Funded Replication.

Valuing with Stochastic Funding.

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Abstract

Funding costs are the costs to a (risky) institution "A" of providing and managing its future cash flows in excess of, say, some risk free funding. For a single deterministic cash flow with maturity T these costs are essentially given by the ratio $\frac{P^{\mathsf{A}}(T)}{P^{\circ}(T)}$ of the risky bond $P^{\mathsf{A}}(T)$ and the risk free bond $P^{\circ}(T)$. They can be expressed by a "funding spread" $s(T)$, which represents an additional interest rate e.g. with continuous compounding: $\exp(-s(T)\,T) := \frac{P^{\mathsf{A}}(T)}{P^{\circ}(T)}$.

For stochastic cash flows funding can be interpreted as a dynamic hedging of future cash flows. The problem can also lead to a complex portfolio problem, see [3].

We will consider two possible valuations for a product consisting of future cash flows:

- The valuation which ignores any funding cost and calculates a liquidation value of the product. This valuation could be termed "mark-to-marked".
- The valuation including all funding costs from managing all future cash flows. This valuation could be termed "funded replication".

In this paper we will thoroughly introduce the valuation with funding, i.e. the "funded replication". We will also consider partial funding costs. For products with liquid market price we will derive a single consistent valuation: Having a liquid market price offers the option of canceling all future funding costs. This option can be valued.

The key points of this paper can be summarized as follows:

- The valuation of a derivative including funding corresponds to the valuation of a quanto product.
- Even a collateralized deal may bear some funding costs. These funding costs come from a correlation effect analog to the quanto adjustment.
- A liquid instrument having a market value represents the option to cancel all future funding. This option can be valued.

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1 Introduction

The classical method of valuing a derivative product is to consider a replication using market prices, including some market interest rate. By construction, this valuation ignores funding costs, which we define as the difference between a specific, issuer dependent, interest rate (the rate for borrowing money) and the market interest rate.

Alternatively the derivative product can be valued including the cost of funding. We call this "funded replication". If the rates for borrowing and lending depend on the net cash position (i.e. factoring in funding benefit or not), then this funded replication may become a complex portfolio problem, see [3]. However, assuming that we are always in need of funding, we may consistently apply the funding curve to all incoming and outgoing cash flows.

Nevertheless, calculating the funded replication value, i.e. the cost of market replication including funding cost, may require a bit more than exchanging a discounting curve, e.g. if we consider a consistent model with stochastic funding costs.

1.1 Relation to Cross-Currency Cash Flow Modeling

In this paper we derive a consistent valuation for funded replication costs by constructing a model from scratch and describing the valuation algorithm for funded cash flows. The essence of this paper can be boiled down to the following points:

- A model with stochastic funding is essentially a cross-currency model where the funding rate is a rate in a (virtual) currency.
- The valuation of a derivative including funding corresponds to the valuation of a quanto product.

From this analogy, the modeling can simply reuse derivations and formulas from cross-currency models, e.g. a cross-currency LIBOR market model, [2].

1.2 Relation to Risky Cash Flow Modeling

There is an alternative interpretation of a funded cash flow, namely that of a risky (defaultable) cash flow. Hence, we could start with a model for non-risky and risky cash flows, e.g. a defaultable LIBOR market model, [6]. Indeed, this interpretation will lead to exactly the same model. The cross-currency interpretation may be a bit more attractive since it avoids considering default in the context of funding.

1.3 Some Conclusions

The framework derived will bring clarity to some subtle points in the valuation of collateralized deals and liquid instruments:

- Even a collateralized deal may bear some funding costs. These funding costs come from a correlation effect analogous to the quanto adjustment.
- A mark-to-market valuation of a collateralized deal which ignores funding corresponds to neglecting the quanto adjustment.

- A liquid instrument having a market value represents an option to cancel all future funding. This option can be valued. However, the mark-to-market is not the correct value for the P&L unless the product is liquidated immediately.

1.4 Literature

The paper actually does not introduce any new modeling approaches. Instead we show that the stochastic funding model is simply a re-interpretation of existing models, e.g. a cross-currency model or a defaultable cash flow model. We will give two model-specific examples, namely the corresponding LIBOR market model formulations. These can be found in the literature, e.g. [2, 6]. We show how these models can be adapted for stochastic funding and how to construct the valuation algorithm. We also consider the funding costs of collateralized deals, partial funding and optimal liquidation.

The convexity adjustment between a collateralized trade and funding, which we unmasked as a quanto adjustment, had already been mentioned in [5] and its references. This paper also discussed a simple (short rate like) model for stochastic funding, but without giving the exact relationship to the valuation measure.

The foreign currency analogy and convexity adjustments are also discussed in [1]. In this paper the primary focus is the multi-curve setup which accounts for a basis swap spread between forward rates of different tenors. As for funding, it also leads to discounting on a different curve, see [1] and references therein.

On the other hand, funding costs are often associated with one's own credit risk, suggesting credit modeling. We will present the analogies between all three interpretations: cross-currency modeling, stochastic funding, and risky curve modeling. In addition we discuss the relation of mark-to-market valuation and funded replication.

2 Foreign Currency Numraire Revisited

We review the valuation of cross-currency and quantoed cash flows. We assume the usual risk neutral valuation theory and adopt the notation of [2].

Let N^{dom} denote some domestic numéraire. Note that N^{dom} is not a traded asset for the foreign investor. But $N^{\text{for}}(t) := FX(t) N^{\text{dom}}(t)$ is a traded asset for the foreign investor. Here FX is the spot fx rate and has the unit $[FX] = 1 \frac{\text{for}}{\text{dom}}$.

