Derivatives Pricing under Collateralization *

Masaaki Fujii†, Kenichiro Shiraya‡, Akihiko Takahashi¶

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† Graduate School of Economics, The University of Tokyo

‡ Graduate School of Economics, The University of Tokyo

¶ Speaker, Graduate School of Economics, The University of Tokyo

Introduction

New market realties after the Financial Crisis

- **Wide use of collateralization in OTC**
  Dramatic increase in recent years (ISDA Margin Survey 2011)
  - **30%(2003) → 70%(2010) in terms of trade volume for all OTC.**
  - **Coverage goes up to 79% (for all OTC) and 88% (for fixed income) among major financial institutions.**
  - **More than 80% of collateral is Cash.**
    (About half of the cash collateral is USD.)
- **Persistently wide basis spreads:**
  Much more volatile Cross Currency Swap(CCS) basis spread.
  Non-negligible basis spreads even in the single currency market. (e.g. Tenor swap spread, Libor-OIS spread)

Impact of Collateralization

Impact of collateralization :

- **Reduction of Counter-party Exposure**
- **Change of Funding Cost**
  - **Require new term structure model to distinguish discounting and reference rates.**
  - **Cost of collateral is different from currency to currency.**
  - **Choice of collateral currency (”cheapest-to-deliver” option).**
  - **Significant impact on derivative pricing.**

Topics of this talk

- **Valuation framework under collateralization**
- **Derivatives pricing under the perfect collateralization:**
  - **Symmetric collateralization and choice of collateral currency**
  - **Asymmetric collateralization and potential effects from the difference in collateral management**
  - **Imperfect collateralization and CVA**
- **(a new computational scheme for FBSDEs[1], which seems useful for pricing securities under asymmetric/imperfect collateralization)**
  - **(perturbation scheme)**
  - **(perturbation with interacting particle method)**

For details, please see the series of our papers ([11]- [19]).

[1] forward backward stochastic differential equations

Setup

Pricing Framework ([15])

- Probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F}$ contains all the market information including defaults.
- Consider two firms, $i \in \{1, 2\}$, whose default time is $\tau^i \in [0, \infty]$, and $\tau = \tau^1 \wedge \tau^2$.
- τ^i (and hence τ) is assumed to be totally-inaccessible $\mathbb{F}$-stopping time.
- Indicator functions: $H_t^i = 1_{\{\tau^i \le t\}}$, $H_t = 1_{\{\tau \le t\}}$
- Assume the existence of absolutely continuous compensator for H^i:

$$A_t^i = \int_0^t h_s^i 1_{\{\tau^i > s\}} ds, \quad t \ge 0$$

- Assume no simultaneous defaults, and hence the hazard rate of H is

$$h_t = h_t^1 + h_t^2 .$$

- Money market account: $\beta_t = \exp\left(\int_0^t r_u du\right)$

Collateralization

- When party $i \in \{1,2\}$ has negative mark-to-market, it has to post cash collateral to party $j(\neq i)$, and it is assumed to be done continuously.
- collateral coverage ratio is $\delta_t^i \in \mathbb{R}_+$, and the amount of collateral at time t is given by $\delta_t^i(-V_t^i)$ when party i posts collateral. (V_t^i denotes the mark-to-market value of the contract from the view point of party i.)
  - δ_t^i effectively takes into account under- as well as over-collateralization. Thus $\delta_t^i < 1$, and also $\delta_t^i > 1$ are possible.
- party j has to pay the collateral rate c_t^i on the posted cash continuously.
- c_t^i is determined by the currency posted by party i.
  - market convention is to use overnight (O/N) rate at time t of corresponding currency.
    $\Rightarrow$ Traded through OIS (overnight index swap), which is also collateralized.
  - In general, $c_t^i \neq r_t^i$. (r_t^i is the risk-free interest rate of the same currency.) This is necessary to explain CCS basis spread consistently .

Counterparty Exposure and Recovery Scheme

- Counterparty exposure to party j at time t from the view point of party i is given as:

$$\max(1-\delta_t^j, 0)\max(V_t^i, 0)+\max(\delta_t^i-1, 0)\max(-V_t^i, 0).$$

- Assume party-j's recovery rate at time t as $R_t^j \in [0, 1]$.
- Then, the recovery value at the time of j's default is given as:

$$R_t^j\left([1-\delta_t^j]^+[V_t^i]^+ + [\delta_t^i-1]^+[-V_t^i]^+\right),$$

$x^+ \equiv \max(x, 0)$.

Pricing Formula

- **Pricing from the view point of party 1.**

$$S_t = \beta_t E^Q\Big[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}}\Big\{dD_u + (y_u^1 \delta_u^1 1_{\{S_u<0\}} + y_u^2 \delta_u^2 1_{\{S_u \ge 0\}}) S_u du\Big\} + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \ge u\}} \Big(Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2\Big) \Big| \mathcal{F}_t\Big]$$

- D: **cumulative dividend to party 1.**
- **Default payoff:** Z^i **when party** i **defaults.**

$$Z^1(t,v) = \Big(1 - l_t^1 (1-\delta_t^1)^+\Big) v 1_{\{v<0\}} + \Big(1 + l_t^1(\delta_t^2 - 1)^+\Big) v 1_{\{v \ge 0\}}$$
$$Z^2(t,v) = \Big(1 - l_t^2 (1-\delta_t^2)^+\Big) v 1_{\{v\ge 0\}} + \Big(1 + l_t^2(\delta_t^1 - 1)^+\Big) v 1_{\{v < 0\}},$$

$l_t^i \equiv (1 - R_t^i)$, $i = 1, 2$

- $y_t^i = r_t^i - c_t^i$, $(i \in \{1,2\})$ **denotes the instantaneous return for** j **or funding cost for** i **at time** t **from the cash collateral posted by party** i**.**

Pricing Formula

• (Remark) The return from risky investments, or the borrowing cost from the external market can be quite different from the risk-free rate, of course.
However, if one wants to treat this fact directly, an explicit modeling of the associated risks is required.
Here, we use the risk-free rate as net return/cost after hedging these risks.
As we shall see, under full collateralization the final formula does not require any knowledge of the risk-free rate, and hence there is no need of its estimation, which is crucial for the practical implementation.

Pricing Formula

Following the method in Duffie&Huang (1996), pre-default value of the contract V_t such that $V_t 1_{\{\tau>t\}} = S_t$ is given by

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))du\right) dD_s \middle| \mathcal{F}_t\right], \quad t \leq T,$$

where

$$\begin{aligned} \mu(t,v) &= \tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}} \\ \tilde{y}_t^i &= \delta_t^i y_t^i - (1-\delta_t^i)^+(l_t^i h_t^i) + (\delta_t^i - 1)^+(l_t^j h_t^j), \end{aligned}$$

if some technical condition(so called *no jump* condition for V at default) [2] is satisfied, which is assumed hereafter.

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[2] This technical condition ($\Delta V_\tau = 0$) becomes important when we consider credit derivatives: the condition is violated in general when the contagious effects induce jumps to variables contained in pre-default value process. (e.g. Schönbucher(2000), Collin-Dufresne-Goldstein-Hugonnier(2004), Brigo-Capponi(2009), [17])

Symmetric Case

Effective discount factor is non-linear:

$$r_t - \mu(t,v) = r_t - (\tilde{y}_t^1 1_{\{v<0\}} + \tilde{y}_t^2 1_{\{v\geq 0\}}),$$

which makes the portfolio value non-additive.
If $\tilde{y}_t^1 = \tilde{y}_t^2 = \tilde{y}_t$, then we have

$$\mu(t,v) = \tilde{y}_t \; .$$

Further, if $\tilde{y}$ is not explicitly dependent on V, we can recover the linearity.

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

Portfolio valuation can be decomposed into that of each payment.

$\Downarrow$

A good characteristic for market benchmark price.

Symmetric Perfect Collateralization

Special Cases

Case 1 : Benchmark for single currency product

- **bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)**
- **both parties use the same currency (i) as collateral, which is also the payment (evaluation) currency.**

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

The valuation method for single currency swap adopted by LCH Swapclear (2010) is the same with this equation. [3]

[3] See also Piterbarg (2010) for other derivation of this equation.

Symmetric Perfect Collateralization

Special Cases

Case 2 : Collateral in a Foreign Currency

- **bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)**
- **both parties use the same currency (k) as collateral, which is different from the payment (evaluation) currency (i)**

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,k)})du\right) dD_s \middle| \mathcal{F}_t\right]$$

Funding Spread between the two currencies

$$y^{(i,k)} = y^{(i)} - y^{(k)} = \left(r^{(i)} - c^{(i)}\right) - \left(r^{(k)} - c^{(k)}\right)$$

- **This is necessary to explain CCS basis spreads consistently.**

Collateral Rate

Overnight Index Swap (OIS)

- exchange fixed rate(F) with compounded overnight rate periodically.
- collateralized by domestic currency
- Par rate at t for T_0 ($> t$)-start T_N-maturing OIS with currency (i):

$$\mathrm{OIS}_N(t) = F^{par}(t) = \frac{D^{(i)}(t, T_0) - D^{(i)}(t, T_N)}{\sum_{n=1}^{N} \Delta_n D^{(i)}(t, T_n)},$$

(Δ_n : daycount fraction).

- $D^{(i)}(t, T) = E^{Q^{(i)}}\left[e^{-\int_t^T c_u^{(i)} du} \middle| \mathcal{F}_t \right]$ is a value of domestically collateralized zero-coupon bond.

