from our simplifying assumption arise from either the convexity correction in $E\left[e \int_t^{\tau} y^{(k)}(s) ds\right]$ or from the covariance between $e \int_t^{\tau} y^{(k)}(s) ds$ and another stochastic variable such as Libor or FX rates. Considering the absolute size of the spread y and its volatility, one can reasonably expect that the corrections are quite small. Actually, the fact that separate quotes of these instruments for each collateral currency are unobservable indicates that the corrections induced from the assumptions are well within the current market bid/offer spreads. As will be seen in the following sections, the above assumption will allow a flexible enough framework to address the issues described in the Introduction without causing unnecessary complications.

3.2 Model with Dynamic Basis Spreads in Multiple Currencies

Now, let us finally preset the modeling framework in the multi-currency environment under the simplified assumption given in Section 3.1.3. We have already set up the dynamics for the forward collateral rate Libor–OIS spread for each tenor, and an equity with effective dividend yield q for each currency as in Section 2.3:

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s)\cdot\left(\int_t^s \sigma_c^{(i)}(t,u)du\right)dt + \sigma_c^{(i)}(t,s)\cdot dW^{Q_i}(t), \tag{3.24}$$

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t,T;\tau)\cdot\left(\int_t^T \sigma_c^{(i)}(t,s)ds\right)dt + \sigma_B^{(i)}(t,T;\tau)\cdot dW^{Q_i}(t), \tag{3.25}$$

$$dS^{(i)}(t)/S^{(i)}(t) = (c^{(i)}(t) - q^{(i)}(t))dt + \sigma_S^{(i)}(t)\cdot dW^{Q_i}(t). \tag{3.26}$$

We have the above set of stochastic differential equations for each currency i. The foreign exchange dynamics between currency i and j is given by

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW^{Q_i}(t), \tag{3.27}$$

where $y^{(i,j)}(t)$ is defined as

$$y^{(i,j)}(t) = y^{(i)}(t) - y^{(j)}(t) \tag{3.28}$$

$$= (r^{(i)}(t) - r^{(j)}(t)) - (c^{(i)}(t) - c^{(j)}(t)), \tag{3.29}$$

which is a deterministic function of time.

If a specific currency i is chosen to be a home currency for simulation, the stochastic differential equations for other currencies $j \neq i$ are given by

$$dc^{(j)}(t,s) = \sigma_c^{(j)}(t,s)\cdot\left[\left(\int_t^s \sigma_c^{(j)}(t,u)du\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_c^{(j)}(t,s)\cdot dW^{Q_i}(t), \tag{3.30}$$

$$\frac{dB^{(j)}(t,T;\tau)}{B^{(j)}(t,T;\tau)} = \sigma_B^{(j)}(t,T;\tau)\cdot\left[\left(\int_t^T \sigma_c^{(j)}(t,s)ds\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_B^{(j)}(t,T;\tau)\cdot dW^{Q_i}(t), \tag{3.31}$$

$$dS^{(j)}(t)/S^{(j)}(t) = \left[\left(c^{(j)}(t) - q^{(j)}(t)\right) - \sigma_S^{(j)}(t)\cdot\sigma_X^{(i,j)}(t)\right]dt + \sigma_S^{(j)}(t)\cdot dW^{Q_i}(t), \tag{3.32}$$

where the relation (3.6) has been used. These are the relevant underlying factors for a multi-currency environment.

3.3 Curve Calibration

This section explains how to set up the initial conditions for the modeling framework explained in the previous section. As we will see, the spread curves $\{y(t)^{(i,j)}\}$ for the relevant currency pairs can be bootstrapped by fitting to the term structure of the CCS basis spread, or equivalently to the FX forwards.