Let $V^{\text{dom}}(T)$ denote some (possibly stochastic) cash flow/value in domestic currency. The domestic investor values this via its numéraire N^{dom} under the measure $\mathbb{Q}^{N^{\text{dom}}}$

$$V^{\text{dom}}(t) \;=\; N^{\text{dom}}(t)\,\mathrm{E}^{\mathbb{Q}^{N^{\text{dom}}}}\left(\frac{V^{\text{dom}}(T)}{N^{\text{dom}}(T)} \mid \mathcal{F}_t\right). \tag{1}$$

The foreign investor values the same cash flow by converting all assets into foreign currency, using her numéraire N^{for} under her measure $\mathbb{Q}^{N^{\text{for}}}$:

$$\begin{aligned} V^{\text{dom}}(t) FX(t) \;&=\; N^{\text{dom}}(t) FX(t)\,\mathrm{E}^{\mathbb{Q}^{N^{\text{dom}} FX}}\left(\frac{V^{\text{dom}}(T) FX(T)}{N^{\text{dom}}(T) FX(T)} \mid \mathcal{F}_t\right) \\ &=\; N^{\text{for}}(t)\,\mathrm{E}^{\mathbb{Q}^{N^{\text{for}}}}\left(\frac{V^{\text{dom}}(T) FX(T)}{N^{\text{for}}(T)} \mid \mathcal{F}_t\right) \end{aligned} \tag{2}$$

Note that the transition from (1) to (2) is not a change of numéraire, because N^{dom} is not a numéraire for the foreign investor. It is a change of market!

Note also, that the two equivalent martingale measures $\mathbb{Q}^{N^{\text{dom}}}$ and $\mathbb{Q}^{N^{\text{for}}}$ agree, i.e.

$$\mathbb{Q}^{N^{\text{dom}}} \equiv \mathbb{Q}^{N^{\text{for}}} \tag{3}$$

because

$$\frac{V^{\text{dom}}}{N^{\text{dom}}} \text{ is a } \mathbb{Q}^{N^{\text{dom}}}\text{-martingale} \quad \Leftrightarrow \quad \frac{V^{\text{dom}} FX}{N^{\text{dom}} FX} \text{ is a } \mathbb{Q}^{N^{\text{dom}}}\text{-martingale}$$

In other words: if we have derived the dynamic of the processes under the measure $\mathbb{Q}^{N^{\text{dom}}}$, then we know their dynamic under $\mathbb{Q}^{N^{\text{for}}}$.

What we are effectively doing is starting with a domestic investor's model with domestic numéraire and measure, and then transforming it into a foreign investor's model with foreign numéraire. The foreign numéraire is defined through FX-conversion of the domestic numéraire.

For didactical reasons, let us assume in the following, that we are a foreign investor, i.e. our currency is 1for.

2.1 Quantos

The quanto is a payment of an amount originally denominated in one currency, paid in the other currency. Assume that we are a foreign investor receiving a domestic quantity in (our) foreign currency. I.e. at one point in (2) we use the unit conversion $1 \frac{\text{for}}{\text{dom}}$ instead of $FX(t)$:

$$V^{\text{quanto}}(t) = N^{\text{for}}(t) \mathrm{E}^{\mathbb{Q}^{N^{\text{for}}}} \left(\frac{V^{\text{dom}}(T) 1 \frac{\text{for}}{\text{dom}}}{N^{\text{for}}(T)} \mid \mathcal{F}_t \right).$$

Since $\mathbb{Q}^{N^{\text{for}}} \equiv \mathbb{Q}^{N^{\text{dom}}}$ we have

$$V^{\text{quanto}}(t) = N^{\text{for}}(t) \mathrm{E}^{\mathbb{Q}^{N^{\text{dom}}}} \left(\frac{V^{\text{dom}}(T) 1 \frac{\text{for}}{\text{dom}}}{N^{\text{for}}(T)} \mid \mathcal{F}_t \right)$$

Put differently: The quanto (from the perspective of a foreign investor) is a replacement of the numéraire (in contrast to a change of the numéraire). The numéraire N^{dom} is replaced by N^{for}, while the measure remains the same.

3 Valuation under Funding - Funded Replication

3.1 Funding Numéraire and its Analogy to Cross Currency

The analogy between fx quanto and funding is as follows: Considering funding we have:

- There is some (virtual) market where funding is performed at a some rate (risk free rate, collateral rate, average funding rate, etc.), resulting in *their* numéraire N $(= N^{\text{dom}})$.
- We are not part of the market, in the sense that we fund at our unique rate, resulting in *our* numéraire N^{fd} $(= N^{\text{for}})$.
- We exchange cash flows with the market and our counterparts do not adjust the cash flow for our funding: no funding adjustment $\leftrightarrow$ quanto.

Translating the above we obtain for the funded replication value $V^{\text{fd}}(t)$ of some time T cash flow/market value $V(T)$

$$V^{\text{fd}}(t) \; = \; N^{\text{fd}}(t) \mathrm{E}^{\mathbb{Q}^N} \left(\frac{V(T)}{N^{\text{fd}}(T)} \mid \mathcal{F}_t \right)$$

where N^{fd} is the numéraire associated with our funding rate. Table 1 summarizes the analogies between cross-currency modeling and valuation with stochastic funding.