Funding Spread

(i, j) Mark-to-Market Cross Currency OIS:
The funding spread(the difference of collateral costs) is directly linked to the corresponding CCOIS, though it seems not liquid in the current market.

- **compounded O/N rate of currency (i) is exchanged by that of currency (j) with additional spread periodically.**
- **notional of currency (j) is kept constant while that of currency (i) is refreshed at every reset time with the spot FX rate. (currency (i) is usually USD.)**
- **collateralized by currency (i) .**

Funding Spread

- **Define**

$$D^{(j,i)}(t,T) = E^{Q^{(j)}}\left[e^{-\int_t^T (c_u^{(j)}+y_u^{(j,i)})du}\Big|\mathcal{F}_t\right] = D^{(j)}(t,T)e^{-\int_t^T y^{(j,i)}(t,s)ds}.$$

$$y^{(j,i)}(t,T) = -\frac{\partial}{\partial T}\ln E^{T^{(j)}}\left[e^{-\int_t^T y_u^{(j,i)}du}\Big|\mathcal{F}_t\right].$$

- **$D^{(j,i)}(t,T)$: the zero coupon bond of currency j collateralized by currency i. $E^{T^{(j)}}[\cdot|\mathcal{F}_t]$: conditional expectation under the fwd measure associated with $D^{(j)}(t,T)$.**

  Then, under a simplifying assumption such as independence between $c^{(j)}$ and $y^{(j,i)}$[4],

- **MtMCCOIS basis spread is obtained by:**

$$B_N = \frac{\sum_{n=1}^N D^{(j,i)}(t,T_{n-1})\left(1-e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)du}\right)}{\sum_{n=1}^N \delta_n D^{(j,i)}(t,T_n)} \sim \frac{1}{T_N-T_0}\int_{T_0}^{T_N} y^{(j,i)}(t,u)du.$$

[4] The assumption seems reasonable for the recent data studied in [13].

Modeling framework of Interest rates

Symmetric perfectly collateralized price is becoming the market benchmark, at least for standardized products.

"Term structure construction procedures": [5]

- **(1), OIS ⇒ $c^{(i)}(0,T)$(T-maturity instantaneous fwd rate at time 0)**
- **(2), results of (1) + IRS + TS ⇒ $B^{(i)}(0,T;\tau)$ (i-currency forward Libor-OIS spread with tenor τ)**
- **(3), results of (1),(2) +CCS ⇒ $y^{(i,j)}(0,T)$(funding spread)**

Given the initial term structures, no-arbitrage dynamics of $c^{(i)}(t,T)$,$B^{(i)}(t,T;\tau)$ and $y^{(i,j)}(t,T)$ in HJM-framework can be constructed. (For the detail, please see our paper [11], [20]. For other approaches, see Bianchetti(2010), Mercurio(2009), Morini(2009), for instance.)

[5] Assume collateralization in domestic currency for OIS, IRS and TS. Assume collateralization in USD for CCS (USD crosses).

Curve Construction

- **Collateralized OIS**

$$\mathrm{OIS}_N(0) \sum_{n=1}^{N} \Delta_n D(0,T_n) = D(0,T_0) - D(0,T_N)$$

- **Collateralized IRS**

$$\mathrm{IRS}_M(0) \sum_{m=1}^{M} \Delta_m D(0,T_m) = \sum_{m=1}^{M} \delta_m D(0,T_m) E^{T_m}[L(T_{m-1},T_m;\tau)]$$

- **Collateralized TS**[6]

$$\sum_{n=1}^{N} \delta_n D(0,T_n) \left(E^{T_n}[L(T_{n-1},T_n;\tau_S)] + TS_N(0) \right)$$

$$= \sum_{m=1}^{M} \delta_m D(0,T_m) E^{T_m}[L(T_{m-1},T_m;\tau_L)]$$

($\Delta_m, \Delta_n, \delta_m, \delta_n$: daycount fractions)

Market quotes of collateralized OIS, IRS, TS, (and a proper spline method) allow us to determine all the relevant $\{D(0,T)\}$, and forward Libors $\{E^{T_m}[L(T_{m-1},T_m,\tau)]\}$.

[6] The short-tenor Leg may be compounded, and then exits additional small corrections.

Curve Construction

Collateralized FX Forward: USD/JPY

- **Suppose USD= (i), JPY= (j) and collateral currency is USD.**
- **Current time: t. Maturity: T**
- **At T , one unit of (i) is exchanged for K (fixed at t) units of (j).**
- **FX forward is the break-even value of K.**

$$KE_t^{Q^{(j)}}\left[e^{-\int_t^T (c_s^{(j)}+y_s^{(i,j)})ds}\right] = f_x^{(j,i)}(t)E_t^{Q^{(i)}}\left[e^{-\int_t^T c_s^{(i)}ds}1\right].^{7}$$

$$\begin{aligned} f_x^{(j,i)}(t,T;(i)) &= f_x^{(j,i)}(t)\frac{D^{(i)}(t,T)}{D^{(j)}(t,T)}\exp\left(\int_t^T y^{(j,i)}(t,u)du\right), \\ y^{(j,i)}(t,T) &= -\frac{\partial}{\partial T}\ln E_t^{T(j)}\left[e^{-\int_t^T y_s^{(j,i)}ds}\right]. \end{aligned}$$

- **FX Forward → Forward curve of funding spread ($\{y^{(j,i)}(t,T)\}$)**
- **CCS for longer maturities.**

[7] $f_x^{(j,i)}(t)$ denotes spot FX rate at t that is, the price of the unit amount of currency (i) in terms of currency (j).

Curve Construction

Remark: Constant Notional CCS vs MtM-CCS

(USD-LIBOR) is exchanged for (X-currency LIBOR + basis spread).

- **Constant Notional CCS (CNCCS)**
  - **Notional of both legs are kept constant.**
- **Mark-to-Market CCS (MtMCCS)**
  - **Notional of currency X is kept constant at N_X.**
  - **Notional of USD is readjusted to $f_x^{(USD;X)} \times N_X$ at every start of LIBOR accrual period.**

Curve Construction

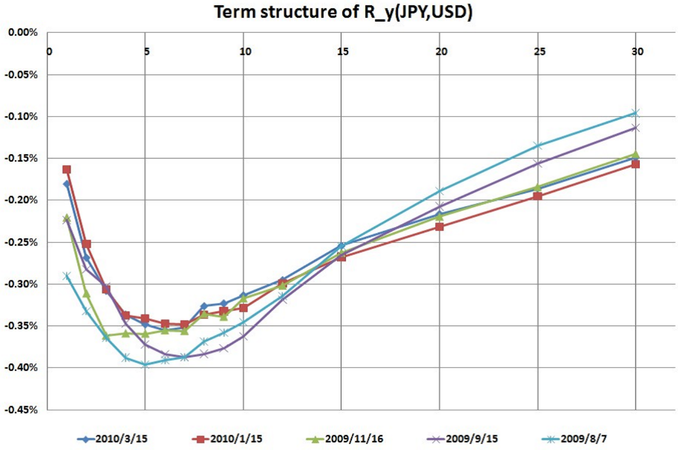
Remark: the difference between MtM and Constant notional basis spreads:

$$B_N^{MtM} - B_N^{CN} = \frac{\sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0,T_n) E^{T_n^{(i)}} \left[\left(\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} - 1 \right) B^{(i)}(T_{n-1},T_n) \right]}{\sum_{n=1}^{N} \delta_n^{(j)} D^{(j,i)}(0,T_n)},$$

where $B^{(i)}(T_{n-1}, T_n)$ stands for the Libor-OIS spread of the currency i at T_{n-1}. This spread is not zero in general.

For the two USDJPY CCSs, the two swaps should have the same basis spreads if USD LIBOR-OIS spreads are all zero. This held approximately well before the Lehman crisis but the spread has been far from zero since then. If USD interest rate level is higher than JPY, as is usually the case, the equation tells us that the spread for MtMCCS is quite likely to be higher than that of CNCCS, $B_N^{MtM} > B_N^{CN}$. The size of spread may not be negligible dependent on situations, and hence it is worthwhile paying enough attention to the difference in this post crisis era.

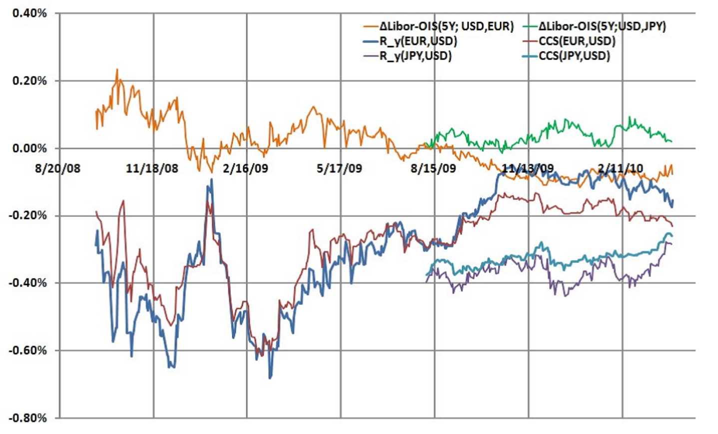
Curve Construction



$R_y(j,i) = \left(\int_0^T y^{j,i}(0,u)du\right)/T$(funding spread curve): posting USD as collateral tends to be expensive for collateral payers.