3.3.1 Single-Currency Instruments

Let us first remember the setup of the single-currency sector of the model. As explained in Section 2.3, the collateralized zero-coupon bonds $D(t,T)$ and Libor expectations $E_t^{\tau_k^c}[L(T_{k-1},T_k;\tau)]$ can be extracted from the following set of equations:

$$\mathrm{OIS}_N^{(i)}(t)\sum_{n=1}^N \Delta_n^{(i)} D^{(i)}(t,\mathrm{T}_n) = D^{(i)}(t,T_0) - D^{(i)}(t,T_N) \tag{3.33}$$

$$\mathrm{IRS}_M^{(i)}(t)\sum_{m=1}^M \Delta_m^{(i)} D^{(i)}(t,T_m) = \sum_{m=1}^M \delta_m^{(i)} D^{(i)}(t,T_m) E_t^{\tau^c_{m,(i)}}[L^{(i)}(T_{m-1},T_m;\tau)] \tag{3.34}$$

$$\sum_{n=1}^N \delta_n^{(i)} D^{(i)}(t,T_n)\left(E_t^{\tau^c_{n,(i)}}\left[L^{(i)}(T_{n-1},T_n;\tau_S)\right] + TS^{(i)}(t)\right)$$
$$= \sum_{m=1}^M \delta_m^{(i)} D^{(i)}(t,T_m) E_t^{\mathcal{T}^c_{m,(i)}}\left[L^{(i)}(T_{m-1},T_m;\tau_L)\right], \tag{3.35}$$

for OIS, IRS, and TS contracts, respectively. Using the relations

$$c^{(i)}(t,s) = -\frac{\partial}{\partial s}\ln D^{(i)}(t,s) \tag{3.36}$$

and

$$B^{(i)}(t,T_n;\tau) = E_t^{\tau^c_{n,(i)}}\left[L^{(i)}(T_{n-1},T_n;\tau)\right] - \frac{1}{\delta_n^{(i)}}\left(\frac{D^{(i)}(t,T_{n-1})}{D^{(i)}(t,T_n)} - 1\right), \tag{3.37}$$

one can get the initial conditions for the collateral rate $c(t,s)$, and the Libor–OIS spreads $B(t,T;\tau)$ for each currency.

3.3.2 FX Forward

Next, let us consider FX forward contracts. In the current setup, a FX forward contract maturing at time T between currencies (i,j) becomes

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)} \tag{3.38}$$

$$= f_x^{(i,j)}(t)\frac{D^{(j)}(t,T)}{D^{(i)}(t,T)} e^{\int_t^T y^{(i,j)}(s)ds}. \tag{3.39}$$

By the quotes of spot and forward FX rates, and the $\{D(t,T)\}$ derived in the previous section, the value of $\int_t^T y^{(i,j)}(s)ds$ can be found. Based on the quotes for various maturities T and the proper spline technique, $y^{(i,j)}(s)$ will be obtained as a continuous function of time s. This can be done for all the relevant pairs of currencies. This will give another important input of the model required in (3.27). If one needs to assume that the collateral rate of a given currency i is actually the risk-free rate, the set of functions $\{y^{(j)}(s)\}j \neq i$ can be obtained by combination of the information of FX forwards with $y^{(i)}(s) \equiv 0$. Note that one cannot assume the several collateral rates are equal to the risk-free rates simultaneously, since the model should be made consistent with FX forwards (and CCS).

As mentioned before, the current setup does not recognize the differences among FX forwards from their choice of collateral currencies. It arises from our simplified assumption that the spread between the risk-free and the collateral rates of a given currency is a deterministic function of time. This seems consistent with the reality, at least in the current market.[12]

3.4 Other Vanilla Instruments

The instruments explained in Sections 3.3.1 and 3.3.2 are sufficient to fix the initial conditions of the curves used in the model. Next, let us check other fundamental instruments and the implications of the model.

