Discounting Revisited.

Valuations under Funding Costs, Counterparty Risk and Collateralization.

Christian P. Fries
email@christian-fries.de

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Abstract

Looking at the valuation of a swap when funding costs and counterparty risk are neglected (i.e. when there is a unique risk free discounting curve), it is natural to ask "What is the discounting curve of a swap in the presence of funding costs, counterparty risk and/or collateralization?".

In this note we try to give an answer to this question. The answer depends on who you are and in general it is "There is no such thing as a unique discounting curve (for swaps)." Our approach is somewhat axiomatic, i.e., we make very few basic assumptions. We shed some light on the use of *own credit risk* in mark-to-market valuations, where the mark-to-market value of a portfolio increases as the owner's credibility decreases.

We present two different valuations. The first is a mark-to-market valuation which determines the liquidation value of a product, excluding our own funding cost. The second is a portfolio valuation which determines the replication value of a product including funding costs.

We will also consider counterparty risk. If funding costs are present, i.e. if we use a replication strategy to value a portfolio, then counterparty risk and funding are tied together:

- In addition to the default risk with respect to our exposure we have to consider the loss of a potential funding benefit, i.e. the impact of default on funding.
- Buying protection against default has to be funded itself and we account for that.

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Contents

1 Introduction

Looking at the valuation of a swap when funding costs and counterparty risk are neglected (i.e. when there is a unique risk free discounting curve), it is natural to ask "What is the discounting curve of a swap in the presence of funding costs, counterparty risk and/or collateralization?".

The answer depends on who you are and your valuation methodology. Factoring in funding costs the answer to that question may be: "There is no such thing as a unique discounting curve (for swaps)."

1.1 Two Different Valuations

In Section 2 we will first take a simplified view when valuing claims and cash flows: We view the market price of a zero coupon bond as the value of a claim and value all claims according to such zero coupon bond prices. However, this is only one possible approach. We could call it the third party (mark-to-market) liquidation view, which is to ask what the value of a portfolio of claims will be if we liquidate it today. This approach does not put a value on our own funding costs.

In Section 3 we will then construct a different valuation which includes the funding costs of net cash requirements. These funding costs occur over the lifetime of the product. They are of course not present if the portfolio is liquidated. This alternative valuation will also give an answer to an otherwise puzzling problem: In the mark-to-market valuation it appears that the value of the portfolio increases if its credit rating decreases (because liabilities are written down). However, if we include the funding cost, then the effect is reversed since funding costs are increased, and since we are not liquidating the portfolio we have to factor them in.

The difference between the two valuations is their point of view. Liquidating a portfolio we value it from "outside" as a third party. Accounting for operational cost we value it from the "inside". This produces two different values for the simple reason that we cannot short our own debt (sell protection on our self)[1]. However, a third party can do this.

1.2 Related Works

The backward algorithm for evaluating under different curves for borrowing and lending has already been given in the textbook [6]. Some of our examples below (e.g., the multi-curve model using deterministic default intensities in Section 2.4.1) are also taken from this book.

The usual setup of other papers discussing "multiple" curves is to consider a two or three curve valuation framework: one (forward) curve for the indices (forward rates and swap rates), one (discounting) curve for collateralized deals and one (discounting) curve for funded deals, see e.g. [2, 12]. However, as we will illustrate, funding costs and netting agreements may imply that there is no such thing as a product independent discounting curve. The switching between different discounting curves has to be considered on a portfolio level (incorporating netting effects) and is in general stochastic

[1] See axiom 2 below for a more precise definition.

(incorporating the effect of the net position being positive or negative). We carry this out in Section 3.

Piterbarg [11] discusses the effects of the correlation term induced by a mismatch of the index and the discounting curve, i.e. convexity adjustments. Those convexity adjustments are reflected by our (numerical) valuation algorithm once corresponding assumptions (e.g. correlations of rates and spreads) are introduced into the underlying model.[2] The formulas derived in [11] are of great importance since they can be used to provide efficient approximations for a multi-curve model's calibration products.

For the modeling of rates (e.g. through basis swap spreads) see e.g. [8].

In [10] Morini and Prampolini considered a zero coupon bond together with its funding and counterparty risk. They showed that, for a zero coupon, the mark-to-market valuation including own credit can be re-interpreted as a funding benefit. However, their argument relies on the fact that there is a premium paid which can be factored in as a funding benefit, i.e. they consider the liquidation or inception of the deal (i.e. they consider a mark-to-market valuation). By doing so, the premium paid for a future liability can always function as funding, hence funding costs beyond the bond-cds basis do not occur. Our approach is more general in the sense that we consider the true net cash position. The net cash position will give rise to a complex discounting algorithm where a funding benefit may or may not occur (in a stochastic way). Morini and Prampolini clarify the relation of the bond-cds-basis, i.e. the difference between bond spread and cds spread. In theory a defaultable bond for which one buys protection should equal a non defaultable bond as long as the protection itself cannot default. That market prices often do not reflect that situation is attributed to liquidity aspects of the products. See also the short comment after equation (7).

For the valuation of CVAs[3] using the modeling of default times / stopping times to value the contract under default see [7, 3] and Section 8 in [9]. In the latter a homogeneity assumption is made resulting in $P^{\mathsf{A}}(T;t) = P^{\mathsf{B}}(T;t)$. In contrast to the CVA approach modeling default times, we considered market prices of zero coupon bonds only and the valuation would require a model of the zero coupon bond process (one way is to model default times, another is to model spreads and survival probabilities). With the possible exception of Section 6 on "Modeling" we do not explicitly model the impact of liquidity or default risk.

For an introduction to counterparty risk, unilateral and bilateral CVA's see [7] and references therein.

[2] Our presentation is essentially "model free".
[3] CVA: Counterparty Valuation Adjustment.

2 Discounting Cash Flows: The Third Party (Mark-to-Market) Liquidation View

Let us first take the point of view of a third party and value claims between two other entities A and B. In this situation we derive the discounting by considering a single financial product: *the zero coupon bond*. Discounting, i.e. discount factors, are given by the price at which a zero coupon bond can be sold or bought. Let us formalize this:

2.1 Discount Factors for Outgoing and Incoming Cash Flows

Assume entity A can issue a bond with maturity T and notional 1. By this we mean that A offers the financial product which offers the payment of 1 in time T. Let us denote the time t market price of this product by $P^{\mathsf{A}}(T;t)$. This is the value at which the market is willing to buy or sell this bond.

Likewise let $P^{\mathsf{B}}(T;t)$ denote the price at which the market is willing to buy or sell bonds issued by some other entity B.

Assume that A receives a cash flow $C(T) > 0$ from entity B. This corresponds to A holding a zero coupon bond from entity B having notional $C(T)$ and maturity T. Hence, the time T value of this isolated cash flow is $C(T)P^{\mathsf{B}}(T;T)$ (seen from A's perspective). Given that $C(T)$ is not stochastic, the time $t < T$ value of this isolated cash flow then is $C(T)P^{\mathsf{B}}(T;t)$. We will call this cash flow “incoming”. However, we want to stress that we view ourselves as a third party independent of A and B trading in bonds. Thus we have: $P^{\mathsf{B}}(T;t)$ is the discount factor of all *incoming cash flows* from entity B.

Consider some other contract featuring a cash flow $C(T) > 0$ from A to B at time T. The time T value of this cash flow is $-C(T)P^{\mathsf{A}}(T;T)$, where we view the value from A's perspective, hence the minus sign. We will call this cash flow outgoing. However, we want to stress again that we view ourselves as a third party independent of A and B trading in bonds. Given that $C(T)$ is not stochastic, the time $t < T$ value of this isolated cash flow is $C(T)P^{\mathsf{A}}(T;t)$. Thus we have: $P^{\mathsf{A}}(T;t)$ is the discount factor of all *outgoing cash flows* of entity A.

Since we view ourselves as a third party, in this framework the discount factor of a cash flow is determined by the value of the zero coupon bonds of the originating entity.

If we view a cash flow $|C(T)|$ between A and B from the perspective of the entity A such that $C(T) > 0$ means that the cash flow is incoming (positive value for A) and $C(T) < 0$ means that the cash flow is outgoing, then its time T value can be written as

$$\min(C(T),0)\ P^{\mathsf{A}}(T;T)\ +\ \max(C(T),0)\ P^{\mathsf{B}}(T;T).$$

2.1.1 Valuation of a Fixed Coupon Bond

Knowing the discount factors allows the valuation of a fixed coupon bond, because here all future cash flows have the same origin.

2.2 Counterparty Risk

The price $P^{\mathsf{A}}(T;t)$ contains the time-value of a cash flow from A (e.g. through a risk free interest rate) and the counterparty risk of A.

Usually we expect

$$0 \leq P^{\mathsf{A}}(T;T) \leq 1,$$

and due to A's credit risk we may have $P^{\mathsf{A}}(T;T,\omega) < 1$ for some path ω. As a consequence, we will often use the symbol $P^{\mathsf{A}}(T;T)$ which would not be present if counterparty risk (and funding) had been neglected.

In Section 2.6 we will see a case where $P^{\mathsf{A_C}}(T;T,\omega) > 1$ will be meaningful for some "virtual" entity $\mathsf{A_C}$, namely for over-collateralized cash flows from A.

2.2.1 Example

If we do not consider "recoveries" then $P^{A}(T;T,\omega) \in \{0,1\}$. For example, if entity A defaults in time $\tau(\omega)$, then $P^{A}(T;t,\omega) = 0$ for $t > \tau(\omega)$.

2.3 Netting

Let us now consider that entities A and B have two contracts with each other: one resulting in a cash flow from A to B, the other resulting in a cash flow from B to A. Let us assume further that both cash flows will occur at the same future time T. Let $C^{\mathsf{A}}(T) > 0$ denote the cash flow originating from A to B. Let $C^{\mathsf{B}}(T) > 0$ denote the cash flow originating from B to A. Individually the time T value of the two contracts is

$$-C^{\mathsf{A}}(T)\; P^{\mathsf{A}}(T;T) \qquad \text{and} \qquad +C^{\mathsf{B}}(T)\; P^{\mathsf{B}}(T;T),$$

where the signs stem from considering the value from A's perspective. From B's perspective we would have the opposite signs. If $C^{\mathsf{A}}(T)$ and $C^{\mathsf{B}}(T)$ are deterministic, then the time t value of these cash flows is

$$-C^{\mathsf{A}}(T)\; P^{\mathsf{A}}(T;t) \qquad \text{and} \qquad +C^{\mathsf{B}}(T)\; P^{\mathsf{B}}(T;t),$$

respectively.