Curve Construction

Close relationship - CCS Basis and Funding Spread -



A significant portion of CCS spreads movement stems from the change in the funding spreads. Libor-OIS spread seems to have minor effect.

HJM-framework under full collateralization

$$
\begin{aligned}
dc^{(i)}(t,s) &= \sigma_c^{(i)}(t,s)\cdot\left(\int_t^s \sigma_c^{(i)}(t,u)du\right)dt + \sigma_c^{(i)}(t,s)\cdot dW_t^{Q^{(i)}} \\
dy^{(i,k)}(t,s) &= \sigma_y^{(i,k)}(t,s)\cdot\left(\int_t^s (\sigma_y^{(i,k)}(t,u)+\sigma_c^{i}(t,u))du\right)dt + \sigma_y^{(i,k)}(t,s)\cdot dW_t^{Q^{(i)}} \\
\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} &= \sigma_B^{(i)}(t,T;\tau)\cdot\left(\int_t^T \sigma_c^{(i)}(t,s)ds\right)dt + \sigma_B^{(i)}(t,T;\tau)\cdot dW_t^{Q^{(i)}} \\
\frac{df_x^{(i,j)}(t)}{f_x^{(i,j)}(t)} &= \left(c^{(i)}(t)-c^{(j)}(t)+y^{(i,j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW_t^{Q^{(i)}},
\end{aligned}
$$

$$
B^{(i)}(t,T_k;\tau) = E_t^{T_k^{(i)}}\left[L^{(i)}(T_{k-1},T_k;\tau)\right] - \frac{1}{\delta_k^{(i)}}\left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)}-1\right)
$$

is forward LIBOR-OIS spread.

Choice of Collateral Currency

Special Cases

Case 3 : Multiple Eligible Collaterals

- bilateral perfect collateralization ($\delta^1 = \delta^2 = 1$)
- both parties choose the optimal currency from the eligible collateral set C. Currency (i) is used as the evaluation currency.

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + \max_{k \in C}[y_u^{(i,k)}])du \right) dD_s \middle| \mathcal{F}_t \right]$$

- The party who needs to post collateral has optionality.
- The cheapest collateral currency is chosen based on CCS information.
  To choose "strong" currency, such as USD,
  is expensive for the collateral payer.

Choice of Collateral Currency

Role of $y^{(j,i)}$

Optimal behavior of collateral payer can significantly change the derivative value.

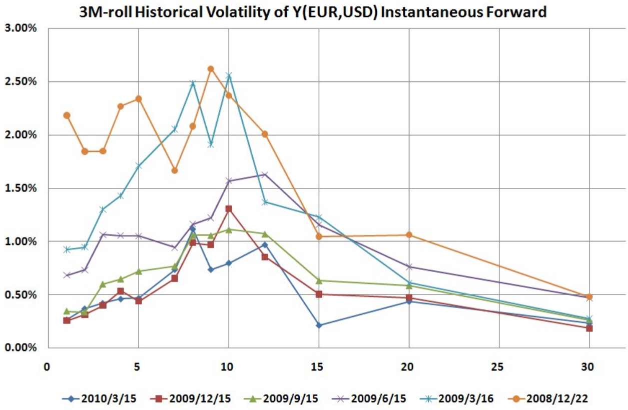
- **Payment currency and USD as eligible collateral is relatively common. Then, the effective discounting factor becomes**

$$D^{(j)}(t,T) \Rightarrow E_t^{T^{(j)}}\left[e^{-\int_t^T \max\{y^{(j,USD)}(s),0\}ds}\right] D^{(j)}(t,T)$$

  except correlation effects.

- **Volatility of $y^{(j,USD)}$ is an important determinant. (Embedded option change effective discounting factor, which crucially depends on the volatility of funding spread.)**

Choice of Collateral Currency



vols tend to be 50 bps in a calm market, but they were more than a percentage point just after the market crisis, which reflects a significant widening of the CCS basis to seek USD cash in the low liquidity market.

Choice of Collateral Currency

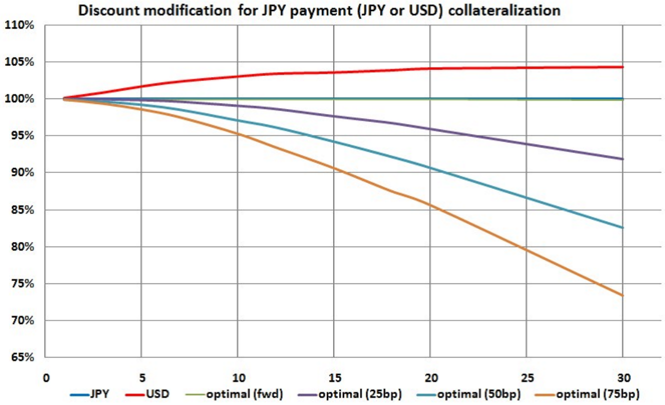


Figure: Modification of JPY discounting factors based on HW model for $y^{(JPY,USD)}$ as of 2010/3/16.

the effective discounting rate is increased by around 50 bps annually even when the annualized vol. of $y^{(JPY,USD)}$ is 50 bps.

More generic situations: marginal impact of asymmetry

$$
\begin{aligned}
V_t &= E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u, V_u))du\right) dD_s \middle| \mathcal{F}_t\right] \\
\mu(t,v) &= \tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}} \\
\tilde{y}_t^i &= \delta_t^i y_t^i - (1-\delta_t^i)^+(l_t^i h_t^i) + (\delta_t^i - 1)^+(l_t^j h_t^j)
\end{aligned}
$$

- Make use of Gateaux derivative(GD) as the first-order Approximation [8]:

$$
\lim_{\epsilon \downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta}; \eta) - \frac{V_t(\bar{\eta} + \epsilon\eta) - V_t(\bar{\eta})}{\epsilon} \right| = 0, \ (\eta, \bar{\eta}\text{: bounded and predictable})
$$

- We want to expand the price around a symmetric benchmark price.

$$
\mu(t,v) = y_t + \Delta\tilde{y}_t^1 \mathbf{1}_{\{v<0\}} + \Delta\tilde{y}_t^2 \mathbf{1}_{\{v\geq 0\}}, \ (\Delta\tilde{y}_t^i = \tilde{y}_t^i - y_t)
$$

- Calculate GD at symmetric $\mu = y$ point.

$$
V_t(\mu) \simeq V_t(y) + \nabla V_t(y, \mu - y)
$$

[8] Duffie&Skiadas (1994), Duffie&Huang (1996)

Asymmetric Collateralization(marginal impact of asymmetry)

- Then, V_t is decomposed as $V_t = \overline{V}_t + \nabla V_t$, where

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\nabla V_t = E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}\overline{V}_s \left(\Delta\tilde{y}_s^1 \mathbf{1}_{\{\overline{V}_s<0\}} + \Delta\tilde{y}_s^2 \mathbf{1}_{\{\overline{V}_s\geq 0\}}\right) ds \middle| \mathcal{F}_t\right]$$

If y is chosen in such a way that it reflects the funding cost of the standard collateral agreements, $\overline{V}$ turns out to be **the market benchmark price**, and ∇V represents the correction for it.

Asymmetric Collateralization(marginal impact of asymmetry)

An example of asymmetric perfect collateralization

- party 1 choose optimal currency from the eligible collateral set $\mathcal{C}$, but the party 2 can only use currency (i) as collateral, either due to the asymmetric CSA or lack of easy access to foreign currency pool. The evaluation (payment) currency is (i).

$$\overline{V}_t = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right]$$

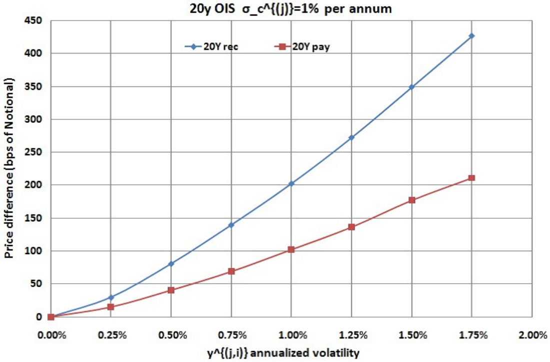
$$\nabla V_t = E^{Q^{(i)}}\left[\int_t^T \exp\left(-\int_t^s c_u^{(i)} du\right) [-\overline{V}_s]^+ \max_{k \in \mathcal{C}}[y_s^{(i,k)}] \middle| \mathcal{F}_t\right]$$

$$V_t \simeq \overline{V}_t + \nabla V_t$$

$\Rightarrow$ Expansion around the symmetric collateralization with currency (i).

Asymmetric Collateralization(marginal impact of asymmetry)

- **Numerical Example of ∇V for JPY-OIS [9].**
- **Eligible collateral are USD and JPY for party-1 but only JPY for party-2.**



- **OIS rate is set to make $\overline{V} = 0$.**
- **Difference between Receiver and Payer comes from up-ward sloping term structure. (the receiver's mark-to-market value tends to be negative in the long end of the contract, which makes the optionality larger.)**

[9] based on the data in early 2010. see [15] for the detail.