3.4.1 European FX Option

Calculation of a European FX option is quite simple. Let us consider the T-maturing FX call option for $f_x^{(i,j)}$ collateralized by the k-currency. The present value can be written as

$$PV(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(k)}(s)ds} \left(f_x^{(i,j)}(T) - K \right)^+ \right] \tag{3.40}$$

$$= D^{(i)}(t,T) e^{\int_t^T y^{(k,i)}(s)ds} E_t^{\mathcal{T}_{(i)}} \left[\left(f_x^{(i,j)}(T,T) - K \right)^+ \right]. \tag{3.41}$$

The FX forward $f_x^{(i,j)}(\cdot, T)$ is a martingale under the forward measure $\mathcal{T}_{(i)}$ (or equivalently $\mathcal{T}_{(i)}^c$ in our assumption), and its stochastic differential equation is given by

$$\frac{df_x^{(i,j)}(t,T)}{f_x^{(i,j)}(t,T)} = \sigma_{FX}^{(i,j)}(t,T) \cdot dW^{\tau_{(i)}^c}(t) \tag{3.42}$$

$$= \left\{ \sigma_X^{(i,j)}(t) + \int_t^T \sigma_c^{(i)}(t,s)ds - \int_t^T \sigma_c^{(j)}(t,s)ds \right\} \cdot dW^{\tau_{(i)}^c}(t), \tag{3.43}$$

under the same forward measure. It is straightforward to obtain an analytical approximation of (3.41).

3.4.2 Constant Notional Cross-Currency Swap

A constant notional CCS (CNCCS) of a currency pair (i,j) is a floating vs floating swap where the two parties exchange the i-Libor flat vs j-Libor plus fixed spread periodically for a certain period. There are both initial and final notional exchanges, and the notional for each leg is kept constant throughout the contract. The currency i, in which Libor is paid in flat, is dominated by USD in the market. CNCCS has been used to convert a loan denominated in a given currency to that of another currency to reduce its funding cost. Owing to its significant FX exposure, mark-to-market CCS (MtMCCS), which will be explained in the next section, has now become quite popular. The information in CNCCS is equivalent to that extracted from FX forwards, since CNCCS combined with IRS and TS with the same collateral currency can replicate an FX forward contract.

Here, we will provide the formula for the CNCCS of a currency pair (i,j), just for completeness. Assume that the collateral is posted in the i-currency. Then, the present value of the i-leg for unit notional is given by

$$PV_i(t) = \sum_{n=1}^N \delta_n^{(i)} D^{(i)}(t,T_n) E_t^{\tau_{n,(i)}^c} \left[L^{(i)}(T_{n-1},T_n;\tau) \right] - D^{(i)}(t,T_0) + D^{(i)}(t,T_N)$$

$$= \sum_{n=1}^N \delta_n^{(i)} D^{(i)}(t,T_n) B^{(i)}(t,T_n;\tau), \tag{3.44}$$

where T_0 is the effective date of the contract. On the other hand, the present value of the j-leg with a spread $B_N^{CCS}(t) = B_N^{CCS}(t,T_0,T_N;\tau)$ for unit notional is

$$PV_j(t) = -E_t^{Q_j}\left[e^{-\int_t^{T_0} \left(r^{(j)}(s) - y^{(i)}(s) \right) ds} \right] + E_t^{Q_j}\left[e^{-\int_t^{T_N} \left(r^{(j)}(s) - y^{(i)}(s) \right) ds} \right]$$

$$= + \sum_{n=1}^N \delta_n^{(j)} E_t^{Q_j}\left[e^{-\int_t^{T_n} \left(r^{(j)}(s) - y^{(i)}(s) \right) ds} \left(L^{(j)}(T_{n-1},T_n;\tau) + B_N^{CCS}(t) \right) \right], \tag{3.45}$$

and using the assumption of the deterministic spread y leads to

$$PV_j(t) = \sum_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \left(B^{(j)}(t,T_n;\tau) + B_N^{CCS}(t) \right)$$

$$+ \sum_{n=1}^N D^{(j)}(t,T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1 \right). \tag{3.46}$$