However, if we have a netting agreement, i.e. the two counter parties A and B agree that only the net cash flow is exchanged, then we effectively have a single contract with a single cash flow of

$$C(T) \;:=\; -C^{\mathsf{A}}(T) + C^{\mathsf{B}}(T).$$

The origin of this cash flow is now determined by its sign. If $C(T) < 0$ then $|C(T)|$ flows from A to B. If $C(T) > 0$ then $|C(T)|$ flows from B to A. The time T value of the netted cash flow $C(T)$, seen from A's perspective, is

$$\min(C(T),0)\; P^{\mathsf{A}}(T;T) \;+\; \max(C(T),0)\; P^{\mathsf{B}}(T;T).$$

Note that if

$$P^{\mathsf{A}}(T;t) = P^{\mathsf{B}}(T;t) =: P(T;t)$$

then there is no difference between the value of a netted cash flow and the sum the individual values, but normally this property does not hold.

2.4 Valuation of Stochastic Cash Flows

If cash flows $C(T)$ are stochastic, then we have to value their time t value using a valuation model, e.g. risk neutral valuation using a numeraire N and a corresponding martingale measure $\mathbb{Q}^N$. The analytic valuation (which actually stems from a static hedge) as $C(T)P^{\mathsf{A}}(T;t)$ no longer holds. Let N denote a numeraire and $\mathbb{Q}^N$ a corresponding martingale measure such that

$$\frac{P^{\mathsf{A}}(T;t)}{N(t)} \;=\; \mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{A}}(T;T)}{N(T)}|\mathcal{F}_t\right).$$

Then a possibly stochastic cash flow $C(T)$, outgoing from A, is evaluated in the usual way, where the value is given as

$$\mathrm{E}^{\mathbb{Q}^N}\left(\frac{C(T)\;P^{\mathsf{A}}(T;T)}{N(T)}|\mathcal{F}_t\right)$$

Note that the factor $P^{\mathsf{A}}(T;T)$ determines the effect of the origin of the cash flow, here A. In theories where counterparty risk (and funding) is neglected, the cash flow $C(T)$ is valued as

$$\mathrm{E}^{\mathbb{Q}^N}\left(\frac{C(T)}{N(T)}|\mathcal{F}_t\right).$$

2.4.1 Example: A simple approach to constructing a multi-curve model

A simple approach to including counterparty risk, i.e. different discounting curves, into a standard single curve interest rate model, is by assuming a deterministic default intensity. To formalize this, let $\mathcal{F}_t$ denote the filtration including counterparty risk and assume that

$$\mathcal{F}_t \;=\; \mathcal{G}_t \times \mathcal{H}_t,$$

where $\mathcal{G}_t$ is the filtration associated with a given counterparty risk-free model. In other words, we assume that default free payoffs are valued as

$$P(T;t) \;=\; N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{1}{N(T)}\;\Big|\;\mathcal{F}_T\right) \;=\; N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{1}{N(T)}\;\Big|\;\mathcal{G}_T\right)$$

and we implicitly define the counterparty risk as

$$\begin{aligned} P^{\mathsf{A}}(T;t) \;&=\; N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{A}}(T;T)}{N(T)}\;\Big|\;\mathcal{F}_T\right) \\ &=\; N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{1}{N(T)}\;\Big|\;\mathcal{G}_T\right)\;P^{\mathsf{A}}(t;t)\;\exp\left(-\int_t^T \lambda^{\mathsf{A}}(\tau)\,\mathrm{d}\tau\right) \\ &=\; P(T;t)\;P^{\mathsf{A}}(t;t)\;\exp\left(-\int_t^T \lambda^{\mathsf{A}}(\tau)\,\mathrm{d}\tau\right). \end{aligned}$$

So in other words, every time T cash flow has to carry a *marker* $P^{\mathsf{A}}(T;T)$ which identifies its counterparty (source), here A. The time t valuation of this cash flow, i.e. the numeraire relative conditional expectation of the cash flow, is given by the conditional expectation of the corresponding counterparty risk-free cash flow (i.e. with respect to the filtration $\mathcal{G}$ of the single curve model) times the survival probability times a new *marker* $P^{\mathsf{A}}(t;t)$. Obviously, the process $P^{\mathsf{A}}(T)$ is a $\mathbb{Q}^N$-martingale (with respect to the full filtration $\mathcal{F}$. Note that the default event, i.e., the filtration $\mathcal{H}$, is not modeled at all. The assumption that the default intensity λ is deterministic is sufficient to value the default indicator function. See also Chapter 28 in [6].

Note that such a model does not explicitly distinguish liquidity effects and default risk effects. They are all subsumed in the market implied default intensity λ.

2.5 Credit Linking

Let us consider a bond $P^{\mathsf{C}}(T;t)$ issued by entity C. Let us consider a contract where A pays $P^{\mathsf{C}}(T;T)$ in $t = T$, i.e. A pays 1 only if C survives, otherwise it will pay C's recovery. However, this cash flow still *is* a cash flow originating from (granted by) A. The time T value of this cash flow is $P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T)$.

This contract can been seen as a credit linked deal.

2.5.1 Examples

If $P^{\mathsf{C}}(T;T,\omega) = 1$ for all $\omega \in \Omega$, then P^{C} has no credit risk. In this case we have

$$N(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T)}{N(T)} \mid \mathcal{F}_t\right) \;=\; P^{\mathsf{A}}(T;t).$$

Also, if $P^{\mathsf{C}}(T;T,\omega) = P^{\mathsf{A}}(T;T,\omega) \in \{0,1\}$ for all $\omega \in \Omega$, then C defaults if and only if A defaults and there are no recoveries. In that case we also have

$$N(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T)}{N(T)} \mid \mathcal{F}_t\right) \;=\; P^{\mathsf{A}}(T;t).$$

If the two random variables $P^{\mathsf{A}}(T;T)$ and $P^{\mathsf{C}}(T;T)$ are independent we have

$$N(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T)}{N(T)} \mid \mathcal{F}_t\right) \;=\; P^{\mathsf{C}}(T;t)\; P^{\mathsf{A}}(T;t)\; \frac{1}{P(T;t)},$$

where $P(T;t) := N(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{1}{N(T)} \mid \mathcal{F}_t\right)$.

To prove the latter we switch to terminal measure (i.e. choose $N = P(T)$ such that $N(T) = 1$) and get

$$\begin{aligned} & N(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T)}{N(T)} \mid \mathcal{F}_t\right) \\ &= N(t)\; \mathrm{E}^{\mathbb{Q}^N}\left(P^{\mathsf{C}}(T;T) \mid \mathcal{F}_t\right)\; \mathrm{E}^{\mathbb{Q}^N}\left(P^{\mathsf{A}}(T;T) \mid \mathcal{F}_t\right) \\ &= P^{\mathsf{C}}(T;t)\; P^{\mathsf{A}}(T;t)\frac{1}{N(t)} \;=\; P^{\mathsf{C}}(T;t)\; P^{\mathsf{A}}(T;t)\frac{1}{P(T;t)} \end{aligned}$$

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2.6 Collateralization

Collateralization is not some "special case" which has to be considered in the above valuation framework. Collateralization is an additional contract with an additional netting agreement (and a credit link). As we will illustrate, we can re-interpret a collateralized contract as a contract with a different discount curve. However, this *is* only a re-interpretation.

For simplicity let us consider the collateralization of a single future cash flow.

Let us assume that counterparty A pays M in time T. Thus, seen from the perspective of A, there is a cash flow

$$-MP^{\mathsf{A}}(T;T) \qquad \text{in } t = T.$$

Hence, the time t value of the non-collateralized cash flow is $-MP^{\mathsf{A}}(T;t)$. Next, assume that counterparty A holds a contract where an entity C will pay K in time T. Thus, seen from the the perspective of A there is a cash flow

$$KP^{\mathsf{C}}(T;T) \qquad \text{in } t = T.$$

Hence, the time t value of this cash flow is $KP^{\mathsf{C}}(T;t)$.

If we value both contracts separately, then the first contract evaluates to $-M$, the second contract evaluates to K. If we use the second contract to "collateralize" the first contract we bundle the two contracts in the sense that the second contract is passed to the counterparty B as a pawn. This can be seen as letting the second contract default if the first contract defaults. The time T value thus is

$$\left(KP^{\mathsf{C}}(T;T) - M\right) P^{\mathsf{A}}(T;T),$$

where we assume that the net cash flow is non-positive, which is the case if $K < M$ and $P^{\mathsf{C}}(T;T) \leq 1$, so we do not consider over-collateralization. The random variable $P^{\mathsf{C}}(T;T)$ accounts for the fact that the collateral may itself default over the time; see "credit linked" above.

We have

$$\begin{aligned}&\left(KP^{\mathsf{C}}(T;T) - M\right) P^{\mathsf{A}}(T;T)\\ &\qquad = KP^{\mathsf{C}}(T;T) - MP^{\mathsf{A}}(T;T) \;+\; K\left(P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) - P^{\mathsf{C}}(T;T)\right)\end{aligned}$$

Thus, the difference of the value of the collateralized package and the sum of the individual deals ($M - K$) is

$$K\left(P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) - P^{\mathsf{C}}(T;T)\right).$$

It is possible to view this change in the value of the portfolio as a change in the value of the outgoing cash flow. Let us determine the implied zero coupon bond process $P^{\mathsf{A}_{\mathsf{C}}}$ such that

$$\begin{aligned}-M\; P^{\mathsf{A}_{\mathsf{C}}}(T;T) \overset{!}{=} &- M\; P^{\mathsf{A}}(T;T)\\ &+ K\;\left(P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) - P^{\mathsf{C}}(T;T)\right).\end{aligned}$$

It is

$$P^{\mathsf{A}_\mathsf{C}}(T;T) \ := \ P^{\mathsf{A}}(T;T) - \frac{K}{M}\left(P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) - P^{\mathsf{C}}(T;T)\right). \qquad (1)$$

We refer to $P^{\mathsf{A}_\mathsf{C}}$ as the discount factor for collateralized deals. It should be noted that a corresponding zero coupon bond does not exist (though it may be synthesized) and that this discount factor is simply a computational tool. In addition, note that the discount factor depends on the value (K) and quality ($P^{\mathsf{C}}(T;T)$) of the collateral.