Imperfect Collateralization

CVA as the Deviation from the Perfect Collateralization

- **Assume the both parties use the same currency for simplicity, and hence $y^1 = y^2 = y$.**

$$
\begin{aligned}
&\mu(t,v) = y_t - \\
&\Big\{\Big((1-\delta_t^1)y_t + (1-\delta_t^1)^+(l_t^1 h_t^1) - (\delta_t^1 - 1)^+(l_t^2 h_t^2)\Big)\, 1_{\{v<0\}} \\
&+ \Big((1-\delta_t^2)y_t + (1-\delta_t^2)^+(l_t^2 h_t^2) - (\delta_t^2 - 1)^+(l_t^1 h_t^1)\Big)\, 1_{\{v\geq 0\}}\Big\}
\end{aligned}
$$

- **GD(Gateaux derivative) around $\mu = y$ decomposes the price into three parts:**
  - **Symmetric perfectly collateralized benchmark price**
  - **$(1-\delta^i)y1_{\{v\lessgtr 0\}} \Rightarrow$ Collateral Cost Adjustment (CCA)**
  - **Remaining h dependent terms $\Rightarrow$ Credit Value Adjustment (CVA)**

$$
\begin{aligned}
V_t &\simeq \overline{V}_t + \nabla V_t \\
&= \overline{V}_t + \mathrm{CCA} + \mathrm{CVA}
\end{aligned}
$$

Imperfect Collateralization

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\text{CCA} = E^Q\left[\int_t^T e^{-\int_t^s (r_u-y_u)du} y_s \left((1-\delta_s^1)[-\overline{V}_s]^+ - (1-\delta_s^2)[\overline{V}_s]^+\right) ds \middle| \mathcal{F}_t\right]$$

$$\text{CVA} =$$

$$E^Q\Bigg[\int_t^T e^{-\int_t^s (r_u-y_u)du}(l_s^1 h_s^1)\left[(1-\delta_s^1)^+[-\overline{V}_s]^+ + (\delta_s^2 - 1)^+[\overline{V}_s]^+\right] ds$$

$$- \int_t^T e^{-\int_t^s (r_u-y_u)du}(l_s^2 h_s^2)\left[(1-\delta_s^2)^+[\overline{V}_s]^+ + (\delta_s^1 - 1)^+[-\overline{V}_s]^+\right] ds \Bigg| \mathcal{F}_t\Bigg]$$

- **The discounting rate is different from the risk-free rate and reflects the funding cost of collateral, while the terms in CVA are pretty similar to the usual result of bilateral CVA.**
- **Dependence among y, δ and other variables such as $\overline{V}, h^i$ is particularly important. ⇒ New type of Wrong (Right)-way Risk. (e.g. y is closely related to the CCS basis spread. Hence, y is expected to be highly sensitive to the market liquidity, and is also strongly affected by the overall market credit conditions.)**

Collateral Thresholds

- **Thresholds: $\Gamma^i > 0$ for party-i: A threshold is a level of exposure below which collateral will not be called, and hence it represents an amount of uncollateralized exposure. Only the incremental exposure will be collateralized if the exposure is above the threshold.**

Case of perfect collateralization above the thresholds

$$S_t = \beta_t E^Q \left[\int_{]t,T]} \beta_u^{-1} 1_{\{\tau > u\}} \{dD_u + q(u, S_u) S_u du\} + \int_{]t,T]} \beta_u^{-1} 1_{\{\tau \geq u\}} \left\{ Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \right\} \middle| \mathcal{F}_t \right]$$

$$q(t, S_t) = y_t^1 \left(1 + \frac{\Gamma_t^1}{S_t}\right) 1_{\{S_t < -\Gamma_t^1\}} + y_t^2 \left(1 - \frac{\Gamma_t^2}{S_t}\right) 1_{\{S_t > \Gamma_t^2\}}$$

$$Z^1(t, S_t) = S_t \left[\left(1 + l_t^1 \frac{\Gamma_t^1}{S_t}\right) 1_{\{S_t < -\Gamma_t^1\}} + R_t^1 1_{\{-\Gamma_t^1 \leq S_t < 0\}} + 1_{\{S_t \geq 0\}} \right]$$

$$Z^2(t, S_t) = S_t \left[\left(1 - l_t^2 \frac{\Gamma_t^2}{S_t}\right) 1_{\{S_t \geq \Gamma_t^2\}} + R_t^2 1_{\{0 \leq S_t < \Gamma_t^2\}} + 1_{\{S_t < 0\}} \right]$$

Collateral Thresholds

Assume the domestic currency as collateral $y^1 = y^2 = y$.

$$\overline{V}_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s c_u du\right) dD_s \middle| \mathcal{F}_t\right]$$

$$\mathrm{CCA} = -E^Q\left[\int_t^T e^{-\int_t^s c_u du} y_s \overline{V}_s \mathbf{1}_{\{-\Gamma_s^1 \le \overline{V}_s < \Gamma_s^2\}} ds \middle| \mathcal{F}_t\right]$$

$$+E^Q\left[\int_t^T e^{-\int_t^s c_u du} y_s \left\{\Gamma_s^1 \mathbf{1}_{\{\overline{V}_s < -\Gamma_s^1\}} - \Gamma_s^2 \mathbf{1}_{\{\overline{V}_s \ge \Gamma_s^2\}}\right\} ds \middle| \mathcal{F}_t\right]$$

$$\mathrm{CVA} =$$

$$E^Q\left[\int_t^T e^{-\int_t^s c_u du}\left\{(l_s^1 h_s^1)[-\overline{V}_s \mathbf{1}_{\{-\Gamma_s^1 \le \overline{V}_s < 0\}} + \Gamma_s^1 \mathbf{1}_{\{\overline{V}_s < -\Gamma_s^1\}}]\right\} ds \middle| \mathcal{F}_t\right]$$

$$-E^Q\left[\int_t^T e^{-\int_t^s c_u du}\left\{(l_s^2 h_s^2)[\overline{V}_s \mathbf{1}_{\{0 < \overline{V}_s \le \Gamma_s^2\}} + \Gamma_s^2 \mathbf{1}_{\{\overline{V}_s > \Gamma_s^2\}}]\right\} ds \middle| \mathcal{F}_t\right]$$

The terms in CCA reflect the fact that no collateral is posted in the range $\{-\Gamma_t^1 \le V_t \le \Gamma_t^2\}$, and that the posted amount of collateral is smaller than $|V|$ by the size of threshold.

The terms in CVA represent bilateral uncollateralized credit exposure, which is capped by each threshold.

FBSDE Approximation Scheme

([17])

- **The forward backward stochastic differential equations (FBSDEs) have been found particularly relevant for various valuation problems (e.g. pricing securities under asymmetric/imperfect collateralization, optimal portfolio and indifference pricing issues in incomplete and/or constrained markets).**
- **Their financial applications are discussed in details for example, El Karoui, Peng and Quenez [1997], Ma and Yong [2000], a recent book edited by Carmona [2009], Crépey [2011], and references therein.**
- **We will present a simple analytical approximation with perturbation scheme for the non-linear FBSDEs.**

FBSDE Approximation Scheme - Setup-

- We consider the following FBSDE:

$$dV_t = -f(X_t, V_t, Z_t)dt + Z_t \cdot dW_t \quad (6.1)$$
$$V_T = \Phi(X_T), \quad (6.2)$$

where V takes the value in $\mathbb{R}$, W is a r-dimensional Brownian motion, and $X_t \in \mathbb{R}^d$ is assumed to follow a diffusion which is the solution to the (forward) SDE:

$$dX_t = \gamma_0(X_t)dt + \gamma(X_t) \cdot dW_t; \ X_0 = x\,. \quad (6.3)$$

- We assume that the appropriate regularity conditions are satisfied for the necessary treatments.

Perturbative Expansion for Non-linear Generator

- In order to solve the pair of (V_t, Z_t) in terms of X_t, we extract the linear term from the generator f and treat the residual non-linear term as the perturbation to the linear FBSDE.
- We introduce the perturbation parameter ϵ, and then write the equation as

$$
\begin{aligned}
dV_t^{(\epsilon)} &= c(X_t)V_t^{(\epsilon)}dt - \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)} \cdot dW_t \\
V_T^{(\epsilon)} &= \Phi(X_T),
\end{aligned}
\tag{6.4}
$$

where $\epsilon = 1$ corresponds to the original model by [10]

$$
f(X_t, V_t, Z_t) = -c(X_t)V_t + g(X_t, V_t, Z_t). \tag{6.5}
$$

[10] Or, one can consider $\epsilon = 1$ as simply a parameter convenient to count the approximation order. The actual quantity that should be small for the approximation is the residual part g.

Perturbative Expansion for Non-linear Generator

- One should choose the linear term $c(X_t)V_t^{(\epsilon)}$ in such a way that the residual non-linear term g becomes as small as possible to achieve better convergence.
- Now, we are going to expand the solution of BSDE (6.4) in terms of ϵ: that is, suppose $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ are expanded as

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \quad (6.6)$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots . \quad (6.7)$$

Perturbative Expansion for Non-linear Generator

- **Once we obtain the solution up to the certain order, say k for example, then by putting $\epsilon = 1$,**

$$\tilde{V}_t = \sum_{i=0}^{k} V_t^{(i)}, \qquad \tilde{Z}_t = \sum_{i=0}^{k} Z_t^{(i)} \tag{6.8}$$

  is expected to provide a reasonable approximation for the original model as long as the residual term g is small enough to allow the perturbative treatment.
- **$V_t^{(i)}$ and $Z_t^{(i)}$, the corrections to each order can be calculated recursively using the results of the lower order approximations.**

Recursive Approximation

Zero-th Order

- For the zero-th order of ϵ, one can easily see the following equation should be satisfied:

$$dV_t^{(0)} = c(X_t)V_t^{(0)}dt + Z_t^{(0)} \cdot dW_t \tag{6.9}$$

$$V_T^{(0)} = \Phi(X_T)\,. \tag{6.10}$$

- It can be integrated as

$$V_t^{(0)} = E\left[e^{-\int_t^T c(X_s)ds} \Phi(X_T) \middle| \mathcal{F}_t \right] \tag{6.11}$$

  which is equivalent to the pricing of a standard European contingent claim, and $V_t^{(0)}$ is a function of X_t.
- Applying It$\hat{o}$'s formula (or Malliavin derivative), we obtain $Z_t^{(0)}$ as a function of X_t, too.