Let us denote the notional of the i-leg per unit amount of j-notional as $N^{(i)}$. Usually, it is fixed by the forward FX at the time of inception of the contract as $N^{(i)} = f_x^{(i,j)}(t,T_0)$, and then the total present value of the i-leg in terms of currency j is given by

$$\frac{N^{(i)}}{f_x^{(i,j)}(t)} PV_i(t) = \sum_{n=1}^N \delta_n^{(i)} \frac{N^{(i)}}{f_x^{(i,j)}(t)} D^{(i)}(t,T_n) B^{(i)}(t,T_n;\tau) \tag{3.47}$$

$$= \sum_{n=1}^N \delta_n^{(i)} \frac{N^{(i)}}{f_x^{(i,j)}(t,T_n)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} B^{(i)}(t,T_n;\tau). \tag{3.48}$$

Hence, the following expression of the T_0-start T_N-maturing CNCCS basis spread is obtained:

$$B_N^{CCS}(t,T_0,T_N;\tau)$$

$$= \left[\sum_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \left\{ \frac{\delta_n^{(i)}}{\delta_n^{(j)}} \frac{N_{(i)}}{f_x^{(i,j)}(t,T_n)} B^{(i)}(t,T_n;\tau) - B^{(j)}(t,T_n;\tau) \right\} \right.$$

$$\left. - \sum_{n=1}^N D^{(j)}(t,T_{n-1}) \, e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1 \right) \right] \Big/$$

$$\sum_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \tag{3.49}$$

One can also get a formula for a different collateral currency by repeating a similar calculation.

Note that the $B_N^{CCS}(t,T_0,T_N;\tau)$ in (3.49) is a martingale under the annuity measure $\hat{\mathcal{A}}$ where the i-collateralized j-annuity $\Sigma_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}$ is used as numeraire. Therefore, the present value of a T_0-start T_N-maturing constant notional cross-currency payer swaption with strike spread K is given as

$$PV(t) = \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_n^{T_n} y^{(i,j)}(s)ds} E_t^{\hat{A}} \left[\left(B_N^{\mathrm{CCS}}(T_0,T_0,T_N;\tau) - K \right)^+ \right], \tag{3.50}$$

where the notional of the j-leg is assumed to be the unit amount of a corresponding currency. Once every volatility process is specified, it will be tedious but possible to derive an analytic approximation by, for example, applying an asymptotic expansion technique.

3.4.3 Mark-to-Market Cross-Currency Swap

A mark-to-market cross-currency swap (MtMCCS) is a similar contract to the aforementioned CNCCS except that the notional of the leg which pays Libor flat is refreshed at the start of every Libor calculation period based on the spot FX at that time. The notional for the other leg is kept constant throughout the contract. More specifically, let us consider a MtMCCS for the (i,j) currency pair where a j-Libor plus spread is exchanged for i-Libor flat. In this case, the notional of the i-leg is going to be set at $f_x^{(i,j)}(t)$ times the notional of the j-leg at the beginning of every period and the amount of notional change is exchanged at the same time. Owing to the notional refreshment, an (i,j)-MtMCCS can be considered as a portfolio of one-period (i,j)-CNCCS, where the notional of the j-leg of every contract is the same. Here, the net effect from the final notional exchange of the nth CNCCS and the initial exchange of the $(n+1)$th CNCCS is equivalent to the notional adjustment at the start of the $(n+1)$th period of the MtMCCS.

Let us assume the collateral currency is i as before. The present value of the j-leg can be calculated in exactly the same way as the CNCCS, and is given by

$$\begin{aligned} PV_j(t) &= \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \left(B^{(j)}(t,T_n;\tau) + B_N^{\mathrm{MtM}}(t) \right) \\ &\quad + \sum_{n=1}^{N} D^{(j)}(t,T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1 \right), \end{aligned} \tag{3.51}$$

where $B_N^{\mathrm{MTM}}(t, T_0, T_N; \tau)$ is the time-t value of the MtMCCS basis spread for this contract. On the other hand, the present value of the i-leg can be calculated as

$$\begin{aligned} PV_i(t) &= -\sum_{n=1}^{N} E_t^{Q_i} \left[e^{-\int_t^{T_{n-1}} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1}) \right] \\ &\quad + \sum_{n=1}^{N} E_t^{Q_i} \left[e^{-\int_t^{T_n} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1}) \left(1 + \delta_n^{(i)} L^{(i)}(T_{n-1},T_n;\tau) \right) \right] \\ &= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t,T_n) E_t^{\tau_{n,(i)}^c} \left[f_x^{(i,j)}(T_{n-1}) B^{(i)}(T_{n-1},T_n;\tau) \right]. \end{aligned} \tag{3.52}$$