2.6.1 Interpretation

From the above, we can check some limit cases:

- For $P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) = P^{\mathsf{C}}(T;T)$ we find that

  $$P^{\mathsf{A}_\mathsf{C}}(T;T) = P^{\mathsf{A}}(T;T).$$

  Note that this equations holds, for example, if $P^{\mathsf{A}}(T;T) < 1 \ \Rightarrow \ P^{\mathsf{C}}(T;T)$. This can be interpreted as: If the quality of the collateral is "less or equal" to the quality of the original counterpart, then collateralization has no effect.

- For $P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) = P^{\mathsf{A}}(T;T)$ and $0 \leq K \leq M$ we find that

  $$P^{\mathsf{A}_\mathsf{C}}(T;T) = \alpha P^{\mathsf{C}}(T;T) + (1-\alpha)P^{\mathsf{A}}(T;T),$$

  where $\alpha = \frac{K}{M}$, i.e. if the collateral does not compromise the quality of the bond as a credit linked bond, then collateralization constitutes a mixing of the two discount factors at the ratio of the collateralized amount.

- For $P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) = P^{\mathsf{A}}(T;T)$ and $K = M$ we find that

  $$P^{\mathsf{A}_\mathsf{C}}(T;T) = P^{\mathsf{C}}(T;T),$$

  i.e. if the collateral does not compromise the quality of the bond as a credit linked bond and the collateral matches the future cash flow, then the collateralized discount factor agrees with the discount factor of the collateral.

Given that $P^{\mathsf{C}}(T;T)P^{\mathsf{A}}(T;T) = P^{\mathsf{A}}(T;T)$ we find that collateralization has a positive value for the entity receiving the collateral. The reason can be understood as a matter of funding: The interest paid on the collateral is less than the interest paid on an issued bond, hence, the entity receiving collateral can save funding costs.

2.6.2 Example

Let us consider entities A and C where

$$N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{A}}(T;T)}{N(T)}|\mathcal{F}_t\right) \ = \ \exp(-r(T-t))\exp(-\lambda^{\mathsf{A}}(T-t))$$

$$N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{C}}(T;T)}{N(T)}|\mathcal{F}_t\right) \ = \ \exp(-r(T-t))\exp(-\lambda^{\mathsf{C}}(T-t))$$

and

$$N(t)\,\mathrm{E}^{\mathbb{Q}^N}\left(\frac{P^{\mathsf{C}}(T;T)\,P^{\mathsf{A}}(T;T)}{N(T)}|\mathcal{F}_t\right)$$
$$= \exp(-r(T-t))\exp(-\lambda^{\mathsf{C}}(T-t))\exp(-\lambda^{\mathsf{A}}(T-t)).$$

The first two equations could be viewed as a definition of some base interest rate level r and the counterparty dependent default intensities λ. So to some extent these equations are definitions and do not constitute an assumption. However, given that the base level r is given, the third equation constitutes an assumption, namely that the defaults of A and C are independent.

From this we get for the impact of collateralization that

$$P^{\mathsf{A}_{\mathsf{C}}}(T;t) = \exp(-r(T-t))\exp(-\lambda^{\mathsf{A}}(T-t))\cdot$$
$$\cdot\left(1+\frac{K}{M}\exp(-\lambda^{\mathsf{C}}(T-t))\left(\exp(\lambda^{\mathsf{A}}(T-t))-1\right)\right).$$

Note that for $K > M$ this discount factor could have $P^{\mathsf{A}_{\mathsf{C}}}(T;T) > 1$. This would correspond to the case where the original deal is over-collateralized. We excluded this case from the derivation and in fact the formula above does not hold in general for an over-collateralized deal since we would need to consider the discount factor of the counterpart receiving the collateral (the over-collateralized part is at risk now). Nevertheless, a similar formula can be derived.

2.6.3 Full Bilateral Collateralization

Full bilateral collateralization with collateral having the same discount factor, i.e., $P^{\mathsf{A}_{\mathsf{C}}}(T) = P^{\mathsf{B}_{\mathsf{D}}}(T)$, will result in a single discounting curve (namely that of the collateral) regardless of the origin of the cash flow.

2.7 Collateralization with Different Maturities

2.7.1 Collateralization with Shorter Maturity

The formulas above considered the simple case of two cashflows K, M being expected at the same time. Next, consider that we collateralize a cashflow of 1 paid by A in T_2 (i.e. $P^{\mathsf{A}}(T_2)$) by a cashflow of α received from C in T_1, i.e. $\alpha P^{\mathsf{C}}(T_1)$. We consider $T_1 < T_2$ and assume that only part of the time can be collateralized.

The value $P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_1)}}(T_2;t)$ of this contract in $t = T_1$ is thus

$$P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_1)}}(T_2;T_1) := P^{\mathsf{A}}(T_2;T_1)+\alpha\left(P^{\mathsf{C}}(T_1;T_1)-P^{\mathsf{C}}(T_1;T_1)P^{\mathsf{A}}(T_1;T_1)\right),$$

where $P^{\mathsf{A}}(T_1;T_1)$ functions as survival indicator function for the cashflow from A. Equation (2.7.1) holds because at maturity T_1 we either receive(d) the collateral or still hold a bond $P^{\mathsf{A}}(T_2)$.

For $t < T_1$ the time t value is given by

$$P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_1)}}(T_2;t) = P^{\mathsf{A}}(T_2;t)+\alpha\left(P^{\mathsf{C}}(T_1;t)-\mathrm{E}\left[P^{\mathsf{C}}(T_1)P^{\mathsf{A}}(T_1),T_1;t\right]\right). \quad (2)$$

where we use the short-hand notation

$$\mathrm{E}\left[X,T;t\right] \;:=\; \mathrm{E}^{\mathbb{Q}^N}\left(X(T)\frac{N(t)}{N(T)} \mid \mathcal{F}_t\right)$$

for the valuation of the correlation term $P^{\mathsf{C}}(T_1)P^{\mathsf{A}}(T_1)$. If

$$P^{\mathsf{C}}(T_1;T_1)P^{\mathsf{A}}(T_1;T_1) \;=\; P^{\mathsf{A}}(T_1;T_1), \tag{3}$$

then (2) simplifies to

$$P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_1)}}(T_2;t) \;:=\; P^{\mathsf{A}}(T_2;T_1) + \alpha\left(P^{\mathsf{C}}(T_1;T_1) - P^{\mathsf{A}}(T_1;T_1)\right). \tag{4}$$

Note that if $P^{\mathsf{C}}(T_1)$ is truly riskless, which we denote by $P^{\mathsf{C}}(T_1) = P^{\circ}(T_1)$, then (3) holds.

2.7.2 Collateralization with Longer Maturity

Consider now collateralizing a cashflow of 1 paid by A in T_1, i.e. $P^{\mathsf{A}}(T_1;T_1)$ by a cashflow of α received from C in T_2, i.e. $\alpha P^{\mathsf{C}}(T_2;T_2)$, where $T_1 < T_2$.

The value $P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_2)}}(T_1;t)$ of this contract in $t = T_1$ then is

$$P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_2)}}(T_1;T_1) \;:=\; P^{\mathsf{A}}(T_1;T_1)+\alpha\left(P^{\mathsf{C}}(T_2;T_1) - P^{\mathsf{C}}(T_2;T_1)P^{\mathsf{A}}(T_1;T_1)\right), \tag{5}$$

where - as before - $P^{\mathsf{A}}(T_1;T_1)$ functions as survival indicator function for the cashflow from A. Equation (5) holds because at maturity T_1 we either receive the cashflow from A or receive(d) the collateral.

For $t < T_1$ the time t value is given by

$$P^{\mathsf{A}_{\alpha P^{\mathsf{C}}(T_2)}}(T_1;t) \;=\; P^{\mathsf{A}}(T_1;t) + \alpha\left(P^{\mathsf{C}}(T_2;t) - \mathrm{E}\left[P^{\mathsf{C}}(T_2)P^{\mathsf{A}}(T_1), T_1; t\right]\right). \tag{6}$$

where we use the short hand notation

$$\mathrm{E}\left[X,T;t\right] \;:=\; \mathrm{E}^{\mathbb{Q}^N}\left(X(T)\frac{N(t)}{N(T)} \mid \mathcal{F}_t\right).$$

Note that if $P^{\mathsf{C}}(T_2)$ is truly riskless, i.e. $P^{\mathsf{C}}(T_2) = P^{\circ}(T_2)$, then this assumption is not sufficient to simplify (6). Unlike in the previous case, we are still left with a correlation term. That correlation term expresses the possibility that the collateral value may change due to a market movement of the forward rates in $[T_1, T_2]$.

3 Discounting Cash Flows: The Hedging View

We now take a different approach to valuing cash flows and we change our point of view. We now assume that *we are* entity A and as a consequence postulate

Axiom 1: (Going Concern)

Entity A wants to stay in business (i.e. liquidation is not an option) and cash flows have to be "hedged" (i.e., neutralized, replicated). In order to stay in business, future outgoing cash flows have to be ensured.

The axiom means that we do not value cash flows by relating them to market bond prices (the liquidation view). Instead we value future cash flows by trading in assets such that future cash flows are hedged (neutralized). The costs or proceeds of this trading define the value of future cash flows. An entity must do this because all uncovered negative net cash flow will mean default. Future *net* cash flow is undesirable. A future cash flow has to be managed.

This is to some extent reasonable since cash itself is a bad thing. It does not earn interest. It needs to be invested. We will take a replication / hedging approach to value future cash. In addition we take a conservative point of view: A liability in the future (which cannot be neutralized by trading in some other asset (netting)) has to be secured in order to ensure that we stay in business. Not paying is not an option. This relates to the "going concern" as a fundamental principle in accounting.

At this point one may argue that a future outgoing cash flow is not a problem. Once the cash has to flow we just sell an asset. However, we would then be exposed to risk in that asset. This is not the business model of a bank. A bank hedges its risk and so future cash flow has to be hedged as well. Valuation is done by valuing the hedging cost.

Changing the point of view, i.e. assuming that *we* are entity A has another consequence, which we also give as an "Axiom":

Axiom 2:

We cannot short our own bond.