Cross-Currency Interpretation		Funding Interpretation
Domestic Rate Curve	$\leftrightarrow$	Market Rate Curve
Foreign Rate Curve	$\leftrightarrow$	Funding Curve
Domestic Cash Flow	$\leftrightarrow$	Market Cash Flow
Quantoed Domestic Cash Flow in Foreign Currency	$\leftrightarrow$	Funded Market Cash Flow.

Table 1: Analogies between cross-currency modeling and stochastic funding.

3.1.1 Example: Constant Deterministic Funding Spread

Consider the example where we have a constant deterministic funding spread of the form

$$\frac{N^{\text{fd}}(T)}{N(T)} \; = \; \exp(sT)$$

where s denotes the funding spread. Then the valuation above is given by

$$V^{\text{fd}}(t) \; = \; N(t) \mathrm{E}^{\mathbb{Q}^N} \left(\frac{V(T)}{N(T)} \exp\left(-s(T-t)\right) \mid \mathcal{F}_t \right)$$

Note: The FX rate accrues according to the funding spread. We have

$$\mathrm{d}FX(t) \; = \; sFX(t)\mathrm{d}t.$$

3.1.2 Example: Deterministic Term Structure Funding Spread

A term structure of funding spreads is given by

$$\frac{N^{\mathrm{fd}}(T)}{N(T)} = \exp\left(\int_0^T s(\tau)\mathrm{d}\tau\right),$$

i.e.

$$\frac{N(t)}{N^{\mathrm{fd}}(t)}\frac{N^{\mathrm{fd}}(T)}{N(T)} = \exp\left(\int_t^T s(\tau)\mathrm{d}\tau\right).$$

Then the valuation via funded replication is given by

$$V^{\mathrm{fd}}(t) = N(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{V(T)}{N(T)}\exp\left(-\int_t^T s(\tau)\mathrm{d}\tau\right) \mid \mathcal{F}_t\right).$$

Note: The FX rate accrues according to the funding spread. We have

$$\mathrm{d}FX(t) = s(t)FX(t)\mathrm{d}t.$$

3.2 Interpreting Funding Spreads as Default Intensities

Obviously we have an alternative interpretation: We can interpret $s(\tau)$ as a default intensity and (hence)

$$\exp\left(-\int_t^T s(\tau)\mathrm{d}\tau\right)$$

as a survival probability. With this interpretation N is the risk free numéraire while N^{fd} is the risky numéraire. From N we obtain risk free forward rates $L(T_i, T_{i+1})$ and from N^{fd} we obtain risky forward rates $L^{\mathrm{d}}(T_i, T_{i+1})$.

The rigorous modeling of defaultable (risky) cash flows requires more care: We need to define a richer filtration and a default time process, see, e.g., [6]:

Let τ denote the default time process and let $\mathcal{H} = \mathcal{F} \otimes \mathcal{G}$ denote the augmented filtration, where $\mathcal{G}$ contains the information of the default time and $\mathcal{F} = \cup_t \mathcal{F}_t$ contains the information of the non-defaultable model. Then

$$\mathrm{E}\left(\mathbf{1}_{\tau>T} \mid \mathcal{F}\otimes\mathcal{G}_t\right) = \exp\left(-\int_t^T s(\tau)\mathrm{d}\tau\right)\mathbf{1}_{\tau>t}$$

In this setup the valuation of a defaultable cash flow can be interpreted as an additional discounting:

$$\begin{aligned}\mathrm{E}\left(\frac{V(T)}{N(T)}\mathbf{1}_{\tau>T} \mid \mathcal{H}_t\right) &= \mathrm{E}\left(\frac{V(T)}{N(T)}\mathrm{E}\left(\mathbf{1}_{\tau>T} \mid \mathcal{F}\otimes\mathcal{G}_t\right) \mid \mathcal{H}_t\right)\\ &= \mathrm{E}\left(\frac{V(T)}{N(T)}\exp\left(-\int_t^T s(\tau)\mathrm{d}\tau\right) \mid \mathcal{H}_t\right)\mathbf{1}_{\tau>t}\end{aligned}$$

with $s(t)$ being $\mathcal{F}_t$-measurable. The additional discounting corresponds to the inner expectation of an iterated expectation. Note: The two interpretations should not be mixed.[1]

[1] The double counting mentioned in [4] actually comes from mixing a default indicator and discounting a risky cash flow with funding.

3.3 Comparing Foreign-Currency, Risky Cashflow and Funding

Table 2 summarizes the interpretation of the three frameworks.

Cross-Currency Interpretation		Funding Interpretation		Risky Curve Interpretation
Model Primitives:				
Domestic Rate Curve	$\leftrightarrow$	Market Rate Curve	$\leftrightarrow$	Risk Free Curve
Foreign Rate Curve	$\leftrightarrow$	Funding Curve	$\leftrightarrow$	Risky Curve
Domestic Cash Flow	$\leftrightarrow$	Market Cash Flow	$\leftrightarrow$	Risk Free Cash Flow
FX Rate	$\leftrightarrow$	Not needed (quantoed).	$\leftrightarrow$	Instantaneous Survival Probability
Valuation:				
FX Quantoed Cash Flow Valued by Foreign Investor	$\leftrightarrow$	Funded Market Cash Flow	$\leftrightarrow$	Risky Discounting of Risk Free Cash Flow
Modeling:				
Cross-Currency Model (CCY LMM)	$\leftrightarrow$	Stochastic Funding Model	$\leftrightarrow$	Stochastic Credit Spread Model (Defaultable LMM)
$FX(t) N^{\mathrm{dom}}$, $\mathbb{Q}^{FX(t) N^{\mathrm{dom}}} = \mathbb{Q}^{N^{\mathrm{dom}}}$	$\leftrightarrow$	N^{df}, $\mathbb{Q}^{N^{fd}} = \mathbb{Q}^{N}$	$\leftrightarrow$	$\exp(\int \lambda(t) \mathrm{d}t) N^{\mathrm{dom}}$, $\mathbb{Q}^{N}$

Table 2: Analogies between cross-currency modeling, stochastic funding, and credit risk modeling.