As a result, the MtMCCS basis spread is given by

$$\begin{aligned} &B_N^{\mathrm{MtM}}(t,T_0,T_N;\tau) = \\ &\left[\sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \left\{ \frac{\delta_n^{(i)}}{\delta_n^{(j)}} E_t^{\mathcal{T}_{n,(i)}^c} \left[\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(t,T_n)} B^{(i)}(T_{n-1},T_n;\tau) \right] - B^{(j)}(t,T_n;\tau) \right\} \right. \\ &\left. - \sum_{n=1}^{N} D^{(j)}(t,T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1 \right) \right] \Big/ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}, \end{aligned} \tag{3.53}$$

and, after some calculation, we get

$$\begin{aligned} &B_N^{\mathrm{MtM}}(t,T_0,T_N;\tau) = \\ &\left[\sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \left\{ \frac{\delta_n^{(i)} f_x^{(i,j)}(t,T_{n-1})}{\delta_n^{(j)} f_x^{(i,j)}(t,T_n)} B^{(i)}(t,T_n;\tau) Y_n^{(i,j)}(t) - B^{(j)}(t,T_n;\tau) \right\} \right. \\ &\left. - \sum_{n=1}^{N} D^{(j)}(t,T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1 \right) \right] \Big/ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}, \end{aligned} \tag{3.54}$$

Here, $Y_n^{(i,j)}(t)$ is defined by

$$\begin{aligned} Y_n^{(i,j)}(t) = E_t^{\tau_{n,(i)}^c} \Bigg[exp \Bigg\{ & \int_t^{T_{n-1}} \sigma_{FX}^{(i,j)}(s,T_{n-1}) \cdot \left(\sigma_B^{(i)}(s,T_n;\tau) - \int_{T_{n-1}}^{T_n} \sigma_c^{(i)}(s,u)du \right) ds \\ & + \int_t^{T_{n-1}} \left(\sigma_{X_n}(s) \cdot dW^{c}_{n,(i)}(s) - \frac{1}{2} \sigma_{X_n}(s)^2 ds \right) \Bigg\} \Bigg], \end{aligned} \tag{3.55}$$

where

$$\sigma_{X_n}(t) = \sigma_{FX}^{(i,j)}(t,T_{n-1}) + \sigma_B^{(i)}(t,T_n;\tau). \tag{3.56}$$

If we have liquid markets for FX forwards and the CNCCS, volatility and correlation parameters involved in the expression of $Y_n^{(i,j)}$ need to be adjusted to make the model consistent with the MtMCCS. However, considering the popularity of the MtMCCS and the limited liquidity of FX forwards with long maturities, it may be more practical to calibrate $\{y^{(i,j)}(t)\}$ using the MtMCCS directly. One can see easily that approximating $Y_n^{(i,j)} \simeq 1$ allows us straightforward bootstrapping of $\{y^{(i,j)}(t)\}$.

As is the case in CNCCS, the forward MtMCCS basis spread given in (3) is a martingale under the annuity measure $\hat{\mathcal{A}}$, where the i-collateralized j-annuity, $\Sigma_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}$, is used as numeraire. Therefore, a T_0-start T_N-maturing mark-to-market cross-currency payer swaption with strike spread K is calculated as

$$PV(t) = \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} E_t^{\hat{\mathcal{A}}} \left[\left(B_N^{\mathrm{MtM}}(T_0,T_0,T_N;\tau) - K \right)^+ \right] \tag{3.57}$$

where we have used the unit amount of the j-leg notional. A similar formula for a different collateral currency case can also be derived. One can see that the forward MtMCCS basis spread has a much smaller volatility than that of the CNCCS, due to the cancellation of FX exposure thanks to its notional refreshments.