Let us clarify the wording here: By "shorting our own bond" we mean "shorting our own debt", i.e. entering a transaction which would pay us 1 in T if we are not in default, but 0 otherwise. This is the offsetting transaction as opposed to issuing a bond, so we use the word "short". Synthetically such a cash flow can be constructed by buying a risk free bond (a cash flow of $+1$ in T always) and selling protection (on oneself) (a cash flow of -1 in T upon default). The cash flows of such a transaction would correspond to the cash flows of $-P^{\mathsf{A}}(T;t)$.

The rationale of Axiom 2 is clear: While a third party E can in fact short a bond issued by A by selling protection on A, A itself cannot offer such an instrument. It would be offering an insurance on its own default, but if the default occurred, the insurance would not cover the event. Hence such a product is worthless.

3.1 Interpretation

Axiom 1 will result in a valuation where cash flows are valued according to their funding cost and funding cost can be understood as hedging cost. Axiom 2 will limit the set of hedging instruments in the following sense: while we can issue a bond $P^{\mathsf{A}}(T)$ resulting in proceeds $P^{\mathsf{A}}(T;t)$, Axiom 2 implies that we cannot deposit money risklessly at the same interest rate. To deposit money riskless we have to buy $P^{\circ}(T;t)$.

If we could short our own bond, i.e. if we could enter a transaction paying 1 only upon our own survival, this transaction would constitute a better investment than a risk free bond: We would deposit money in terms of a cash flow $-P^{\mathsf{A}}(T;t)$ in t, generating a cash flow $+1$ in T upon our own survival and we simply wouldn't care about a loss in case of our own default, because we have defaulted anyway. In this case we would have a unique interest rate for borrowing and depositing money given by $P^{\mathsf{A}}(T;t)$.

On the other hand, it is clear that the existence of such a transaction and it's use as a cash flow hedge instrument would create the incentive to deteriorate our own credit rating.

Let us assume that it would be possible to replicate the cash flows of $-P^{\mathsf{A}}$ by trading (buying) a bond $P^{\mathsf{D}}(T;t)$ of some counterpart D whose default risk is highly correlated to A, such that - by some magic - D always defaults after A defaults. In that case the above argument would correspond to arguing that it is not necessary to value D's default risk (buy protection) because we (A) cannot suffer from D's default due to our own default. Put differently, conditional upon our own survival, $P^{\mathsf{D}}(T;t)$ is a risk free bond which is cheaper than the (unconditional) risk free bond $P^{\circ}(T;t)$. Obviously, such a replication strategy breaks down if D defaults before A's default. This example shows that axiom 2 corresponds to considering all counterparty risk regardless of our own default, which actually is a consequence of axiom 1, namely that our own default is not an option.

The situation here is similar to the difference between unilateral and bilateral counterparty valuation adjustments (CVA's): While unilateral CVAs consider all counterparty risks without factoring in one's own default, a bilateral CVA factors in one's own default. See [7].

3.2 Moving Cash Flow through Time

Axiom 1 requires that we consider transactions where all future cash flow is converted into todays cash flow, so that a netting of cash occurs. Axiom 2 then restricts the means by which we can move cash flows around.

Let us explore the means of "moving" cash through time. To illustrate the concept we first consider deterministic cash flows only. These allow for static hedges through the construction of appropriate forward bonds in Section 3.3. Dynamic hedges will be considered in Section 3.4.

Let us assume that there is a risk free entity issuing risk free bonds $P^{\circ}(T;t)$ by which we may deposit cash.[4]

[4] Counterparty risk will be considered at a later stage.

3.2.1 Moving Positive Cash to the Future

If entity A (i.e. *ourselves*) has cash N in time t it can invest it and buy bonds $P^{\circ}(T;t)$. The cash flow received in T then is $N\frac{1}{P^{\circ}(T;t)}$.

This is the way by which positive cash can be moved from t to a later time $T > t$ (risk less, i.e. suitable for hedging). Note that investing in to a bond $P^{\mathsf{B}}(T;t)$ of some other entity B is not admissible since then the future cash flow would be credit linked to B.

3.2.2 Moving Negative Cash to the Future

If entity A (i.e. *ourselves*) has cash $-N$ in time t it has to issue a bond in order to cover this cash. In fact, there is no such thing as negative cash. Either we have to sell assets or issue a bond. Assume for now that selling assets is not an option. Issuing a bond generating cash flow $+N$ (proceeds) the cash flow in T then is $-N\frac{1}{P^{\mathsf{A}}(T;t)}$ (payment).

This is the way by which negative cash can be moved from t to a later time $T > t$.

3.2.3 Hedging Negative Future Cash Flow

If entity A (i.e. *ourselves*) is confronted with a cash flow $-N$ in time T it needs to hedge (guarantee) this cash flow by depositing $-NP^{\circ}(T;t)$ today (in bonds $P^{\circ}(T;t)$).

This is the way by which negative cash can be moved from T to an earlier time $t < T$.

A remark is in order now: An alternative to net a future outgoing cash flow is by buying back a corresponding $-P^{\mathsf{A}}(T;t)$ bond. However, let us assume that buying back bonds is currently not an option, perhaps because there are no such bonds. Note that due to Axiom 2 it is not admissible to short sell our own bond.[5]

3.2.4 Hedging Positive Future Cash Flow

If entity A (i.e. *ourselves*) is confronted with a cash flow $+N$ in time T it needs to hedge (net) this cash flow by issuing a bond with proceeds $NP^{\mathsf{A}}(T;t)$ today (in bonds $P^{\circ}(T;t)$).

This is the way by which positive cash can be moved from T to an earlier time $t < T$.

3.3 Construction of Forward Bonds

The basic instrument to manage cash flows will be the forward bond transaction, which we consider next. To comply with Axiom 2 we simply assume that we can only enter into one of the following transactions, but we cannot enter the reversed transactions. Hence we have to consider two different forward bonds, as we do in Section 3.3.1 and 3.3.2.

Before we begin, let us review the usual forward bond construction, which violates Axiom 2: Assume we can trade in bonds of entity E and we can also short those bonds. Then buying one unit of $P^{\mathsf{E}}(T_2)$ in t and shorting (selling) $\alpha = \frac{P^{\mathsf{E}}(T_2;t)}{P^{\mathsf{E}}(T_1;t)}$ units of $P^{\mathsf{E}}(T_1)$

[5] We will later relax this assumption and allow for (partial) funding benefits by buying back bonds.

in t results in a zero net cashflow in time t, an outgoing cashflow of $\alpha = \frac{P^{\mathsf{E}}(T_2;t)}{P^{\mathsf{E}}(T_1;t)}$ in time T_1 and an incoming cashflow of 1 in time T_2. The quantity α is then called the *forward bond* and we denote it by

$$P^{\mathsf{E}}(T_1, T_2) \ := \ \frac{P^{\mathsf{E}}(T_2)}{P^{\mathsf{E}}(T_1)}.$$

The flaw in this construction is that, if we consider trading in our own bond, then we (entity A) cannot short the bond.

However, we will show in the next section that a forward bond can still be constructed and the expressions obtained are still related to the forward bond $P^{\mathsf{E}}(T_1, T_2)$. The argument involves collateralization.

3.3.1 Forward Bond 1: Hedging Future Incoming Cash with Outgoing Cash

Assume we have an incoming cash flow M from some counterparty risk free entity to entity A in time T_2 and an outgoing cash flow cash flow $-N$ from entity A (i.e. *ourselves*) to some other entity in time T_1. Then we perform the following transactions:

- We neutralize (secure) the outgoing cash flow by an incoming cash flow N by investing $-NP^{\circ}(T_1;t)$ in time 0.
- We net the incoming cash flow by issuing a bond in $t = 0$ paying back $-M$, where we securitize the issued bond by the investment in $NP^{\circ}(T_1;t)$. Note that the issued bond is securitized only over the period $[0, T_1]$.

This transaction has zero costs if $N = M\frac{P^{\mathsf{A}}(T_2;t)}{P^{\mathsf{A}}(T_1;t)}$. Let

$$P^{\mathsf{A}}(T_1, T_2) \ := \ \frac{P^{\mathsf{A}}(T_2)}{P^{\mathsf{A}}(T_1)}.$$

There is a gap in the argument above: It is not immediately clear that the collateralization results in a cashflow $MP^{\mathsf{A}}(T_2;t)\frac{P^{\circ}(T_1;t)}{P^{\mathsf{A}}(T_1;t)}$. Put differently, it is not clear that the proceeds of the collateralized bond are sufficient to buy enough collateral. The answer will be given by the following proof.

Proof: We issue a zero coupon bond $P^{\mathsf{A}}(T_2)$ having maturity T_2 and collateralize it by α units of $P^{\circ}(T_1)$. According to (4) the collateralized bond has a value of

$$P^{\mathsf{A}_{\alpha P^{\circ}(T_1)}}(T_2) \ := \ P^{\mathsf{A}}(T_2) + \alpha\left(P^{\circ}(T_1) - P^{\mathsf{A}}(T_1)\right).$$

The proceeds should match the costs of the collateral, i.e.

$$P^{\mathsf{A}_{\alpha P^{\circ}(T_1)}}(T_2;t) \ \stackrel{!}{=} \ \alpha P^{\circ}(T_1;t),$$

which then implies

$$P^{\mathsf{A}}(T_2) - \alpha P^{\mathsf{A}}(T_1) \ = \ 0,$$

i.e.

$$\alpha \ = \ \frac{P^{\mathsf{A}}(T_2;t)}{P^{\mathsf{A}}(T_1;t)}.$$

3.3.2 Forward Bond 2: Hedging Future Outgoing Cash with Incoming Cash

Assume we have an outgoing cash flow $-M$ in T_2 and an incoming cash flow N from some counterparty risk free entity to entity A (i.e. *ourselves*) in T_1. Then we perform the following transactions:

- We neutralize (secure) the outgoing cash flow by an incoming cash flow N by investing $-NP^{\circ}(T_2;t)$ in time $t=0$.
- We net the incoming cash flow by issuing a bond in $t=0$ paying back $-M$ in T_1, where we securitize the issued bond $P^{\mathsf{A}}(T_1)$ by the investment in $NP^{\circ}(T_2;t)$.