4 Modeling a Stochastic Funding Curve

We take the route of adapting the cross-currency model to funding. While the interpretation of a risky curve model can be equivalently adapted, it carries the additional aspect of default, which may be confusing. While the possibility of default is indeed the source of higher funding, funded replication will not factor in the default event itself. The replication is performed under the assumption of non-default (*going concern*). Hence, it is not necessary to model the default event itself.

As example, we consider a LIBOR market type modeling. The model for stochastic funding is derived as follows

- We start with a model for the market, considering a market numéraire N related to market forward rates L_i.
- We specify the model for the forward rates L_i^{fd} modeling the funding curve. The model is derived under the numéraire $N^{\mathrm{fd}} := FX N$ and the associated measure $\mathbb{Q}^{N^{\mathrm{fd}}}$. We use the fact that $\mathbb{Q}^{N^{\mathrm{fd}}} = \mathbb{Q}^N$, i.e. the dynamic of L_i under $\mathbb{Q}^{N^{\mathrm{fd}}}$ is the same as the dynamic of L_i under $\mathbb{Q}^N$.
- The valuation with funded replication is then performed by putting market cash flows relative to N^{fd} and performing expectation under $\mathbb{Q}^N$.

Note again, that there is no change of numéraire involved in this setup. The numéraire N^{fd} is defined via the numéraire N and the two relate by the “change of market”.

4.1 LIBOR Market Model with Stochastic Funding

We model the market forward rates L_i and under a market numéraire. For example the spot measure numéraire

$$N(t) \ := \ P(T_{m(t)+1};t) \prod_{j=0}^{m(t)} (1 + L_j(T_j)\ \delta_j), \tag{4}$$

where $m(t) := \max\{i \ : \ T_i \leq t\}$, $\delta_j := T_{j+1} - T_j$.

In addition, let us model funding spread S_i^{fd} and with them funding forward rates $L_i^{\mathrm{fd}} = L_i + S_i^{\mathrm{fd}}$.

From (3) we find that the dynamic of the forward rates L_i^{fd} agrees with the dynamic of “foreign rates” in a cross currency LIBOR market model under $\mathbb{Q}^N$, i.e. we have ($\tilde{L}_i := L_i^{\mathrm{fd}}$)

$$\begin{aligned} \mathrm{d}L_i(t) &= L_i(t)\ \mu_i(t)\ \mathrm{d}t \ + \ L_i(t)\sigma_i(t)\ \mathrm{d}W_i^{\mathbb{Q}^N}(t), \qquad (0 \leq i \leq n-1) \\ \mathrm{d}FX(t) &= FX(t)\ \mu^{FX}(t)\ \mathrm{d}t \ + \ FX(t)\sigma^{FX}(t)\ \mathrm{d}W_{FX}^{\mathbb{Q}^N}(t) \\ \mathrm{d}\tilde{L}_i(t) &= \tilde{L}_i(t)\ \tilde{\mu}_i(t)\ \mathrm{d}t \ + \ \tilde{L}_i(t)\tilde{\sigma}_i(t)\ \mathrm{d}\tilde{W}_i^{\mathbb{Q}^N}(t) \qquad (0 \leq i \leq n-1), \end{aligned} \tag{5}$$

with

$$\tilde{\mu}_i(t) \ = \ \sum_{j=m(t)+1}^{i} \frac{\delta_j \tilde{L}_j(t)}{(1+\delta_j \tilde{L}_j)(t)} \tilde{\sigma}_i(t)\tilde{\sigma}_j(t)\tilde{\rho}_{i,j}(t) \ - \ \tilde{\sigma}_i(t)\sigma^{FX}(t)\rho_{\tilde{i},\mathrm{FX}}(t).$$

and

$$\int_{T_i}^{T_{i+1}} \mu^{FX}(t)\, \mathrm{d}t \;=\; \log\left(\frac{P(T_{i+1};T_{i+1})}{\tilde{P}(T_{i+1};T_{i+1})}\right) - \log\left(\frac{P(T_{i+1};T_i)}{\tilde{P}(T_{i+1};T_i)}\right)$$
$$\;=\; -\log\left(\frac{P(T_{i+1};T_i)}{\tilde{P}(T_{i+1};T_i)}\right) \;=\; \log\left(\frac{1+\tilde{L}(T_i,T_{i+1};T_i)}{1+L(T_i,T_{i+1};T_i)}\right).$$

For a derivation of these drifts see [2].

This is also the dynamic of the model under $\mathbb{Q}^{FX\ N}$.