By comparing the expression in (3.49), we can also derive the difference between the i-collateralized CNCCS and the MtMCCS basis spread as follows:

$$B_N^{\mathrm{MtM}}(t,T_0,T_n;\tau) - B_N^{\mathrm{CCS}}(t,T_0,T_n;\tau)$$
$$= \frac{\sum_{n=1}^N \delta_n^{(i)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} \left\{ \frac{f_x^{(i,j)}(t,T_{n-1})}{f_x^{(i,j)}(t,T_n)} B^{(i)}(t,T_n;\tau) Y_n^{(i,j)}(t) - \frac{N^{(i)}}{f_x^{(i,j)}(t,T_n)} B^{(i)}(t,T_n;\tau) \right\}}{\sum_{n=1}^N \delta_n^{(i)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}} \tag{3.58}$$

One can check that the difference between the FX exposure and the correction term $Y_n^{(i,j)}$ gives rise to the gap between the two CCSs.

4 Comments on Inflation Modeling

Before closing this paper, let us briefly comment on the inflation modeling in the presence of collateral. Although it is straightforward to use the multi-currency framework as proposed in the work of Jarrow and Yildirim [13], it requires the simulation of unobservable real interest rates. It is quite difficult to estimate the real rate volatility and its correlations to the other underlying factors. Here, let us present the method by which the collateralized forward CPI is directly simulated in the same way as for the Libor−OIS spreads. This is a simple extension of the model proposed by Belgrade and Benhamou (2004) for collateralized contracts.

First, define the forward CPI as the fixed amount of payment which is exchanged for $I(T)$ units of the corresponding currency at time T. Here, $I(T)$ is the time-T CPI index. Let us consider a CPI of the i-currency continuously collateralized by the j-currency. Then, the forward CPI $I^{(i)}(t,T)$ should satisfy

$$I^{(i)}(t,T) E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(j)}(s)ds} \right] = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(j)}(s)ds} I(T) \right]. \tag{4.1}$$

Under the assumption of deterministic spread $y^{(j)}$, this becomes

$$I^{(i)}(t,T) = E_t^{\tau^{(i)}}[I(T)] = E_t^{\mathcal{T}^c_{(i)}}[I(T)], \tag{4.2}$$

and is independent of the collateralized currency as for the multi-currency example in the previous section. The present value of a future CPI payment of the currency i collateralized by the foreign currency j is expressed using the forward CPI as

$$PV_i(t) = D^{(i)}(t,T) e^{\int_t^T y^{(j,i)}(s)ds} I^{(i)}(t,T), \tag{4.3}$$

where $y^{(j,i)}(s)$ is available after the multi-currency curve calibration.

The forward CPI can easily be extracted from a set of zero coupon inflation swap (ZCIS), which is the most liquid inflation product in the current market. The break-even rate K_N of the N-year zero coupon inflation swap satisfies

$$\left[(1+K_N)^N - 1\right] D(t,T_N) = \left(\frac{E_t^{\tau^c}[I(T_N)]}{I(t)} - 1 \right) D(t,T_N), \tag{4.4}$$

and hence

$$I(t,T_N) = I(t)(1+K_N(t))^N. \tag{4.5}$$

Here, the collateral currency is assumed to be the same as the payment currency. It is straightforward to construct a smooth forward CPI curve using the appropriate spline technique. Although we do not go into details, it is also quite important to estimate month-on-month (MoM) seasonality factors using historical data. As is clear from its properties, it should not be treated as a diffusion process and hence it should be added on top of the simulated forward CPI based on the smooth YoY trend process.