This transaction has zero costs if $N = M\frac{P^{\mathsf{A}}(T_1;t)}{\mathrm{E}\left[P^{\circ}(T_2)P^{\mathsf{A}}(T_1),T_1;t\right]}$. In the special case where $P^{\circ}(T_2;T_1)$ and $P^{\mathsf{A}}(T_1;T_1)$ are independent, we find $N = MP^{\circ}(T_1,T_2)$ with

$$P^{\circ}(T_1,T_2) \ := \ \frac{P^{\circ}(T_2)}{P^{\circ}(T_1)}.$$

Proof: If we collateralize a bond $P^{\mathsf{A}}(T_1)$ by α units of $P^{\circ}(T_2)$ then, according to (6), the collateralized bond has a market value (i.e. proceeds of) $P^{\mathsf{A}_{\alpha P^{\circ}(T_2)}}(T_1)$ with

$$P^{\mathsf{A}_{\alpha P^{\circ}(T_2)}}(T_1;t) \ = \ P^{\mathsf{A}}(T_1;t) + \alpha\left(P^{\circ}(T_2;t) - \mathrm{E}\left[P^{\circ}(T_2)P^{\mathsf{A}}(T_1),T_1;t\right]\right).$$

The proceeds should match the costs of the collateral, i.e.

$$P^{\mathsf{A}_{\alpha P^{\circ}(T_2)}}(T_1;t) \ \overset{!}{=} \ \alpha P^{\circ}(T_2;t),$$

which then implies

$$P^{\mathsf{A}}(T_1;t) - \alpha\mathrm{E}\left[P^{\circ}(T_2)P^{\mathsf{A}}(T_1),T_1;t\right] \ = \ 0,$$

i.e.

$$\alpha \ = \ \frac{P^{\mathsf{A}}(T_1;t)}{\mathrm{E}\left[P^{\circ}(T_2)P^{\mathsf{A}}(T_1),T_1;t\right]}.$$

Note that for the special case where $P^{\circ}(T_2;T_1)$ and $P^{\mathsf{A}}(T_1;T_1)$ are independent we find

$$\alpha \ = \ P^{\circ}(T_1,T_2) \ := \ \frac{P^{\circ}(T_2)}{P^{\circ}(T_1)},$$

but in any case, for $t=T_1$ we recover

$$\alpha \ = \ P^{\circ}(T_1,T_2;T_1) \ = \ P^{\circ}(T_2;T_1).$$

3.3.3 Forward Bond 1': Hedging Future Credit Linked Incoming Cash with Credit Linked Outgoing Cash

In the presence of counterparty risk, the construction of the forward bond (discounting of an incoming cash flow) is a bit more complex. Assume we have an incoming cash flow M from entity B to entity A in time T_2. We assume that we can buy protection on

M received from B at $-\text{CDS}^{\mathsf{B}}(T_2;t)$, where this protection fee is paid in T_2. Assume further that we can sell protection on N received from B at $\text{CDS}^{\mathsf{B}}(T_1;t)$, where this protection fee is paid in T_1. Then we repeat the construction of the forward bond with the net protected amounts $M(1-\text{CDS}^{\mathsf{B}}(T_2;t))$ and $N(1-\text{CDS}^{\mathsf{B}}(T_1;t))$, replacing M and N in 3.3.1 respectively. In other words we perform the following transactions:

- We buy protection on B resulting in a cash flow $-M\text{CDS}^{\mathsf{B}}(T_2;t)$ in T_2.
- We issue a bond to net the T_2 net cash flow $M(1-\text{CDS}^{\mathsf{B}}(T_2;t))$.
- We sell protection on B resulting in a cash flow $N\text{CDS}^{\mathsf{B}}(T_1;t)$ in T_1.
- We invest $N(1-\text{CDS}^{\mathsf{B}}(T_1;t))P^{\circ}(T_1;0)$ in t to generate a cash flow of $N(1-\text{CDS}^{\mathsf{B}}(T_1;t))$ in T_1.
- We collateralize the issued bond using the risk free bond over $[0,T_1]$ resulting in proceeds $M(1-\text{CDS}^{\mathsf{B}}(T_2;t))P^{\mathsf{A}}(T_2;0)\frac{P^{\circ}(T_1;0)}{P^{\mathsf{A}}(T_1;0)}$ in t.

This transaction has zero costs if $N = M\frac{P^{\mathsf{A}}(T_2;0)}{P^{\mathsf{A}}(T_1;0)}\frac{1-\text{CDS}^{\mathsf{B}}(T_2;t)}{1-\text{CDS}^{\mathsf{B}}(T_1;t)}$. Let

$$P^{\mathsf{A}|\mathsf{B}}(T_1,T_2) \;:=\; \frac{P^{\mathsf{A}}(T_2)}{P^{\mathsf{A}}(T_1)}\,\frac{1-\text{CDS}^{\mathsf{B}}(T_2;t)}{1-\text{CDS}^{\mathsf{B}}(T_1;t)}.$$

However, while this construction will give us a cash flow in T_1, which is (because of selling protection) under counterparty risk, the netting of such a cash flow will require more care. Note that for $t=T_1$ selling protection does not apply, being only necessary if we consider a forward bond as "static hedge" in $t<T_1$.

In Section 3.5 we apply this construction with $t=T_1$ such that selling protection does not apply.

3.3.4 Price of Counterparty Risk Protection

We denote the price of one unit of counterparty risk protection until T_2 as contracted in t by $\text{CDS}^{\mathsf{B}}(T_2;t)$. If we do not consider liquidity effects a fair (mark-to-market) valuation will give

$$P^{\mathsf{B}}(T_2;t) + \text{CDS}^{\mathsf{B}}(T_2;t)P^{\circ}(T_2;t) \;=\; P^{\circ}(T_2;t),$$

where we assume that the protection fee flows in T_2 (and independently of the default event). This gives

$$1-\text{CDS}^{\mathsf{B}}(T_2;t) \;=\; \frac{P^{\mathsf{B}}(T_2;t)}{P^{\circ}(T_2;t)}.$$

The latter can be interpreted as as a market implied survival probability.

3.3.5 Example: Expressing the Forward Bond in Terms of Rates

If we define

$$\begin{aligned}
\frac{P^{\circ}(T_2;t)}{P^{\circ}(T_1;t)} &=: \exp\left(-\int_{T_1}^{T_2} r(\tau;t)\mathrm{d}\tau\right) \\
\frac{P^{\mathsf{A}}(T_2;t)}{P^{\mathsf{A}}(T_1;t)} &=: \exp\left(-\int_{T_1}^{T_2} r(\tau;t) + s^{\mathsf{A}}(\tau;t)\mathrm{d}\tau\right) \\
\frac{1-\mathrm{CDS}^{\mathsf{B}}(T_2;t)}{1-\mathrm{CDS}^{\mathsf{B}}(T_1;t)} &=: \exp\left(-\int_{T_1}^{T_2} \lambda^{\mathsf{B}}(\tau;t)\mathrm{d}\tau\right)
\end{aligned}$$

we have

$$P^{\mathsf{A}|\mathsf{B}}(T_1,T_2) \;=\; \exp\left(-\int_{T_1}^{T_2} r(\tau;t) + s^{\mathsf{A}}(\tau;t) + \lambda^{\mathsf{B}}(\tau;t)\mathrm{d}\tau\right) \qquad (7)$$

and we see that discounting an outgoing cash flow backward in time by buying the corresponding forward bonds generates additional costs of $\exp\left(-\int_{T_1}^{T_2} s^{\mathsf{A}}(\tau;t)\mathrm{d}\tau\right)$ (funding) compared to discounting an incoming cash flow. In addition, incoming cash flows carry the counterparty risk by the additional discounting (cost of protection) at $\exp\left(-\int_{T_1}^{T_2} \lambda^{\mathsf{B}}(\tau;t)\mathrm{d}\tau\right)$.

The expression (7) is similar to a corresponding term in [10], except that there

$$\exp\left(-\int_{T_1}^{T_2} r(\tau;t) + \gamma^{\mathsf{A}}(\tau;t) + \lambda^{\mathsf{B}}(\tau;t)\mathrm{d}\tau\right)$$

had been derived, where γ^{A} is the bond-cds basis (i.e., $\gamma^{\mathsf{A}} = s^{\mathsf{A}} - \lambda^{\mathsf{A}}$). The difference stems from the fact that [10] always factors in own-credit, which we do not do according to Axiom 1. Depending on the net cash position, we may recover the situation of [10] in our setup, since the effective funding required applies only to the net amount (after all netting has been performed).[6]

Note that these single time step static hedges do not consider any correlation, such as the correlation between A's funding and B's counterparty risk. This correlation will come in once we assume a dynamic model for the corresponding rates and consider dynamic hedges (e.g. in a time-discrete model). We will do so next, in Section 3.4.

3.3.6 Interpretation: Funding Cost as Hedging Costs in Cash Flow Management

From the above, we see that moving cash flows around generates costs and the *cash flow replication value* of future cash flows will be lower than the *portfolio liquidation value* of future cash flows. The difference between the two corresponds to the "operating costs of managing un-netted cash flows", which are just the funding costs. Clearly, with

[6] Netting will play an important role in funding and counterparty risk will enter. Note that for the net exposure we have to net all cash flows related to a single counterpart; for the net funding we have to net all cash flows over all counterparts.

liquidation there are no operating cost - we save them. However these costs are real. If we have cash lying around we can invest it only at a risk free rate; however we fund ourselves at a higher rate. The only way to reduce cashs cost is to net them with other cash flows (e.g. with assets generating higher returns).

3.3.7 Role of Forward Bonds - From Static to Dynamic Hedging

The forward bonds derived in this section differ slightly from the case where funding costs are neglected. However, as noted before, for $t = T_1$ the situation is simplified. In this case our building blocks are

$$\begin{aligned} P^{\circ}(T_1, T_2; T_1) &= P^{\mathsf{A}}(T_2; T_1) \\ P^{\mathsf{A}}(T_1, T_2; T_1) &= P^{\mathsf{A}}(T_2; T_1) \\ P^{\mathsf{A}|\mathsf{B}}(T_1, T_2) &= P^{\mathsf{A}}(T_2; T_1)(1 - \mathrm{CDS}^{\mathsf{B}}(T_2; T_1)). \end{aligned}$$

These constitute our "discount factors" for outgoing (net) cash, incoming risk free (net cash) and incoming (net) cash with counterparty risk. From them a dynamic hedge can be constructed using an obvious backward induction. We consider this in the next section.