4.2 Modeling the Spreads

Our application suggests that it is more intuitive to model the spreads as log-normal processes, thus avoiding negative spreads. Assuming

$$S_i \;:=\; L_i^{\mathrm{fd}} - L_i$$

with

$$\begin{aligned} \mathrm{d}S_i(t) \;&=\; S_i(t)\mu_i^S(t)\mathrm{d}t \;+\; S_i(t)\sigma_i^S(t)\mathrm{d}W_{S,i}^{\mathbb{Q}^N}(t) \qquad (0 \leq i \leq n-1) \\ S_i(0) \;&=\; S_{i,0}. \end{aligned}$$

Obviously we have

$$\tilde{L}_i(t)\tilde{\sigma}_i(t)\,\mathrm{d}\tilde{W}_i^{\mathbb{Q}^N}(t) \;=\; L_i(t)\sigma_i(t)\,\mathrm{d}W_i^{\mathbb{Q}^N}(t) + S_i(t)\sigma_i^S(t)\mathrm{d}W_{S,i}^{\mathbb{Q}^N}(t).$$

which allows us to calculate the model parameters of $\tilde{L}_i$ from the model parameters of S_i.

4.3 Relationship to the Defaultable LIBOR Market Model and Role of the FX

For $\sigma^{FX} = 0$ the model above agrees with the corresponding defaultable LIBOR market model. The assumption $\sigma^{FX} = 0$ corresponds to the assumption that the survival probability is locally risk free, i.e. it is determined by the spreads only.

The assumption $\sigma^{FX} = 0$ is also natural in the interpretation of a funding curve, since such a risk cannot be replicated or observed as all cash flows are quantoed.

4.4 Funded Replication and Mark-to-Market

Funded valuation is performed using the numéraire N^{fd}, while a market valuation uses the numéraire N. There are however some products, for which funded replication and mark-to-market (approximately) agree: products without cash flows.

To some extent, a cash collateralized deal is such a product.[2] Here, each product cash flow is offset by a corresponding flow of the collateral. A collateralized deal can

[2] Of course, the collateral margin calls do represent cash flows, however the N-relative expected value of these flows is zero. The approximation here is that there is no correlation between net margin calls and funding, see Section 5.2.

also be set up without initial cash flow: We pay it's market value and receive the same amount as collateral. For a collateralized deal the funding rate is the collateral rate. In other words the collateral contract is an agreement on the funding of the deal.

Another example is repos. These products can be exchanged for cash for which a repo rate is paid, providing cheaper funding.

However, those products are not different from others. Valued under funded replication we find that their value agrees (almost) with the mark-to-market because additional cash flows are coming from an additional contract (collateral contract, repo).

In Section 5.2 we will further investigate the funded replication of collateralized products and show that under certain conditions additional funding costs may indeed apply.

5 Valuation with Funding

5.1 Repo, Collateralization, OIS Discounting

Assume that S can be used in a repo[3] contract to obtain a cheaper funding. In this case S is endowed with its own funding. Put differently: S is traded in its own virtual market with numéraire N^{rp}. From this we see that $\frac{S}{N^{\mathrm{rp}}}$ is a $\mathbb{Q}^{N^{\mathrm{rp}}}$-martingale. Using the change of market we see that $\frac{S}{N^{\mathrm{rp}}}$ is a $\mathbb{Q}^{N^{\mathrm{fd}}}$-martingale.

Let us consider the repo/collateral contract: The repo contract exchanges S for a cheaper funding. It consists of the following cash flows / transactions:

$$\begin{array}{llllr} S(0) & \text{(cashflow)} & -S(0) & \text{(asset)} & \text{in } 0 \\ S(T) & \text{(asset)} & -S(0) & \frac{1}{P^{\mathrm{rp}}(T;0)}\text{(cashflow)} & \text{in } T, \end{array}$$

where $P^{\mathrm{rp}}(T;t) := N^{\mathrm{rp}}(t)\mathrm{E}^{\mathbb{Q}^{N^{\mathrm{rp}}}}\left(\frac{1}{N^{\mathrm{rp}}(T)} \mid \mathcal{F}_t\right)$. The term $\frac{1}{P^{\mathrm{rp}}(T;t)}$ equals one plus the interest earned through the repo rate.

Under funded replication the repo contract evaluates to

$$\begin{aligned} & N^{\mathrm{fd}}(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{S(T)}{N^{\mathrm{fd}}(T)} - \frac{S(t)\frac{1}{P^{\mathrm{rp}}(T;t)}}{N^{\mathrm{fd}}(T)} \mid \mathcal{F}_t\right) \\ = \; & N^{\mathrm{fd}}(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{S(T)}{N^{\mathrm{fd}}(T)} \mid \mathcal{F}_t\right) - S(t)\frac{P^{\mathrm{fp}}(T;t)}{P^{\mathrm{rp}}(T;t)} \\ = \; & N^{\mathrm{rp}}(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{S(T)}{N^{\mathrm{rp}}(T)}\frac{N^{\mathrm{rp}}(T)}{N^{\mathrm{fd}}(T)}\frac{N^{\mathrm{fd}}(t)}{N^{\mathrm{rp}}(t)} \mid \mathcal{F}_t\right) - S(t)\frac{P^{\mathrm{fp}}(T;t)}{P^{\mathrm{rp}}(T;t)}. \end{aligned}$$

If we neglect the convexity adjustment, say for example if $\frac{N^{\mathrm{rp}}(T)}{N^{\mathrm{fd}}(T)}$ is non-stochastic, then the repo contract evaluates to 0. In other words: If a possible convexity is neglected, the repo contract has no funding effect.

In sum, we find that together with a repo contract we can replicate S with the cheaper funding costs as

$$N^{\mathrm{rp}}(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{S(T)}{N^{\mathrm{rp}}(T)} \mid \mathcal{F}_t\right). \tag{6}$$

We recover the result that the drift of a repo-able asset is the repo rate (see also [5]).