Since $I(t,T)$ is a martingale under the τ^c measure, its stochastic differential equation under the MM measure Q can be specified as follows:

$$dI(t,T) = \sigma_I(t,T) \cdot \left(\int_t^T \sigma_c(t,s)ds \right) dt + \sigma_I(t,T) \cdot dW^Q(t). \tag{4.6}$$

This should be understood as the trend forward CPI process, and needs to be adjusted properly by the use of seasonality factors to derive a forward CPI with odd period. As a summary, necessary stochastic differential equations for IR−inflation hybrids are given by

$$dc(t,s) = \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du \right) dt + \sigma_c(t,s) \cdot dW^Q(t), \tag{4.7}$$

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t,T;\tau) \cdot \left(\int_t^T \sigma_c(t,s)ds \right) dt + \sigma_B(t,T;\tau) \cdot dW^Q(t), \tag{4.8}$$

$$dI(t,T) = \sigma_I(t,T) \cdot \left(\int_t^T \sigma_c(t,s)ds \right) dt + \sigma_I(t,T) \cdot dW^Q(t). \tag{4.9}$$

5 Conclusions

This paper has presented a new framework of interest rate models which reflects the existence as well as the dynamics of various basis spreads in the market. It has also explicitly taken the impacts from the collateralization into account, and provided an extension for a multi-currency environment consistent with FX forwards and MtMCCS. It has also commented on inflation modeling in the presence of collateral. Finally, let us provide a possible order of calibration in this framework:

1. Calibrate domestic swap curves and extract $\{D(t,T)\}$ and $\{B(t,T;\tau)\}$ following the method in Fujii et al. (2009) for each currency.

>

2. Calibrate domestic interest rate options, such as swaptions and caps/floors, and determine the volatility curves (or surface) of the IR sector for each currency. For the correlation structure setup, option implied information or historical data can be used. If one has a set of calibrated swap curves for a certain period of history, it is straightforward to carry out the principal component analysis and extract the several dominant factors. See the explanation given, for example, in the work of Rebonato [15].
3. Calibrate FX forwards (or CNCCS) and extract the set of $\{y^{(i,j)}(s)\}$ for all the relevant currency pairs.
4. Calibrate the vanilla FX options and determine the spot FX volatility for all the relevant currency pairs. The resultant spot FX volatility does depend on the correlation structure between the spot FX and collateral rates of the two currencies. It should be estimated using quanto products and/or historical data.
5. Calibrate MtMCCS and determine the correlation curve between spot FX and Libor–OIS spread. Considering the size of correction, one will have quite a good fit after the calibration of FX forwards, though.

There remain various interesting topics for the practical implementation of this new framework; analytic approximation for vanilla options will be necessary for fast calibration and for use as regressors for Bermudan/American-type exotics. Because of the separation of the discounting curve and the Libor–OIS spread, there will be some important implications for the price of convexity products, such as constant maturity swaps (CMS). It is also an important problem to consider the method of obtaining stable attribution of vega (kappa) exposure to each vanilla option for generic exotics.[13]

Appendix A: Compounding a Tenor Swap

As mentioned in Section 2.2.3, there is a slight complication in TS due to compounding in the leg with the short tenor. For example, in a USD 3m/6m tenor swap, coupon payments from the 3m leg occur semi-annually where the previous coupon (3m Libor plus tenor spread) is compounded by 3m Libor flat. As a result, the present value of the 3m leg is calculated as

$$
\begin{aligned}
PV_{\tau_S}(t) = \sum_{m=1}^{M} E_t^{Q}\Bigg[e^{-\int_t^{T_{2m}} c(s)ds} & \{\delta_{2m-1}\,(L(T_{2m-2}, T_{2m-1};\, \tau_S) \\
& + TS(t))\,(1 + \delta_{2m} L(T_{2m-1}, T_{2m});\, \tau_S)) \\
& +\ \delta_{2m}\,(L(T_{2m-1}, T_{2m};\, \tau_S) + TS(t))\}\Bigg] \\
= \sum_{n=1}^{2M} D(t, T_n)\delta_n & \left(E_t^{\tau_n^c}\,[L(T_{n-1}, T_n;\, \tau_S)] + TS(t)\right) \\
& + \sum_{m=1}^{M} \delta_{2m-1}\delta_{2m} D(t, T_{2m}) TS(t) B(t, T_{2m};\, \tau_S)
\end{aligned}
\tag{A.1}
$$

where τ_S=3m. Note that the second and third terms are a correction to the left-hand side of (2.7). Since the Libor–OIS and tenor spreads have similar sizes, the correction term cannot affect the calibration meaningfully. Considering the bid/offer spread, one can safely neglect the compounding effects in most situations.