3.4 Valuation with Hedging Costs in Cash flow Management (Funding)

The cash flow replication value is now given by the optimal combination of forward bonds to hedge (replicate) the future cash flow, where nearby cash flows are "netted" as best we can. This is given by a backward algorithm, discounting each cash flow according to the appropriate forward bond and then netting the result with the next cash flow.

Let us first consider the valuation neglecting counterparty risk, or, put differently, all future cash flows exposed to counterparty risk have been insured by buying the corresponding protection first (and reducing net the cash flow accordingly). We will detail the inclusion of counterparty risk in Section 3.5.

3.4.1 Interest for Borrowing and Lending

Obviously, using our two assumptions (Axiom 1 and Axiom 2) we have arrived at a very natural situation. The interest to be paid for borrowing money over a period $[T_1, T_2]$ is (as seen in $t = T_1$)

$$\frac{1}{T_2 - T_1} \left(\frac{1}{P^{\mathsf{A}}(T_2; T_1)} - 1 \right),$$

i.e. our funding rate. The interest earned by depositing money (risk free) over a period $[T_1, T_2]$ is

$$\frac{1}{T_2 - T_1} \left(\frac{1)}{P^{\circ}(T_2; T_1)} - 1 \right),$$

i.e. the risk free rate. Hence we are in a setup where interest for borrowing and lending are different. Valuation theory in setups when rates for lending and borrowing are different is well understood, see for example [4].

3.4.2 From Static to Dynamic Hedging

The forward bonds, e.g. $P^{\circ}(T_1, T_2; t)$ or $P^{\mathsf{A}}(T_1, T_2; t)$ define a static hedge in the sense that their value is known in t. If we are discounting / hedging stochastic cash flows $C(T)$ a dynamic hedge is required and $C(T_i)P^{\mathsf{A}}(T_{i-1}, T_i; T_{i-1})$ is replaced by

$$N(T_{i-1}) \, \mathrm{E}^{\mathbb{Q}^N} \left(\frac{C(T_i) P^{\mathsf{A}}(T_i; T_i)}{N(T_i)} \right).$$

Note: It is

$$P^{\mathsf{A}}(T_{i-1}, T_i; T_{i-1}) \ = \ P^{\mathsf{A}}(T_i; T_{i-1}) \ = \ N(T_{i-1}) \, \mathrm{E}^{\mathbb{Q}^N} \left(\frac{P^{\mathsf{A}}(T_i; T_i)}{N(T_i)} \right),$$

i.e. for $t = S$ the forward bond $P(S, T; t)$ is just a bond.

Example: Implementation using Euler simulation of Spreads and Intensities

If we express the bonds in terms of the risk free bond P°, e.g.

$$\frac{P^{\mathsf{A}}(T_i; t)}{P^{\circ}(T_i; t)} \ =: \ \exp\left(- \int_{T_{i-1}}^{T_i} s^{\mathsf{A}}(\tau; t) \mathrm{d}\tau \right)$$

and model the process on the right hand side such that $s^{\mathsf{A}}(\tau; t)$ is $\mathcal{F}_{T_{i-1}}$-measurable for $t \in [T_{i-1}, T_i]$ (which is usually case if we employ a numerical scheme like the Euler scheme for the simulation of spreads s and default intensities λ), then we find that for any cash flow $C(T_i)$

$$\begin{aligned} & N(T_{i-1}) \, \mathrm{E}^{\mathbb{Q}^N} \left(\frac{C(T_i) P^{\mathsf{A}}(T_i; T_i)}{N(T_i)} \right) \\ & = \ N(T_{i-1}) \, \mathrm{E}^{\mathbb{Q}^N} \left(\frac{C(T_i)}{N(T_i)} \right) \ \exp\left(- \int_{T_{i-1}}^{T_i} s^{\mathsf{A}}(\tau; t) \mathrm{d}\tau \right) \ P^{\mathsf{A}}(T_{i-1}; T_{i-1}) \end{aligned}$$

and likewise for P^{B}.

3.4.3 Valuation of a Single Product including Cash Flow Costs

Let us consider the valuation of a collateralized swap which is the only product held by entity A. Let us assume the swap is collateralized by cash. The package consisting of the swap's (collateralized) cash flows and the collateral flows has a mark-to-market value of zero (valued according to Section 2), by definition of the collateral. However, the package represents a continuous flow of cash (margining). If valued using the collateral curve $P^{\mathsf{C}} = P^{\circ}$ these marginal collateral cash flows have mark-to-market value zero (by definition of the collateral).

Let us assume that this swap constitutes the only product of entity A and that we value cash flows by a hedging approach, i.e. by dynamically using the two forward bonds from Section 3.3. Taking into account that we have different interest rates for borrowing and lending cash, the collateral cash flow will generate additional costs. These are given by the following recursive definition:

$$\begin{aligned}\frac{V_i^{\mathrm{d}}(T_{i-1})}{N(T_{i-1})} \ = \ \mathrm{E}^{\mathbb{Q}^N}\Bigg(& \frac{\max(X_i + V_{i+1}^{\mathrm{d}}(T_i), 0)}{N(T_i)} \, P^{\mathsf{A}}(T_i;T_i) \\ & + \frac{\min(X_i + V_{i+1}^{\mathrm{d}}(T_i), 0)}{N(T_i)} \, P^{\circ}(T_i;T_i) \ \Big| \ \mathcal{F}_{T_{i-1}} \Bigg) .\end{aligned} \tag{8}$$

Here $V_{i+1}^{\mathrm{d}}(T_i)$ is the net cash position required in T_i to finance (e.g. fund) the future cash flows in $t > T_i$. X_i is the collateral margin call occurring in time $t = T_i$. Hence we have to borrow or lend the net amount

$$V_i^{\mathrm{d}}(T_i) \ = \ X_i + V_{i+1}^{\mathrm{d}}(T_i)$$

over the period $(T_{i-1}, T_i]$. This amount is then transferred to T_{i-1} using the appropriate forward bond (discounting) for netting with the next margining cash flow X_{i-1}.

This is the valuation under the assumption that the X_i is the *net cash flow* of entity A, e.g. as if the collateralized swap were our only product.

3.4.4 Valuation within a Portfolio Context - Valuing Funding Benefits

The situation of Section 3.4.3 now carries over to the valuation of a portfolio of products held by entity A. In this the algorithm (8) will determine the discount factor to be used in the period $[T_{i-1}, T_i]$ from the portfolio's net cash position $V_i^{\mathrm{d}}(T_i)$, regardless of a product having an outgoing or incoming cash flow.

Let us discuss a product having a cash flow $C(T_i)$ in T_i being part of entity A's portfolio resulting in a net (cash) position $V_i^{\mathrm{d}}(T_i)$.

Incoming Cash Flow, Positive Net Position

Given that our net position $V_i^{\mathrm{d}}(T_i)$ is positive in T_i, an incoming (positive) cash flow $C(T_i)$ can be factored in at an earlier time only by issuing a bond. Hence it is discounted with P^{A} (resulting in a smaller value at $t < T_i$, compared to a discounting with P°). Note that for $t > T_i$ this cash flow can provide a funding benefit for a future negative cash flow which would be considered in the case of a negative net position, see 3.4.4).

Outgoing Cash Flow, Positive Net Position

Given that our net position $V_i^{\mathrm{d}}(T_i)$ is positive in T_i, an outgoing (negative) cash flow $C(T_i)$ can be served from the positive (net) cash position. Hence it does not require funding for $t < T_i$ as long as our net cash position is positive. Factoring in the *funding benefit*, $C(T_i)$ is discounted with P^{A} (resulting in a larger value at $t < T_i$, compared to discounting with P°).

Incoming Cash Flow, Negative Net Position

Given that our net position $V_i^{\mathrm{d}}(T_i)$ is negative in T_i, an incoming (positive) cash flow $C(T_i)$ reduces the funding cost for the net position. Hence it represents a funding benefit and is discounted with P° (resulting in a larger value at $t < T_i$, compared to discounting with P^{A}).

Outgoing Cash Flow, Negative Net Position

Given that our net position $V_i^{\mathrm{d}}(T_i)$ is negative in T_i, an outgoing (negative) cash flow $C(T_i)$ has to be funded on its own (as long as our net position remains negative). Hence it is discounted with P° (resulting in a smaller value at $t < T_i$, compared to discounting with P^{A}).

3.5 Valuation with Counterparty Risk and Funding Cost

So far Section 3 has not considered counterparty risk in incoming cash flow. Of course, it can be included using the forward bond which includes the cost of protection of a corresponding cash flow.

3.5.1 Static Hedge of Counterparty Risk in the Absence of Netting

Assume that all incoming cash flows from entity B are subject to counterparty risk (default), but all outgoing cash flow to entity B have to be served. This would be the case if we considered a portfolio of (zero) bonds only and there were no netting agreement.

If there is no netting agreement we need to buy protection on each individual cash flow obtained from B. It is not admissible to use the forward bond $P^{\mathsf{A}|\mathsf{B}}(T_1, T_2)$ to net a T_2 incoming cash flow for which we have protection over $[T_1, T_2]$ with a T_1 outgoing (to B) cash flow and then buy protection only on the net amount.

If we assume, for simplicity, that all cash flows $X_{i,k}^{\mathsf{B}_j}$ received in T_i from some entity B_j are known in T_0, i.e. $\mathcal{F}_{T_0}$-measurable, then we can account for the required protection fee in T_0 (i.e. we have a static hedge against counterparty risk) and our valuation algorithm becomes

$$\begin{aligned} \frac{V_i^{\mathrm{d}}(T_{i-1})}{N(T_{i-1})} \;=\; \mathrm{E}^{\mathbb{Q}^N} \Bigg(& \frac{\max(X_i + V_{i+1}^{\mathrm{d}}(T_i), 0)}{N(T_i)} \, P^{\mathsf{A}}(T_i; T_i) \\ & + \frac{\min(X_i + V_{i+1}^{\mathrm{d}}(T_i), 0)}{N(T_i)} \, P^{\circ}(T_i; T_i) \;\Big|\; \mathcal{F}_{T_{i-1}} \Bigg) . \end{aligned} \tag{9}$$

where the time T_i net cash flow X_i is given as

$$X_i \;:=\; X_i^{\circ} + \sum_j \sum_k \Big(\min(X_{i,k}^{\mathsf{B}_j}, 0) + (1 - \mathrm{CDS}^{\mathsf{B}_j}(T_i; T_0)) \max(X_{i,k}^{\mathsf{B}_j}, 0) \Big) .$$

where B_j denotes different counterparties and $X_{i,k}^{\mathsf{B}_j}$ is the k-th cash flow (outgoing or incoming) between us and B_j at time T_i. So obviously we are attributing full protection costs for all incoming cash flows and gurantee to serve all outgoing cash flows. Note again, that in any cash flow $X^{\mathsf{B}_j}_{i,k}$ we take into account the full protection costs from T_0 to T_i.