Recursive Approximation

First Order

- Now, let us consider the process $V^{(\epsilon)} - V^{(0)}$. One can see that its dynamics is governed by

$$\begin{aligned} d(V_t^{(\epsilon)} - V_t^{(0)}) &= c(X_t)(V_t^{(\epsilon)} - V_t^{(0)})dt \\ &- \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + (Z_t^{(\epsilon)} - Z_t^{(0)}) \cdot dW_t \\ V_T^{(\epsilon)} - V_T^{(0)} &= 0\,. \end{aligned} \tag{6.12}$$

- Now, by extracting the ϵ-first order term, we can once again recover the linear FBSDE

$$\begin{aligned} dV_t^{(1)} &= c(X_t)V_t^{(1)}dt - g(X_t, V_t^{(0)}, Z_t^{(0)})dt + Z_t^{(1)} \cdot dW_t \\ V_T^{(1)} &= 0\,, \end{aligned} \tag{6.13}$$

which leads to

$$V_t^{(1)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} g(X_u, V_u^{(0)}, Z_u^{(0)})du \middle| \mathcal{F}_t\right]. \tag{6.14}$$

Recursive Approximation

- Because $V_u^{(0)}$ and $Z_u^{(0)}$ are some functions of X_u, we obtain $Z_t^{(1)}$ as a function of X_t through Itô's formula (or Malliavin derivative).
- In exactly the same way, one can derive an arbitrarily higher order correction. Due to the ϵ in front of the non-linear term g, the system remains to be linear in the every order of approximation. e.g.

$$\begin{aligned} dV_t^{(2)} &= c(X_t)V_t^{(2)}dt - \left(\frac{\partial}{\partial v}g(X_t, V_t^{(0)}, Z_t^{(0)})V_t^{(1)} \right. \\ &+ \left. \nabla_z g(X_t, V_t^{(0)}, Z_t^{(0)}) \cdot Z_t^{(1)}\right) dt + Z_t^{(2)} \cdot dW_t \\ V_T^{(2)} &= 0 \end{aligned}$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- Suppose we have succeeded to express backward components (V_t, Z_t) in terms of X_t up to the $(i-1)$-th order. Now, in order to proceed to a higher order approximation, we have to give the following form of expressions with some deterministic function $G(\cdot)$ in terms of the forward components X_t, in general:

$$V_t^{(i)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} G(X_u)du \middle| \mathcal{F}_t\right] \qquad (6.15)$$

Evaluation of $(V^{(i)}, Z^{(i)})$ in terms of X

- **Even if it is impossible to obtain the exact result, we can still obtain an analytic approximation for $(V_t^{(i)}, Z_t^{(i)})$.**
- **For instance, an asymptotic expansion method allows us to obtain the expression. (See [28]-[29], [38]-[41] for the detail of the asymptotic expansion method.)**
  In fact, applying the method, [17] has provided some explicit approximations for $V_t^{(i)}$ and $Z_t^{(i)}$.
- **Also, [18] has explicitly derived an approximation formula for the dynamic optimal portfolio in an incomplete market and confirmed its accuracy comparing with the exact result by Cole-Hopf transformation. (Zariphopoulou [2001])**

Remark on Approximation of Coupled FBSDEs

- Let us consider the following generic *coupled* non-linear FBSDE:

$$
\begin{aligned}
dV_t &= -f(t, X_t, V_t, Z_t)dt + Z_t \cdot dW_t \\
V_T &= \Phi(X_T) \\
dX_t &= \gamma_0(t, X_t, V_t, Z_t)dt + \gamma(t, X_t, V_t, Z_t) \cdot dW_t; \;\; X_0 = x \,.
\end{aligned}
$$

- We can treat this case in the similar way as before(decoupled case) by introducing the following perturbation to the forward process:

$$
\begin{aligned}
d\tilde{V}_t &= c(t, \tilde{X}_t)\tilde{V}_t dt - \epsilon g(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)dt + \tilde{Z}_t \cdot dW_t \\
\tilde{V}_T &= \Phi(\tilde{X}_T) \\
d\tilde{X}_t &= \Big(r(t, \tilde{X}_t) + \epsilon\mu(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)\Big)dt \\
&\quad + \Big(\sigma(t, \tilde{X}_t) + \epsilon\eta(t, \tilde{X}_t, \tilde{V}_t, \tilde{Z}_t)\Big) \cdot dW_t
\end{aligned}
$$

- We can also apply the same method under PDE(partial differential equation) formulation based on *four step scheme* (e.g. Ma-Yong [2000]).

Please consult [17] for the details.

A forward agreement with bilateral default risk

- As the first example, we consider a toy model for a forward agreement on a stock with bilateral default risk of the contracting parties, the investor (party-1) and its counter party (party-2). The terminal payoff of the contract from the view point of the party-1 is

$$\Phi(S_T) = S_T - K \tag{6.16}$$

  where T is the maturity of the contract, and K is a constant.

- We assume the underlying stock follows a simple geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{6.17}$$

  where the risk-free interest rate r and the volatility σ are assumed to be positive constants.

- The default intensity of party-i h_i is specified as

$$h_1 = \lambda, \qquad h_2 = \lambda + h \tag{6.18}$$

  where λ and h are also positive constants.

A forward agreement with bilateral default risk

- In this setup, the pre-default value of the contract at time t, V_t, follows

$$\begin{aligned} dV_t &= rV_t dt - h_1 \max(-V_t, 0)dt + h_2 \max(V_t, 0)dt + Z_t dW_t \\ &= (r+\lambda)V_t dt + h \max(V_t, 0)dt + Z_t dW_t \qquad (6.19) \\ V_T &= \Phi(S_T) \,. \qquad (6.20) \end{aligned}$$

- Now, following the previous arguments, let us introduce the expansion parameter ϵ, and consider the following FBSDE:

$$\begin{aligned} dV_t^{(\epsilon)} &= \mu V_t^{(\epsilon)} dt - \epsilon g(V_t^{(\epsilon)}) dt + Z_t^{(\epsilon)} dW_t \qquad (6.21) \\ V_T^{(\epsilon)} &= \Phi(S_T) \qquad (6.22) \\ dS_t &= S_t(r dt + \sigma dW_t) \,, \qquad (6.23) \end{aligned}$$

where we have defined $\mu = r + \lambda$ and $g(v) = -hv\mathbf{1}_{\{v \geq 0\}}$.

A forward agreement with bilateral default risk

- The next figure shows the numerical results of the forward contract with bilateral default risk with various maturities with the direct solution from the PDE (as in Duffie-Huang [1996]).
- We have used

$$r = 0.02, \quad \lambda = 0.01, \quad h = 0.03, \tag{6.24}$$

$$\sigma = 0.2, \quad S_0 = 100\ , \tag{6.25}$$

where the strike K is chosen to make $V_0^{(0)} = 0$ for each maturity.
- We have plot $V^{(1)}$ for the first order, and $V^{(1)} + V^{(2)}$ for the second order. (Note that we have put $\epsilon = 1$ to compare the original model.)

A forward agreement with bilateral default risk

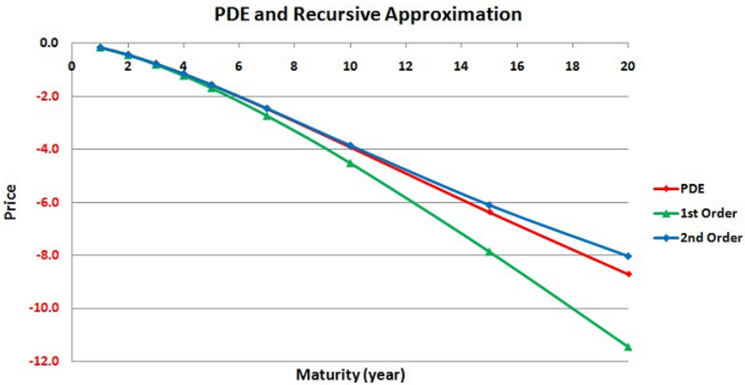


Figure: Numerical Comparison to PDE

A forward agreement with bilateral default risk

- **One can observe how the higher order correction improves the accuracy of approximation.**
- **In this example, the counter party is significantly riskier than the investor, and the underlying contract is volatile.**
- **Even in this situation, the simple approximation to the second order works quite well up to the very long maturity.**
- **In another example of [17] [11], our second order approximation has obtained a fairly close value(2.953) to the one(2.95 with std 0.01) by a regression-based Monte Carlo simulation of Gobet-Lemor-Warin[2005].**

[11] a self-financing portfolio under the situation where there exists a difference between the lending and borrowing interest rates

Example: CVA

([12])

- **When this technique is applied to evaluation of a pre-default contract value with bilateral counter party risk, Its first order approximation term can be regarded as CVA(credit value adjustment) [12].**
- **We present a simple example of an analytic approximation for this term by our 3rd order asymptotic expansion method.**
- **In particular, we consider a forex forward contract with bilateral counter party risk, where both parties post their collateral perfectly with the constant time-lag (Δ) by the same currency as the payment currency. We also assume the risk-free interest rate is equal to the collateral rate.**

[12] Our convention of CVA is different from other literatures by sign where it is defined as the “charge” to the clients. Thus, our CVA = -CVA.