Acknowledgments

This research is supported by CARF (Center for Advanced Research in Finance) and the global COE program "The research and training center for new development in mathematics." All the contents expressed in this research are solely those of the authors and do not represent the view of Shinsei Bank Ltd or any other institution. The authors are not responsible or liable in any manner for any losses and/or damages caused by the use of any contents in this research. M. Fujii is grateful to friends and former colleagues at Morgan Stanley, especially at the IDEAS, IR option, and FX Hybrid desks in Tokyo for fruitful and stimulating discussions. The contents of the paper do not represent any views or opinions of Morgan Stanley.

Masaaki Fujii was graduated from the Department of Physics, Faculty of Science, University of Tokyo. He received his Ph.D. from the Graduate School of Science, University of Tokyo in 2004. After working for the Interest Rate and Currency Analytical Modeling group of Morgan Stanley and studying in the Ph.D. course of Graduate School of Economics, University of Tokyo, he joined as Assistant Professor at the Center for Advanced Research in Finance, Graduate School of Economics, University of Tokyo in 2011.

Yasufumi Shimada was graduated from the Faculty of Economics, University of Tokyo. With over 20 years' experience in the derivatives market, he is presently the Head of Treasury Sub-Group, General Manager of Treasury Funding Division at Shinsei Bank, Limited, and a councilor of Japanese Association of Financial Econometrics and Engineering.

Akihiko Takahashi was graduated from the Faculty of Economics, University of Tokyo. He received his Ph.D. from the Haas School of Business, University of California at Berkeley. After working for the Industrial Bank of Japan and Long Term Capital Management, he started as Associate Professor at the Graduate School of Mathematical Sciences, University of Tokyo and later joined the Graduate School of Economics.

FOOTNOTES

1. A basis spread generally means the interest rate differentials between two different floating rates.
2. It is a floating-vs-floating swap that exchanges Libors with two different tenors with a fixed spread in one side.
3. As for the cross currency basis spread, it has been an important issue for global financial institutions for many years. However, there exists no literature that directly takes its dynamics into account consistently in a multi-currency setup of an interest rate model.
4. It is easy to apply a similar methodology to the unsecured (or uncollateralized) trade by approximately taking into account the credit risk by using Libor as the effective discounting rate.
5. In this section, the collateral currency is the same as the payment currency.
6. Typically, there is only one payment at the very end for the very end for the swap with short maturity (<1 yr) case, and otherwise there are periodical payments, quarterly for example.
7. Basically, OIS quotes allow us to fix the collateralized zero-coupon bond values, and then the combinations of IRS and TS will give us the Libor forward expectations.
8. We need to use a proper spline technique to get smooth, continuous result. See Hagan and West (2006), for example.

9. Exactly the same idea has also been adopted in inflation modeling [2], as will be seen later.
10. The effective dividend yield is given by $q(t) = q_{org}(t) - (r(t) - c(t))$ with the original dividend yield q_{org}. In later sections, we will use a simplified assumption that $(r(t) - c(t))$ is a deterministic function of time.
11. An FX forward contract is usually included in the list of trades for which netting and collateral postings are to be made.
12. Note, however, that the choice of collateral currency does affect the present value of a trade. As can be seen from (3.23), the present value of a payment at time T in the j-currency collateralized with the i-currency is proportional to $D^{(j)}(t,T)\, e^{\int_t^T y^{(i,j)}(s)ds}$, and hence the payer of collateral may want to choose the collateral currency i for each period in such a way that it minimizes $\left(\int_t^T y^{(i,j)}(s)ds\right)$.
13. After completion of the original version of this paper, we published several new works (Fujii and Takahashi, 2010a, 2011, 2010b), which include improvements and further extensions as well as some numerical examples.

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