Although this is one special case we can already see that counterparty risk cannot be separated from funding since (9) makes clear that we have to attribute funding costs for the protection fees.[7]

3.5.2 Dynamic Hedge of Counterparty Risk in the Presence of Netting

However, many contracts feature agreements which result in a "temporal netting" of cash flows exchanged between two counterparts and only the net cash flow carries the counterparty risk.It appears as if we could then use the forward bond $P^{\mathsf{A}|\mathsf{B}}(T_{i-1}, T_i)$ and $P^{\mathsf{B}}(T_{i-1}, T_i)$ on our future T_i net cash flow and then net this one with all T_{i-1} cash flows, i.e. apply additional discounting to each netted set of cash flows between two counterparts. However, this value will usually not coincide with the value finally exposed to counterparty risk. In (8) we are netting T_i cash flows with T_{i-1} cash flows in a specific way which accounts for our own funding costs. The netting agreement between two counterparties may (and will) be different from our funding adjusted netting. For example, the contract may specify that upon default the close out of a product (i.e. the outstanding claim) be determined using the risk free curve for discounting.

Let us denote by $V^{\mathsf{B}}_{\mathrm{CLSOUT},i}(T_i)$ the time T_i cash being exchanged / being at risk at T_i if counterparty B defaults according to all the netting agreements (the deals close out). This value is usually a part of the contract / netting agreement. Usually it will be a mark-to-market valuation of V^{B} at T_i. One approach to account for the mismatch is to buy protection over $[T_{i-1}, T_i]$ for the positive part of $V^{\mathsf{B}}_{\mathrm{CLSOUT},i}(T_i)$ (i.e. the exposure), then additionally buy protection for the mismatch of the contracted default value $V^{\mathsf{B}}_{\mathrm{CLSOUT},i}(T_i)$ and netted non-default value $V^{\mathsf{B}}_i(T_i)$.

As before let $X^{\mathsf{B}_j}_{i,k}$ denote the k-th cash flow (outgoing or incoming) exchanged between us and entity B_j at time T_i. Let

$$R^{\mathsf{B}_j}_i(T_i) \ := \ \sum_k X^{\mathsf{B}_j}_{i,k} + V^{\mathsf{B}_j}_{i+1}(T_i)$$

denote the value contracted with B_j, valued in T_i, including our future funding costs. Then

$$\begin{aligned} V^{\mathsf{B}_j}_i(T_i) \ := \ R^{\mathsf{B}_j}_i(T_i) \ - \ & \underbrace{p_j \max(R^{\mathsf{B}_j}_i(T_i), 0)}_{\substack{\text{protection on netted value} \\ \text{including our funding}}} \\ & + p_j \underbrace{\left(\min(V^{\mathsf{B}_j}_{\mathrm{CLSOUT},i}(T_i), 0) - \min(R^{\mathsf{B}_j}_i(T_i), 0) \right)}_{\substack{\text{mismatch of liability} \\ \text{in case of default}}} \end{aligned}$$

[7] We assumed that the protection fee is paid at the end of the protection period. It is straight-forward to include periodic protection fees.

is the net value including protection fees over the period $[T_{i-1}, T_i]$, where p_j is the price of buying or selling one unit of protection against B_j over the period $[T_{i-1}, T_i]$, i.e.,

$$p_j \; := \; \mathrm{CDS}^{\mathsf{B}_j}(T_i; T_{i-1}).$$

Let

$$V_i(T_i) \; := \; \sum_j V_i^{\mathsf{B}_j}(T_i).$$

The general valuation algorithm including funding and counterparty risk is then given as

$$\begin{aligned}\frac{V_i(T_{i-1})}{N(T_{i-1})} \; = \; \mathrm{E}^{\mathbb{Q}^N}\Bigg(& \frac{\max(V_i(T_i), 0)}{N(T_i)} \; P^{\mathsf{A}}(T_i; T_i) \\ & + \frac{\min(V_i(T_i), 0)}{N(T_i)} \; P^{\circ}(T_i; T_i) \;\Big|\; \mathcal{F}_{T_{i-1}} \Bigg).\end{aligned} \tag{10}$$

In our valuation the time T_{i-1} value being exposed to entities B_j counterparty risk is given by

$$\begin{aligned}\frac{V_i^{\mathsf{B}_j}(T_{i-1})}{N(T_{i-1})} \; := \mathrm{E}^{\mathbb{Q}^N} \Bigg(& \frac{\max(V_i(T_i), 0) - \max(V_i(T_i) - V_i^{\mathsf{B}_j}(T_i), 0)}{N(T_i)} \; P^{\mathsf{A}}(T_i; T_i) \\ & + \frac{\min(V_i(T_i), 0) - \min(V_i(T_i) - V_i^{\mathsf{B}_j}(T_i), 0)}{N(T_i)} \; P^{\circ}(T_i; T_i) \;\Big|\; \mathcal{F}_{T_{i-1}} \Bigg).\end{aligned}$$

In other words, $V_i^{\mathsf{B}_j}(T_{i-1})$ is the true portfolio impact (including the side effects of funding) if the flows $X_{l,k}^{\mathsf{B}_j}$, $l \geq i$, are removed from the portfolio.

In the case where the mismatch of contracted default value $V^{\mathsf{B}}_{\mathrm{CLSOUT},i}(T_i)$ and netted non-default value $R_i^{\mathsf{B}_j}(T_i)$ is zero, we arrive at

$$\begin{aligned} V_i^{\mathsf{B}_j}(T_i) \; &:= \; \min\left(R_i^{\mathsf{B}_j}(T_i), 0\right) + (1 - p_j) \max\left(R_i^{\mathsf{B}_j}(T_i), 0\right) \\ &= \; R_i^{\mathsf{B}_j}(T_i) + p_j \max\left(R_i^{\mathsf{B}_j}(T_i), 0\right) \\ &= \; R_i^{\mathsf{B}_j}(T_i) - p_j \max\left(R_i^{\mathsf{B}_j}(T_i), 0\right) \\ &= \; \sum_k X_{i,k}^{\mathsf{B}_j} + V_{i+1}^{\mathsf{B}_j}(T_i) - p_j \max\left(\sum_k X_{i,k}^{\mathsf{B}_j} + V_{i+1}^{\mathsf{B}_j}(T_i), 0 \right). \end{aligned}$$

- in this case we get an additional discount factor on the positive part (exposure) of the netted value, accounting for the protection costs.

3.5.3 Interpretation

From algorithm (10) it is obvious that the valuation of funding (given by the discounting using either P° or P^{A}) and the valuation of counterparty risk (given by buying/selling protection at p) cannot be separated. Note that

- We not only account for the default risk with respect to our exposure[8], but also to the loss of a potential funding benefit, i.e. the impact of default on funding.
- Buying protection against default has to be funded itself and we account for that.

The algorithm (10) values what is called the *wrong-way-risk*, i.e., the correlation between counterparty default and couterparty exposure via the term $p_j \max(R_i^{\mathsf{B}^j}(T_i), 0)$.

3.5.4 The Collateralized Swap with Funding Costs

Let us consider the collateralized swap again. While the mark-to-market value of a collateralized (secured) cash flow can be calculated from the curve implied by the collateral, the presence of funding cost changes the picture significantly:

- The discount curve determining the collateral may be different from the discount curve accounting for the funding cost. Hence, the collateral will not match the expected value of the future cash flows including funding costs.
- Collateral may require funding or may represent a funding benefit. In case of default these cash flows are lost.

The surprising result is that the presence of funding costs may introduce counterparty risk into a collateralized (secured) deal. As a consequence, the collateralized deal has to be valued within (10) to account for these effects.

4 The Relation of the Different Valuations

Let us summarize the relationship between the two different kinds of discounting in the presence of counterparty risk. The mark-to-market value of a time T_i cash flow is

cash flow	discount factor over $[T_{i-1}, T_i]$
outgoing ($C(T_i) < 0$)	$P^{\mathsf{A}}(T_{i-1}, T_i)$
incoming ($C(T_i) > 0$)	$P^{\mathsf{B}}(T_{i-1}, T_i)$

In this situation a liability of A is written down, because in case of liquidation of A its counterparts agree to receive less than the risk free discounted cash flow today.

The hedging (replication) value of a time T_i cash flow depends, however, on our net cash position, because the net cash position decides if funding costs apply or a funding benefit can be accounted for. The net cash position has to be determined with the backward algorithm (8) where each cash flow X_i is adjusted according to its (netted) counterparty risk. Using $P^{\mathsf{B}}(T_i; T_i) P^{\circ}(T_{i-1}, T_i) = P^{\mathsf{B}}(T_{i-1}, T_i)$ we can write formally

	discount factor over $[T_{i-1}, T_i]$	
cash flow	positive net cash in T	negative net cash in T_i
outgoing ($C(T_i) < 0$)	$P^{\mathsf{A}}(T_{i-1}, T_i)$	$P^{\circ}(T_{i-1}, T_i)$
incoming ($C(T_i) > 0$)	$P^{\mathsf{B}}(T_i; T_i) P^{\mathsf{A}}(T_{i-1}, T_i)$	$P^{\mathsf{B}}(T_{i-1}, T_i)$

[8] I.e., $V^{\mathsf{B}}_{\mathrm{CLSOUT},i}(T_i)$.

The two valuation concepts coincide when $P^{\mathsf{A}}(T_{i-1}, T_i) = P^{\circ}(T_{i-1}, T_i)$, i.e. we do not have funding costs. They also coincide if an outgoing cash flow appears only in the situation of positive net cash in T_i (funding benefit) and an incoming cash flow appears only in the situation of negative net cash T_i (funding benefit).