Let us now consider what happens if we replicate a derivative consisting of a cash flow $S(T)$ in T without putting the derivative under S's repo contract. In this case we have to use funded replication and the value is/the replication costs are

$$N^{\mathrm{fd}}(t)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{S(T)}{N^{\mathrm{fd}}(T)} \mid \mathcal{F}_t\right). \tag{7}$$

The difference between (7) and (6) accounts for the additional hedging costs due to the funding.

In the next section we show that collateralized transactions may still generate funding costs due to convexity in the margin calls.

[3] We use the term repo and collateral somewhat synonymously. We assume that both contracts are endowed with margining.

5.2 Valuation of a Collateralized Product under Funding (convexity)

For a (cash-)collateralized deal, a cash flow in the original contract is always offset by a corresponding cash flow in the collateral. Hence, a collateralized deal does not require funding of the cash flows of the original deal. Its funding rate is given by the interest paid on the cash-collateral, which we assume for simplicity to agree with the market rate. However, a change in the market value of the original deal will result in collateral margin calls which do require funding. The ($\mathbb{Q}^N$-)expected value of the margin calls (together with the interest paid on the margin call and relative to the collateral numéraire) is still zero. Under some assumption this holds also for the funded replication, hence the funded replication of the collateralized deal agrees with the valuation using the collateral rate:

Let us assume that the market values a collateralized deal using $(N, \mathbb{Q}^N)$ and that the cash collateral accrues according to N. In other words, the N-relative cash collateral account is a $\mathbb{Q}^N$-martingale and the time t collateral amount is determined by

$$V^{t_k}(t) \ = \ N(t) \mathrm{E}^{\mathbb{Q}^N} \left(\sum_{t_i \geq t_k} \frac{X_i}{N(t_i)} \mid \mathcal{F}_t \right).$$

Here t_k denote the time of effective outstanding cash flows.

Assume that $\{s_i\}$ denotes collateral margin call dates. Neglecting thresholds, the margin call in s_i will be $V^{t_k}(s_i) - V^{t_k}(s_{i-1})$ and the net collateral cash flow is

$$V^{t_k}(s_i) - V^{t_k}(s_{i-1}) \frac{1}{P(s_i; s_{i-1})},$$

where $t_k = s_i$ and $P(s_i; s_{i-1}) := N(s_{i-1}) \mathrm{E}^{\mathbb{Q}^N} \left(\frac{1}{N(s_i)} \mid \mathcal{F}_{s_{i-1}} \right)$. Here

$$V^{t_k}(s_{i-1}) \left(\frac{1}{P(s_i; s_{i-1})} - 1 \right)$$

is the interest paid on the time s_{i-1} collateral, excluding the time s_{i-1} cash flow.

Obviously the N relative $\mathbb{Q}^N$ expectation of the net collateral cash flow is zero. However, this is not necessarily the case for the N^{fd} relative $\mathbb{Q}^N$ expectation. The funding costs (or benefit) created by the collateralized deal are

$$\begin{aligned} V^{\mathrm{fd}}(0) \ &= \ N^{\mathrm{fd}}(0) \mathrm{E}^{\mathbb{Q}^N} \left(\sum_{s_i} \frac{V^{s_i}(s_i) - V^{s_i}(s_{i-1}) \frac{N(s_i)}{N(s_{i-1})}}{N^{\mathrm{fd}}(s_i)} \mid \mathcal{F}_0 \right) \\ &= \ N^{\mathrm{fd}}(0) \mathrm{E}^{\mathbb{Q}^N} \left(\sum_{s_i} \left(\frac{V^{s_i}(s_i)}{N(s_i)} - \frac{V^{s_i}(s_{i-1})}{N(s_{i-1})} \right) \frac{N(s_i)}{N^{\mathrm{fd}}(s_i)} \mid \mathcal{F}_0 \right). \end{aligned}$$

From this we see that the funding effect of a collateralized deal is due to the covariance of the margin calls and the additional funding discounting factor $\frac{N(s_i)}{N^{\mathrm{fd}}(s_i)}$.

5.2.1 Remark on the Convexity induced by Funding

We see additional convexity in the collateralized product because we view it under funding, i.e. we assume that the collateralized product is hedged with non-collateralized (funded) products. If the collateralized product is hedged with other collateralized products (sharing the same collateral rate), then the derivative's funding convexity will be offset by the hedge's funding convexity. Indeed, in this case there are not even net cash flows coming from the margin calls, because the derivative's margin calls will be offset by the hedge's margin-calls. We can see this effect in the fx quanto interpretation too: If a a quanto product is hedged with quantos, the quanto effect can be ignored by viewing all products in their original currency.

A trader who trades only in collateralized products could take the non-funded valuation to achieve consistent hedge-ratios. However, it is important to note that this valuation relies on some idealized assumptions, most notably:

- The portfolio is fully hedged, i.e. no open position.
- Residual funding costs, e.g. those coming from collateral thresholds, are neglected.
- All collateral contracts agree on the same valuation model and collateral rate.

5.3 Valuing Liquid Products

5.3.1 Mark-to-Market versus Funded Replication

Consider: we hold a product which is liquid, i.e. there is a liquid market and a reliable market price.

- If we hold / hedge the product via funded replication the product evaluates to V^{fd}.
- If we liquidate the product at time t the product evaluates to $V^{\circ}(t)$.[4]

Some accounting rules assign attributes to a product like "held for trading" (trading book) or "held to maturity" (bank book) and allow a mark-to-market valuation for the trading book. However: this implies that the trading book does not take any funding costs into account. On the other hand, the trading book is not liquidated. It is still there for a while. The trades may have changed ID's and some properties, but may still have funding implications.