FBSDE

We consider a forward contract on forex S^δ with strike K and maturity τ; the relevant FBSDE for the pre-default contract value is given as follows: ($h^{j,\delta}$ ($j = 1, 2$): each counter party's hazard rate process; ϵ, δ: perturbative and expansion parameters, respectively.)

$$dV_t^\epsilon = rV_t^\epsilon dt - \epsilon f(h_t^{1,\delta}, h_t^{2,\delta}, V_t^\epsilon, V_{t-\Delta}^\epsilon)dt + Z_t^\epsilon dW_t; \;\; V_\tau = S_\tau^\delta - K,$$

$$f(h_t^{1,\delta}, h_t^{2,\delta}, V_t^\epsilon, V_{t-\epsilon}^\epsilon) = h_t^{1,\delta}(V_{t-\Delta}^\epsilon - V_t^\epsilon)^+ - h_t^{2,\delta}(V_t^\epsilon - V_{t-\Delta}^\epsilon)^+$$

$$dh_t^{j,\delta} = \alpha^j h_t^{j,\delta} dt + \delta\sigma_{h^j} h_t^{j,\delta}(\sum_{\eta=1}^{j} c_{j,\eta} dW_t^\eta); \;\; h_0^{j,\delta} = h_0^j, (j = 1, 2)$$

$$dS_t^\delta = \mu S_t^\delta dt + \delta\nu_t^\delta \left(S_t^\delta\right)^\beta (\sum_{\eta=1}^{3} c_{3,\eta} dW_t^\eta); \;\; S_0^\delta = s_0, \; \mu = r - r_f,$$

$$d\nu_t^\delta = \kappa(\theta - \nu_t^\delta)dt + \delta\xi\nu_t^\delta(\sum_{\eta=1}^{4} c_{4,\eta} dW_t^\eta); \;\; \nu_0^\delta = \nu_0.$$

First order of ϵ

The first order equation is expressed as follows:

$$dV_t^1 = rV_t^1 dt - f(t, V_t^0, V_{t-\Delta}^0)dt + \sum_{\eta=1}^{4} Z_{t,\eta}^1 dW_t^\eta; \ V_\tau^1 = 0$$

Then, our CVA is represented by the following:

$$V_t^1 = \int_t^T e^{-r(u-t)} \mathbb{E}_t \left[f(u, V_u^0, V_{u-\Delta}^0) \right] du$$

$$f(u, V_u^0, V_{u-\Delta}^0) = h_u^{1,\delta} \cdot (V_{u-\Delta}^0 - V_u^0)^+ - h_u^{2,\delta} \cdot (V_u^0 - V_{u-\Delta}^0)^+,$$

where $V_{u-\Delta} = 0$ when $u < t + \Delta$.

$$\begin{aligned} V_u^0 &= e^{-r_f(\tau-u)} S_u^\delta - e^{-r(\tau-u)} K, \\ V_u^0 - V_{u-\Delta}^0 &= e^{-r_f(\tau-u)} S_u^\delta - e^{-r_f(\tau-u+\Delta)} S_{u-\Delta}^\delta - k(u; \Delta, r), \\ k(u; \Delta, r) &:= e^{-r(\tau-u)}(1 - e^{-r\Delta})K. \end{aligned}$$

Numerical Example

- We apply the asymptotic expansion method to evaluation of $IC(t,u) = e^{-r(u-t)}\mathrm{E}_t\left[f(u, V_u^0, V_{u-\Delta}^0)\right]$ up to the third order. Then, the value of CVA is approximated by

$$CVA(t,\tau) = \int_t^\tau IC_{AE}(t,u)du + o(\delta^3). \quad (6.26)$$

- Due to the analytical approximation of each $IC_{AE}(t,u)$, we have no problem in computation, which is very fast.

Numerical Example

The parameters are set as follows:

- **parameters of h^1;**
  $h^1_0 = 0.02, \alpha^1 = -0.02, \sigma_{h1} = 20\%.$
- **parameters of h^2;**
  $h^2_0 = 0.01, \alpha^2 = 0.02, \sigma_{h2} = 30\%.$
- **parameters of S;**
  $S_0 = 10,000, r = \mu = 1\%, \beta = 1.$
- **parameters of ν;**
  $\nu_0 = 10\%, \kappa = 1, \theta = 20\%, \xi = 30\%.$
- **correlation matrix**

	h^1	h^2	S	ν
h^1	1	0.5	-0.3	0.2
h^2	0.5	1	0.1	0.1
S	-0.3	0.1	1	-0.8
ν	0.2	0.1	-0.8	1

Density of CVA

We show the density function of approximate CVA by the asymptotic expansion method with Monte Carlo simulation. The maturity of forward contract is τ, T denotes the future time when CVA is evaluated, and Δ denotes the lag of collateral.

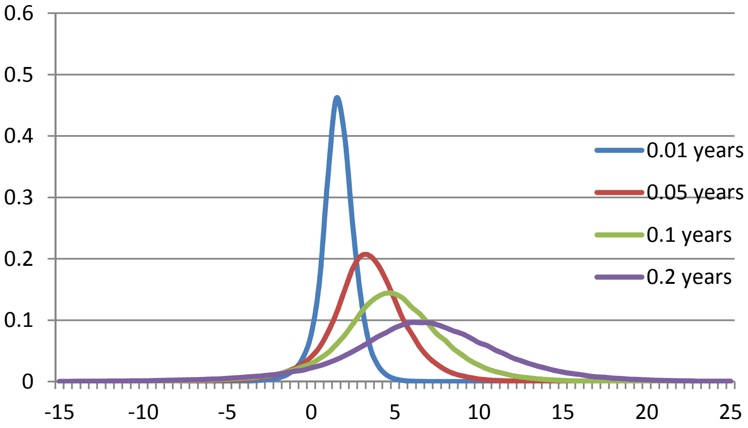
- maturity (τ): 5 years, evaluation date (T): 2.5 years.
- strike: $10,000$.
- time step size: $\frac{1}{400}$year.
- the number of trials: 325,000 with antithetic variates.

Procedure:

1. implement Monte carlo simulation of the state variables (h^1, h^2, S, v) until T.
2. given each realization of the state variables, compute $IC_{AE}(T, u)$.
3. integrate $IC_{AE}(T, u)$ numerically with respect to the time parameter u from T to τ, and plot the values and their frequencies after normalization.

Density of CVA

Figure: Density Functions of CVA with Different Time-Lags



Density of CVA

- The longer the time lag is, the wider the density is.
- The mode (average) moves to the right when the time-lag becomes longer.

$$f(u, V_u^0, V_{u-\Delta}^0) = h_u^{1,\epsilon} \cdot (V_{u-\Delta}^0 - V_u^0)^+ - h_u^{2,\epsilon} \cdot (V_u^0 - V_{u-\Delta}^0)^+.$$

- When the first term increases, the CVA also increases.
- The hazard rate h^1 in the first term tends to be larger than h^2 in the second term in our parameterization.

Perturbation Technique with Interacting Particle Method

([19], [12])

- **We will provide a straightforward simulation scheme to solve nonlinear FBSDEs at each order of perturbative approximation.**
  - **Due to the convoluted nature of the perturbative expansion, it contains multi-dimensional time integrations of expectation values, which make standard Monte Carlo too time consuming.**
  - **To avoid nested simulations, we applied the particle representation inspired by the ideas of branching diffusion models(e.g. McKean (1975), Fujita (1966), Ikeda-Nagasawa-Watanabe (1965,1966,1968), Nagasawa-Sirao (1969)).**
  - **Comparing the direct application of the branching diffusion method, our method is expected to be less numerically intensive since the interested system is already decomposed into a set of linear problems.**

Perturbation Technique with Interacting Particle Method

- **Again, let us introduce the perturbation parameter ϵ:**

$$\begin{cases} dV_s^{(\epsilon)} = -\epsilon f(X_s, V_s^{(\epsilon)}, Z_s^{(\epsilon)})ds + Z_s^{(\epsilon)} \cdot dW_s \\ V_T^{(\epsilon)} = \Psi(X_T), \end{cases} \tag{7.1}$$

where $X_t \in \mathbb{R}^d$ is assumed to follow a generic Markovian forward SDE

$$dX_s = \gamma_0(X_s)ds + \gamma(X_s) \cdot dW_s; \; X_t = x_t. \tag{7.2}$$

- **Let us fix the initial time as t. We denote the Malliavin derivative of X_u $(u \geq t)$ at time t as**

$$\mathcal{D}_t X_u \in \mathbb{R}^{r \times d}. \tag{7.3}$$

Perturbation Technique with Interacting Particle Method

- Its dynamics in terms of the future time u is specified by *stochastic flow*, $(Y_{t,u})^i_j = \partial_{x^j_t} X^i_u$

$$\begin{aligned} d(Y_{t,u})^i_j &= \partial_k \gamma^i_0(X_u)(Y_{t,u})^k_j du + \partial_k \gamma^i_a(X_u)(Y_{t,u})^k_j dW^a_u \\ (Y_{t,t})^i_j &= \delta^i_j \end{aligned} \tag{7.4}$$

  where ∂_k denotes the differential with respect to the k-th component of X, and δ^i_j denotes Kronecker delta. Here, i and j run through $\{1, \cdots, d\}$ and $\{1, \cdots, r\}$ for a. Here, we adopt Einstein notation which assumes the summation of all the paired indexes.