Put differently: The liquidation valuation neglects funding by assuming funding benefits in all possible situations.[9]

4.1 One Product - Two Values

Given the valuation framework discussed above a product has at least two values:

- the product can be evaluated "mark-to-market" as a single product. This value can be seen as a "fair" market price. Here the product is valued according to Section 2. This is the product's *idiosyncratic value*.
- the product can be evaluated within its context in a portfolio of products owned by an institution, i.e. including possible netting agreements *and* operating costs (funding). This will constitute the value of the product for the institution. Here the value of the product is given by the change in the portfolio value when the product is added to the portfolio, where the portfolio is valued with algorithm (8). This is the product's *marginal portfolio value*.

However, both valuations share the property that the sum of the values of the products belonging to the portfolio does not necessarily match the portfolio's value. This is clear for the first value, because netting is neglected altogether. For the second value, removing the product from the portfolio can change the sign of netted cash flow, hence change the choice of the chosen discount factors.

4.2 Valuation of a Bond

To test our frameworks, let us go back to the zero coupon bond with which we started and value it.

4.2.1 Valuation of a Bond at Mark-To-Market

Using the mark-to-market approach we will indeed recover the bonds market value $-P^{\mathsf{A}}(T; 0)$ (this is a liability). Likewise for $P^{\mathsf{B}}(T)$ we get $+P^{\mathsf{B}}(T; 0)$.

4.2.2 Valuation of a Bond at Funding

Factoring in funding costs it appears as if issuing a bond would generate an instantaneous loss, because the bond represents a negative future cash flow which has to be discounted by P° according to the above. This follows from assuming that the proceeds of the issued bond are invested risk fee. Of course, it would be unreasonable to issue a risky bond and invest its proceeds risk free.

[9] This is, assuming zero bond-cds basis spread, the situation in [10].

However, note that the discount factor depends only on the *net* cash position. If the net cash position in T is negative, it would be unreasonable indeed to increase liabilities at this maturity and issue another bond.

Considering the hedging cost approach, for $P^{\mathsf{A}}(T)$ we get the value $-P^{\circ}(T;0)$ if the time T net position is negative. This will indeed represent loss compared to the mark-to-market value. This indicates that we should instead buy back the bond (or not issue it at all). If however our cash position is such that this bond represents a funding benefit (in other words, it could be used for funding), it's value will be $P^{\mathsf{A}}(T;0)$.

For P^{B} we get

$$N(0)\mathrm{E}^{\mathbb{Q}}\left(\frac{P^{\mathsf{B}}(T;T)P^{\mathsf{A}}(T;T)}{N(T)} \mid \mathcal{F}_0\right)$$

Assuming that A's funding and B's counterparty risk are independent of $P^{\circ}(T)$ we arrive at

$$P^{\mathsf{B}}(T;0)\frac{P^{\mathsf{A}}(T;0)}{P^{\circ}(T;0)}$$

which means we should sell B's bond if its return is below our funding requirements (we should not hold risk free bonds if it is not necessary).

4.3 Convergence of the two Concepts

Note that if A runs a perfect business, securing every future cash flow by hedging it (using the risk free counterpart $P^{\circ}(T;T)$), and if there are no risks in its other operations, then the market is likely to value it as risk free and we will come close to $P^{\mathsf{A}}(T;T) = P^{\circ}(T;T)$. In that case, we find that both discounting methods agree and symmetry is restored.

However, there is an even closer link between the two valuations. Let us consider that entity A holds a large portfolio of products $V_1, \ldots, V_N$. Let $V_1(0), \ldots, V_N(0)$ denote the mark-to-market (liquidation) value of those products. Let $\Pi[V_1, \ldots, V_N](0)$ denote the hedging value of the portfolio of those products. If the portfolio's cash flows are hedged in the sense that all future net cash flows are zero, then, neglecting counterparty risk, we have (approximately)

$$V_k(0) \;\approx\; \Pi[V_1, \ldots, V_N](0) - \Pi[V_1, \ldots, V_{k-1}, V_{k+1}, \ldots, V_N](0).$$

Thus, the mark-to-market valuation which includes "own-credit" for liabilities corresponds to the marginal cost or profit generated when removing the product from a large portfolio which includes funding costs. However, portfolio valuation is non-linear in the products and hence the sum of the mark-to-market valuation does not agree with the portfolio valuation with funding costs.

We prove this result only for a product consisting of a single cash flow. A linearization argument shows that it then holds (approximately) for products consisting of many small cash flows.

If the portfolio is fully hedged then all future cash flows are netted. In other words, the entity A accounted for all non-netted cash flows by considering issued bonds or invested money. Then we have $V_j^{\mathrm{d}}(T_j) = 0$ for all j. Let V_k be a product consisting of a single cash flow $C(T_i)$ in T_i, exchanged with a risk free entity. If this cash flow is incoming, i.e. $C(T_i) > 0$ then removing it will leave the portfolio with an un-netted outgoing cash flow $-C(T_i)$, which is, according to our rules, discounted with $P^{\circ}(T_i)$. Likewise, if this cash flow is outgoing, i.e. $C(T_i) < 0$ then when it is removed the portfolio will be left with an un-netted incoming cash flow $C(T_i)$, which is according to our rules discounted with $P^{\mathsf{A}}(T_i)$.

Hence the marginal value of this product corresponds to the mark-to-market valuation.

5 Credit Valuation Adjustments

The valuation defined above includes counterparty risk as well as funding costs. Hence, if a whole portfolio is valued using the above valuation, there is no counterparty risk adjustment.

However, the valuation above is computationally very demanding. First, all products have to be valued together, so it is not straightforward to look at a single product's behavior without valuing the whole portfolio.

Second, even very simple products like a swap have to be evaluated in a backward algorithm using conditional expectations in order to determine the net exposure and their effective funding costs. This is computationally demanding, especially if Monte-Carlo simulations are used.

As illustrated above, the valuation is significantly simplified if all counterparts share the same zero coupon bond curve $P(T;t)$ and/or the curve for lending and borrowing agree. A credit valuation adjustment is simply a splitting of the following form

$$V(t) \ = \ V|_{P^{\cdot}=P^{*}}(t) + \underbrace{(V(t) - V|_{P^{\cdot}=P^{*}}(t))}_{\text{CVA}}$$

where $V|_{P^{\cdot}=P^{*}}(t)$ denotes the simplified valuation assuming a single discounting curve P^{*} (neglecting the origin or collateralization of cash flows).

While the use of a credit valuation adjustment may simplify the implementation of a valuation system it has some disadvantages:

- The valuation using the simplified single curve, in general, is incorrect.
- Hedge parameters (sensitivities) calculated from such valuation, in general, are wrong.

In order to cope with this problem we propose the following setup:

- Construction of proxy discounting curve P^{*} such that the CVA is approximately zero.
- Transfer of sensitivity adjustments calculated from the CVA's sensitivities.

6 Modeling and Implementation

So far we have expressed all valuation in terms of products of cash flows and zero coupon bond processes like $P^{\mathsf{A}}(T;t)$. The value of a stochastic time T cash flow $C(T)$

originating from entity A was given as of time T as

$$C(T)P^{\mathsf{A}}(T;T)$$

and the corresponding time $t < T$ value was expressed using risk neutral valuation

$$N(0)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{C(T)P^{\mathsf{A}}(T;T)}{N(T)} \,\Big|\, \mathcal{F}_0\right).$$

Our presentation has been model-independent so far. Depending on the cash flow the term $C(T)P^{\mathsf{A}}(T;T)$ may give rise to valuation changes stemming from the covariance of $C(T)$ and $\frac{P^{\mathsf{A}}(T;T)}{N(T)}$.

For an implementation of the valuation algorithm, consider a time discretization $0 =: t_0 < t_1 < \cdots < t_n$. We assume that the discretization scheme of the model primitives models the counterparty risk over the interval $[t_i, t_{i+1})$ as an $\mathcal{F}_{t_i}$-measurable survival probability such that we have for all t_{i+1} cash flows $C(t_{i+1})$ that

$$\begin{aligned} & N(t_i)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{C(t_{i+1})P^{\mathsf{A}}(t_{i+1};t_{i+1})}{N(t_i)} \,\Big|\, \mathcal{F}_{t_{i+1}}\right) \\ & \quad = N(t_i)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{C(t_{i+1})}{N(t_{i+1})} \,\Big|\, \mathcal{F}_{t_i}\right) \exp\left(-\int_{t_i}^{t_{i+1}} \lambda^{\mathsf{A}}(s;t_i)\mathrm{d}s\right). \end{aligned} \tag{11}$$

The expression $\exp\left(-\int_{t_i}^{t_{i+1}} \lambda^{\mathsf{A}}(s;t_i)\mathrm{d}s\right)$ represents an additional "discounting" stemming from the issuers credit risk / funding costs. It can be interpreted as an (implied) survival probability (see Chapter 28 in [6]).[10]

Hence any counterparty-risk-free valuation model can be augmented with funding costs, counterparty risk and collateralization effects by two modifications:

- The model is augmented by a simulation of the (conditional) one step (forward) survival probabilities (or funding spreads), e.g. $\exp\left(-\int_{t_i}^{t_{i+1}} \lambda^{\mathsf{A}}(s;t_i)\mathrm{d}s\right)$.
- The valuation is performed using the backward algorithm (8), where in each time step the survival probabilities and/or funding adjusted discount factors are applied according to (11).

More aspects of modeling are found in the associated paper "funded replication", cite [5].

7 Conclusion

We considered the valuation of cash flows and derivatives including counterparty risk and funding. We shed some light on the *own credit risk paradox* that the mark-to-market value of a portfolio increases if its owner's credibility decreases. After our discussion the situation now appears much clearer: This increase in value is reasonable when considering the portfolio's liquidation since lenders are willing to accept a lower return upon liquidation in exchange for the increased credit risk.

[10] A very simple model is to assume that $\exp\left(-\int_{t_i}^{t_{i+1}} \lambda^{\mathsf{A}}(s;t_i)\mathrm{d}s\right)$ is deterministic.

If we value the portfolio from a hedging perspective (where hedging to some extent means funding) we see that the portfolio value decreases if its owner's credibility decreases. This is also reasonable since funding costs (operating costs) have risen.

The two kinds of valuation can co-exist. They are not contradictory. One discounting should be used for mark-to-market valuation. The other should be used for risk management and the governance (managing) of positions.

To see the effect of a single deal on the management of cash flows a bank could use a curve for borrowing and lending and every deal would have to be valued according to algorithm (8).

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Notes

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