Also: If one assumes that the product is liquidated in time t and hence uses its mark-to-market value, then one has to remove the product from all risk calculations, like liquidity risk. If a product is a hedging instrument, then the mark-to-market valuation contradicts the assumption of a hedge: in a dynamic hedge (replication) the replication strategy decides when a product is liquidated at its market value or not. During the dynamic hedge, funding costs or funding benefits may apply.

[4] The formulation is a bit sloppy, as we will see later. More precisely, the cash received in return for the product evaluates to $V^{\circ}(t)$.

Example: Forward Contract. The situation can be illustrated for a simple Black-Scholes model. Let us assume that the market's risk free rate is r. The classical theory considers the dynamic $\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}W$ under the measure $\mathbb{Q}^N$ with $N(t) = \exp(r\ t)$. In our setup we would consider $\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}W$ under the measure $\mathbb{Q}^N$ but put S relative to $N^{\mathrm{fd}}(t) = \exp(r^{\mathrm{fd}}t)$, where $r^{\mathrm{fd}} = r + s$.

There are two drifts: r is the rate at which we expect to buy and sell S, and r^{fd} is the rate of the funding required to buy or sell S.

Assume we want to hedge (replicate) a forward contract, i.e. we are obliged to deliver one unit S in T. We buy one unit $S(0)$ and issue a bond to finance this buying. Thus, obviously the forward contract has to factor in the finance costs and is evaluated as

$$S(0)\frac{\exp(rT)}{\exp(r^{\mathrm{fd}}T)} \;=\; S(0)\exp(-sT).$$

This does not mean that we value the asset $S(0)$ differently. It is the future funding cash flow inside the forward contract that induces the funding costs.

Example: Risk Free Cash Flow. Let us consider a single incoming cash flow of +1 in T, coming from some counterparty risk free entity. How should this cash flow be valued? We will consider two similar cases: a credit and a bond:

Credit: Assume that we have a credit with cash flows $-C(0)$ in $t = 0$ and +1 in T. To fund this cash flow we issue a bond $P^{\mathsf{A}}(T)$, resulting in a cash flow of $P^{\mathsf{A}}(T;0)$ in $t = 0$, and a cash flow of -1 in T. We have effectively hedged the credit. Hence we have $C(0) = P^{\mathsf{A}}(T;0)$.

Bond: Assume that we hold a liquid traded risk free bond $P^{\circ}(T;0)$ paying $+1$ in T. The mark to market value of the bond is $P^{\circ}(T;0)$. However, funded replication will value it as $P^{\mathsf{A}}(T;0)$. Now assume that our owner requests our P&L as cash. There are two options:

- We sell the bond and receive $P^{\circ}(T;0)$ from the market; our P&L is indeed $C(0) = P^{\circ}(T;0)$.
- We hold the bond $P^{\circ}(T)$ and issue a bond $NP^{\mathsf{A}}(T;0)$ having notional $N := \frac{C(0)}{P^{\mathsf{A}}(T;0)}$ to finance the P&L take out of $-C(0)$. If $C(0) = P^{\mathsf{A}}(T;0)$, then $N = 1$ and in T the two cash flows of $NP^{\mathsf{A}}(T)$ and $P^{\circ}(T)$ will match. However, if $C(0) = P^{\circ}(T;0)$ then we will make a loss in T, namely $1 - \frac{P^{\circ}(T;0)}{P^{\mathsf{A}}(T;0)}$, which is our own funding spread.

In other words: If "valuation" means "determining P&L", then a mark-to-market is only admissible if we immediately liquidate the product. Otherwise, a P&L has to be determined by funded replication.

Nevertheless, a trader may still have a trading strategy which differs from a held-to-maturity. The funding costs of such a strategy can be valued and we will discuss two of them in the next sections.

5.3.2 Valuation with Partial Funding

The setup from Section 3 allows to factor in funding costs in a partial, even stochastically partial way. Consider (possibly stochastic) cash flows X_i in t_i. Assume that valuation of these cash flows with respect to N leads to the liquid market price and it is determined that the product is held only up to time s and then liquidated at the time s market value. Prior to s funding costs apply. We can value these partial funding costs by

$$V^{\mathrm{fd}}(0) \ = \ N^{\mathrm{fd}}(0)\mathrm{E}^{\mathbb{Q}^N}\left(\sum_{t_i<s}\frac{X_i}{N^{\mathrm{fd}}(t_i)}+\frac{N(s)}{N^{\mathrm{fd}}(s)}\mathrm{E}^{\mathbb{Q}^N}\left(\sum_{t_i\geq s}\frac{X_i}{N(t_i)}\mid\mathcal{F}_s\right)\mid\mathcal{F}_0\right).$$

This valuation formula also holds if s is a stopping time, allowing for stochastic market closeout of the product.

5.3.3 Valuing Liquidity in Dynamic Hedging

If a product is liquid this constitutes an additional optional right: namely the option to sell the product for the market price V°. In other words, V° is the intrinsic value of the option and V^{fd} is the value if we neglect the option's value (funded replication). Then,

- If we value the product to V° then we have to liquidate it (exercise the option). This means there is no longer a product, only cash.
- If we continue to hold the product we may generate additional funding costs, or a funding benefit.
- If we hold the product, its funding implications have to be considered for the holding period. But this may be a portfolio problem.
- The decision to liquidate the product is given by “optimal exercise” criteria and we have to value it accordingly.