- Then, it is well-known that

$$(\mathcal{D}_t X^i_u)_a = (Y_{t,u}\gamma(x_t))^i_a,$$

  where $a \in \{1, \cdots, r\}$ is the index of r-dimensional Brownian motion.

Perturbation Technique with Interacting Particle Method

- **ϵ-0th order: For the zeroth order, it is easy to see**

$$V_t^{(0)} = \mathbb{E}\Big[\Psi(X_T)\Big|\mathcal{F}_t\Big] \tag{7.5}$$

$$Z_t^{a(0)} = \mathbb{E}\Big[\partial_i\Psi(X_T)(Y_{tT}\gamma(X_t))^i_a\Big|\mathcal{F}_t\Big]. \tag{7.6}$$

- **It is clear that they can be evaluated by standard Monte Carlo simulation. However, for their use in higher order approximation, it is crucial to obtain explicit approximate expressions for these two quantities. We apply asymptotic expansion technique as before.**
- **In the following, let us suppose we have obtained the solutions up to a given order of asymptotic expansion, and write each of them as a function of x_t:**

$$\begin{cases} V_t^{(0)} = v^{(0)}(x_t) \\ Z_t^{(0)} = z^{(0)}(x_t). \end{cases} \tag{7.7}$$

Perturbation Technique with Interacting Particle Method

- **ϵ-1st order:**

$$\begin{aligned} V_t^{(1)} &= \int_t^T \mathbb{E}\Big[f(X_u, V_u^{(0)}, Z_u^{(0)})\Big|\mathcal{F}_t\Big]du \\ &= \int_t^T \mathbb{E}\Big[f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big)\Big|\mathcal{F}_t\Big]du \end{aligned} \tag{7.8}$$

- **Next, define the new process for ($s > t$):**

$$\hat{V}_{ts}^{(1)} = e^{\int_t^s \lambda_u du} V_s^{(1)}, \tag{7.9}$$

where deterministic positive process λ_t (It can be a positive constant for the simplest case.).

Perturbation Technique with Interacting Particle Method

- Then, its dynamics is given by

$$d\hat{V}_{ts}^{(1)} = \lambda_s \hat{V}_{ts}^{(1)} ds - \lambda_s \hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))ds + e^{\int_t^s \lambda_u du} Z_s^{(1)} \cdot dW_s \,,$$

  where

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)).$$

- Since we have $\hat{V}_{tt}^{(1)} = V_t^{(1)}$, one can easily see the following relation holds:

$$V_t^{(1)} = \mathbb{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds} \lambda_u \hat{f}_{tu}(X_u, v^{(0)}(X_u), z^{(0)}(X_u))du \middle| \mathcal{F}_t\right] \quad (7.10)$$

- As in credit risk modeling (e.g. Bielecki-Rutkowski [2002]), it is the present value of default payment where the default intensity is λ_s with the default payoff at $s(> t)$ as $\hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))$. Thus, we obtain the following proposition.

Perturbation Technique with Interacting Particle Method

Proposition

The $\mathbf{V}_t^{(1)}$ in (7.8) can be equivalently expressed as

$$\mathbf{V}_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_{t\tau}\Big(X_\tau, v^{(0)}(X_\tau), z^{(0)}(X_\tau)\Big)\Big|\mathcal{F}_t\right]. \quad (7.11)$$

Here τ is the interaction time where the interaction is drawn independently from Poisson distribution with an arbitrary deterministic positive intensity process λ_t. $\hat{f}$ is defined as

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s}e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)) . \quad (7.12)$$

Perturbation Technique with Interacting Particle Method

- Now, let us consider the component $Z^{(1)}$. It can be expressed as

$$Z_t^{(1)} = \int_t^T \mathbb{E}\left[\mathcal{D}_t f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big)\Big|\mathcal{F}_t\right] du \tag{7.13}$$

Firstly, let us observe the dynamics of Malliavin derivative of $V^{(1)}$ follows

$$\begin{aligned} d(\mathcal{D}_t V_s^{(1)}) &= -(\mathcal{D}_t X_s^i)\nabla_i(x, v^{(0)}, z^{(0)}) f(x, v^{(0)}, z^{(0)}) + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s; \\ \mathcal{D}_t V_t^{(1)} &= Z_t^{(1)}, \end{aligned} \tag{7.14}$$

where

$$\nabla_i(x, v^{(0)}, z^{(0)}) \equiv \partial_i + \partial_i v^{(0)}(x)\partial_v + \partial_i z^{a(0)}(x)\partial_{z^a}, \tag{7.15}$$

$$f(x, v^{(0)}, z^{(0)}) \equiv f(x, v^{(0)}(x), z^{(0)}(x)). \tag{7.16}$$

Perturbation Technique with Interacting Particle Method

- **Define, for $(s > t)$,**

$$\widehat{\mathcal{D}_t V_s^{(1)}} = e^{\int_t^s \lambda_u du} (\mathcal{D}_t V_s^{(1)}). \tag{7.17}$$

Then, its dynamics can be written as

$$\begin{aligned} d(\widehat{\mathcal{D}_t V_s^{(1)}}) &= \lambda_s (\widehat{\mathcal{D}_t V_s^{(1)}}) ds - \lambda_s (\mathcal{D}_t X_s^i) \nabla_i (X_s, v^{(0)}, z^{(0)}) \hat{f}_{ts} (X_s, v^{(0)}, z^{(0)}) ds \\ &+ e^{\int_t^s \lambda_u du} (\mathcal{D}_t Z_s^{(0)}) \cdot dW_s. \end{aligned} \tag{7.18}$$

We again have

$$\widehat{\mathcal{D}_t V_t^{(1)}} = Z_t^{(1)}. \tag{7.19}$$

- **Hence,**

$$Z_t^{(1)} = \mathbb{E} \left[\int_t^T e^{-\int_t^u \lambda_s ds} \lambda_u (\mathcal{D}_t X_u^i) \nabla_i (X_u, v^{(0)}, z^{(0)}) \hat{f}_{tu} (X_u, v^{(0)}, z^{(0)}) du \middle| \mathcal{F}_t \right]. \tag{7.20}$$

Perturbation Technique with Interacting Particle Method

- **Thus, following the same argument for the previous proposition, we have the result below:**

Proposition

$Z_t^{(1)}$ in (7.13) is equivalently expressed as

$$Z_t^{a(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t\tau}\gamma(X_\tau))_a^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\middle|\mathcal{F}_t\right] \quad (7.21)$$

where the definitions of random time τ and the positive deterministic process λ are the same as those in the previous proposition.

Perturbation Technique with Interacting Particle Method

Monte Carlo Method

Now, we have a new particle interpretation of $(V^{(1)}, Z^{(1)})$ as follows:

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}, z^{(0)}\right)\middle|\mathcal{F}_t\right] \quad (7.22)$$

$$Z_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbf{1}_{\{\tau<T\}}(Y_{t,\tau}\gamma(X_\tau))^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\middle|\mathcal{F}_t\right] \quad (7.23)$$

which allows efficient time integration with the following Monte Carlo scheme:

- **Run the diffusion processes of X and Y**
- **Carry out Poisson draw with probability $\lambda_s\Delta s$ at each time s and if "one" is drawn, set that time as τ.**
- **Then stores the relevant quantities at τ, or in the case of $(\tau > T)$ stores 0.**
- **Repeat the above procedures and take their expectation.**

$Z_t^{(2)}$

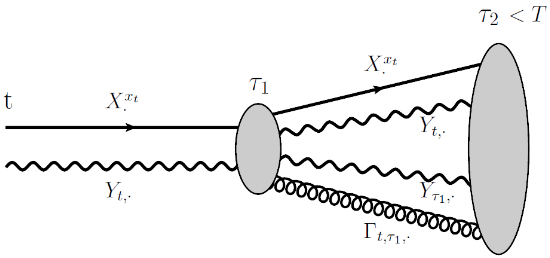


Figure 1: A particle interpretation for $Z_t^{(2)}$.

The second order stochastic flow: for $t < s < u$,

$$(\Gamma_{t,s,u})^i_{jk} := \frac{\partial^2}{\partial x_t^j \partial x_s^k} X_u^i; \ ((\Gamma_{t,s,s})^i_{jk} = 0).$$

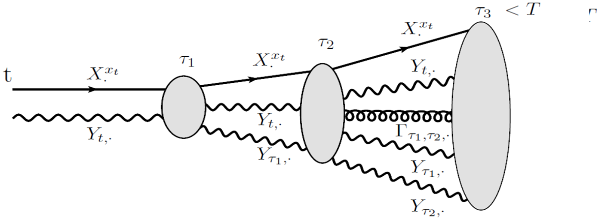


Figure 2: A particle interpretation for the first half of $V_t^{(3)}$.

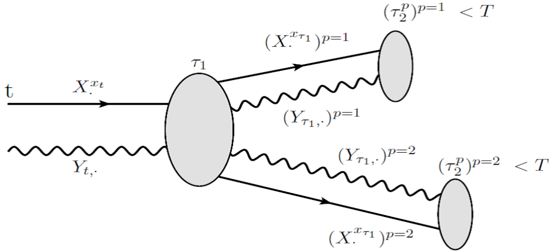


Figure 3: A particle interpretation for the second half of $V_t^{(3)}$.

Numerical Example

- V: pre-default value process,
  h: hazard rate for the option seller,
  S: the underlying asset price,
  ν: the volatility of the asset price.
- Both parties post no collateral or their collateral perfectly with the constant time-lag (Δ) by the same currency as the payment currency. We also assume the risk-free interest rate is equal to the collateral rate as 0 for simplicity.

$$dV_t^\epsilon = -\epsilon f(h_t, V_t^\epsilon, V_{t-\Delta}^\epsilon)dt + Z_t^\epsilon dW_t;$$

$$V_T = (S_T - K)^+$$

$$f(h_t, V_t^\epsilon, V_{t-\Delta}^\epsilon) = -h_t(V_t^\epsilon)^+, \text{ for the no collateral case or,}$$

$$f(h_t, V_t^\epsilon, V_{t-\Delta}^\epsilon) = -h_t(V_t^\epsilon - V_{t-\Delta}^\epsilon)^+.$$

Numerical Example

- **CEV-Heston model for asset price processes,**
- **CIR model for the hazard rate process.**
- **We apply our 3rd order asymptotic expansion method for approximations of V^0. ($\delta \in (0,1]$ below stands for the expansion parameter. See [28]-[29], [38]-[41] for the detail of the asymptotic expansion method.)**

$$
\begin{aligned}
dh_t^\delta &= \kappa\left(\theta - h_t^\delta\right)dt + \delta\gamma\sqrt{h_t^\delta}\,c_1 dW_t^1;\ h_0^\delta = h_0 \\
dS_t^\delta &= \mu_i S_t^\delta dt + \delta\zeta\sqrt{v_t^\delta}\left(S_t^\delta\right)^\beta \left(\sum_{\eta=1}^{2} c_{2,\eta} dW_t^\eta\right);\ S_0^\delta = s_0, \\
dv_t^\delta &= \Lambda\left(\Theta - v_t^\delta\right)dt + \delta\Gamma\sqrt{v_t^\delta}\left(\sum_{\eta=1}^{3} c_{3,\eta} dW_t^\eta\right);\ v_0^\delta = v_0.
\end{aligned}
$$

Numerical Example

We set the parameters following Hull-White(2005), Denault-Gauthier-Simonato(2009) and [37].

	$h(0)$	κ	θ	γ
h	2.38%	0.007	14.65%	7.83%
	$v(0)$	Λ	Θ	Γ
v	1	0.212	1	65.16%
	$S(0)$	μ	ζ	β
S	5.05%	0	3.50%	0.5

- **1 billion notional with 5 year maturity contract .**
- **We will see the pre-default values up to the 2nd order for different correlations of (S vs h), (S vs v), and (h vs v).**
- **(Monte Carlo simulation) time step size : 0.005/year; the number of trials : 2 millions; Poisson parameter: $\lambda_s \equiv 1$(constant).**

Numerical Example(Option Contracts)

Table: Pre-default values of option contracts without collateral (ATM)

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	66.86bp	66.86bp	66.86bp	66.86bp	66.86bp
	1st	-5.11bp	-6.75bp	-8.64bp	-10.78bp	-13.20bp
	2nd	0.27bp	0.48bp	0.77bp	1.14bp	1.59bp
	Total	62.02bp	60.60bp	59.00bp	57.22bp	55.25bp
S and v	0th	65.64bp	66.27bp	66.86bp	67.40bp	67.89bp
	1st	-8.48bp	-8.56bp	-8.64bp	-8.69bp	-8.74bp
	2nd	0.75bp	0.76bp	0.77bp	0.77bp	0.78bp
	Total	57.91bp	58.47bp	59.00bp	59.48bp	59.93bp
h and v	0th	66.86bp	66.86bp	66.86bp	66.86bp	66.86bp
	1st	-7.67bp	-8.14bp	-8.64bp	-9.16bp	-9.70bp
	2nd	0.62bp	0.69bp	0.77bp	0.85bp	0.94bp
	Total	59.81bp	59.41bp	59.00bp	58.56bp	58.10bp

Numerical Example(Option Contracts)

Table: Pre-default values of option contracts with collateral (time-lag: $\Delta = 0.1$)

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	0th	66.86bp	66.86bp	66.86bp	66.86bp	66.86bp
	1st	-0.59bp	-0.72bp	-0.87bp	-1.02bp	-1.19bp
	2nd	0.00bp	0.00bp	0.00bp	0.00bp	0.00bp
	Total	66.27bp	66.14bp	65.99bp	65.84bp	65.67bp
S and v	0th	65.64bp	66.27bp	66.86bp	67.40bp	67.89bp
	1st	-0.74bp	-0.81bp	-0.87bp	-0.92bp	-0.98bp
	2nd	0.00bp	0.00bp	0.00bp	0.00bp	0.00bp
	Total	64.90bp	65.47bp	65.99bp	66.47bp	66.91bp
h and v	0th	66.86bp	66.86bp	66.86bp	66.86bp	66.86bp
	1st	-0.72bp	-0.79bp	-0.87bp	-0.95bp	-1.03bp
	2nd	0.00bp	0.00bp	0.00bp	0.00bp	0.00bp
	Total	66.14bp	66.07bp	65.99bp	65.91bp	65.83bp

Numerical Example(Implied Volatility)

Table: Implied volatility of option contracts without collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	ATM*0.8	12.60%	11.84%	10.98%	10.00%	8.83%
	ATM	13.83%	13.50%	13.15%	12.75%	12.31%
	ATM*1.2	13.98%	13.79%	13.57%	13.30%	13.00%
S and v	ATM*0.8	12.35%	11.76%	10.98%	9.99%	8.71%
	ATM	12.90%	13.03%	13.15%	13.25%	13.35%
	ATM*1.2	11.72%	12.71%	13.57%	14.31%	14.94%
h and v	ATM*0.8	11.21%	11.10%	10.98%	10.85%	10.71%
	ATM	13.33%	13.24%	13.15%	13.05%	12.94%
	ATM*1.2	13.74%	13.66%	13.57%	13.47%	13.37%

Numerical Example(Implied Volatility)

Table: Implied volatility of option contracts with collateral (time-lag:$\mathbf{\Delta = 0.1}$)

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	ATM*0.8	15.66%	15.61%	15.57%	15.52%	15.47%
	ATM	14.78%	14.75%	14.72%	14.68%	14.65%
	ATM*1.2	14.40%	14.38%	14.35%	14.32%	14.29%
S and v	ATM*0.8	16.91%	16.32%	15.57%	14.65%	13.56%
	ATM	14.47%	14.60%	14.72%	14.83%	14.92%
	ATM*1.2	12.34%	13.42%	14.35%	15.16%	15.85%
h and v	ATM*0.8	15.62%	15.59%	15.57%	15.54%	15.51%
	ATM	14.75%	14.74%	14.72%	14.70%	14.68%
	ATM*1.2	14.38%	14.37%	14.35%	14.34%	14.32%

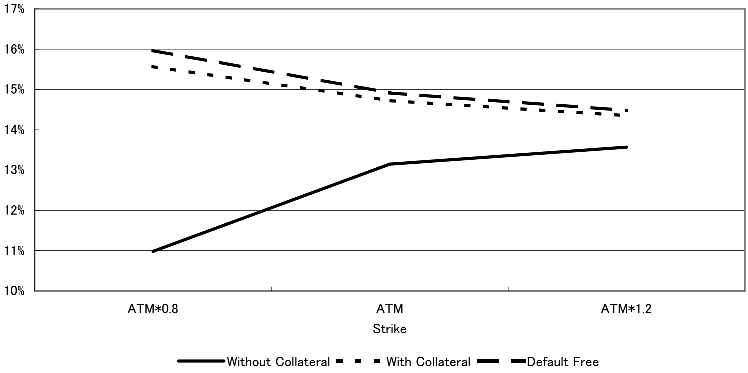
Numerical Example(Implied Volatility)

Table: Implied volatility of default free option contracts without collateral

Correlation		-0.7	-0.35	0	0.35	0.7
S and h	ATM*0.8	15.96%	15.96%	15.96%	15.96%	15.96%
	ATM	14.91%	14.91%	14.91%	14.91%	14.91%
	ATM*1.2	14.48%	14.48%	14.48%	14.48%	14.48%
S and v	ATM*0.8	17.26%	16.69%	15.96%	15.08%	14.02%
	ATM	14.64%	14.78%	14.91%	15.03%	15.14%
	ATM*1.2	12.43%	13.52%	14.48%	15.30%	16.00%
h and v	ATM*0.8	15.96%	15.96%	15.96%	15.96%	15.96%
	ATM	14.91%	14.91%	14.91%	14.91%	14.91%
	ATM*1.2	14.48%	14.48%	14.48%	14.48%	14.48%

Numerical Example(Implied Volatility)

Figure: Implied Volatilities



All the correlations are set to be 0.

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