The intrinsic value V° may not remain the correct value if we continue to hold the product, because removing it later may generate funding requirements due to lost funding benefits. If we factor in the product's future cash flows in our portfolio simulation (e.g. liquidity model, etc.), then V° may not even be a lower bound for the product's value, because removing the product may induce additional costs in our portfolio model (it may have been a funding benefit).

So in summary the real options here are:

- We do not factor in our option to sell at market price but instead factor in the funding for the whole lifetime. We value the product as V^{fd}. This *is* a conservative valuation.
- We factor in our option, and use an optimal exercise strategy to decide when the trade is to be liquidated. The value may or may not coincide with V°.

In the first case, we simply use V^{fd} as the product's valuation. In the second case we need to consider the following algorithm:

$$\begin{aligned} V^{\mathrm{fd}*,s_i}(s_i) &= X_i + \max\left(V^{\mathrm{fd}*,s_{i+1}}(s_i), V^{\circ,s_i}(s_{i+1})\right), \\ V^{\circ,s_{i+1}}(s_i) &= N(s_i)\mathrm{E}^{\mathbb{Q}^N}\left(\sum_{t_k \geq s_{i+1}} \frac{X_k}{N(t_k)} \mid \mathcal{F}_{s_i}\right), \\ V^{\mathrm{fd}*,,s_{i+1}}(s_i) &= N^{\mathrm{fd}}(s_i)\mathrm{E}^{\mathbb{Q}^N}\left(\frac{V^{\mathrm{fd}*,s_{i+1}}(s_{i+1})}{N^{\mathrm{fd}}(s_{i+1})} \mid \mathcal{F}_{s_i}\right). \end{aligned}$$

Note: N is the numéraire associated with the market valuation, while N^{fd} is the numéraire associated with the funded replication valuation.[5]

6 Sensitivities and Hedging with Funding

6.1 Hedging Funding Risk and Market Risk

Briefly this is how the hedging of funding risks and market risks is normally performed:

- First we calculate the funding spread sensitivities. These can be hedged with floating rate bonds (valued under funding). These floating rate bonds have a funding (discounting) sensitivity, but zero market rate sensitivity.
- The remaining portfolio will have market rate sensitivities, but zero funding rate sensitivities (similar to a swap). Market rate sensitivities can be hedged with collateralized deals. These instruments have almost no funding (discounting) sensitivities.

Of course, this two-step process is a simplification. Usually we have to solve a multi-dimensional problem where funding sensitivities and market sensitivities are considered together in a single multi-valued equation.

After the hedging has been performed, the portfolio will have the characteristics of a hedged swap portfolio. Cash flow and market risks are netted via a dynamical hedge. A P&L of this portfolio will always be realized P&L.

6.2 Calculating Hedge Ratios in a Mixed Setup

Basically a collateralized transaction consists of two parts:

- The claims and liabilities from the (otherwise) un-collateralized product.
- The collateral contract agreeing to a specific valuation and a collateral rate.

We have seen that there are two (almost) equivalent methods for calculating the value / replication costs of a collateralized product:

[5] Taking the equivalence of funding and FX quantoing, the product represents a Bermudan option to un-quanto a quanto.

- Discounting the product with the collateral curve (or repo curve), i.e. we value only the claims from the product and the collateral contract is valued 0.
- Discounting the product with the funding curve and explicitly calculating the funding benefit coming from the collateral contract.

Both valuations lead to the same value (apart from the convexity effect). The first method is also known as *OIS discounting* since the collateral rate is often an OIS rate. It is a short cut or approximation to the second method, the valuation of the collateral contract with funded replication.

The hedging of an uncollateralized (funded) product with collateralized product will lead to slightly different hedge ratios due to the different discounting/funding. For example: An uncollateralized forward contract is not hedged by one unit of the collateralized underlying. This effect may be counter-intuitive at first. The following two examples compare the cash flows, sensitivities and hedge-ratios of un-collateralized (i.e. funded) and collateralized products. We keep the example simple and assume deterministic funding rates to illustrate the effect:

6.2.1 Example: Single Claim, Collateralized versus Funded

We can illustrate the effect in a very simple setup, namely a single collateralized claim (e.g. an asset with repo). Let us assume that S is a collateralizable (repo-able) underlying with the following dynamic

$$S(t) \ = \ S_0 \exp\left(rt - \frac{\sigma^2}{2} + \sigma W(t)\right),$$

where r is the collateral rate (repo rate) and the associated numéraire is

$$N(t) \ = \ \exp(rt).$$

Furthermore let the funding numéraire be given via a constant funding rate $r^{\mathrm{fd}} = r + s$ as

$$N^{\mathrm{fd}}(t) \ = \ \exp((r+s)t) \ = \ N(t)\exp(st).$$

Let us first assume that we put one asset S into collateralization over the period $[0,T]$ and sell in T. The collateralized liability evaluates to

$$\mathrm{E}\left(\frac{-S(T)}{N(T)} \mid \mathcal{F}_0\right) \ = \ -S_0. \tag{8}$$

Let us now assume that there is a claim on a cash flow $S(T)$, but the claim is not collateralizable. According to funded replication this claim evaluates to

$$\mathrm{E}\left(\frac{S(T)}{N^{\mathrm{fd}}(T)} \mid \mathcal{F}_0\right) \ = \ S_0 \exp((r - r^{\mathrm{fd}})T) \ = \ S_0 \exp(-sT). \tag{9}$$

We now consider to hedging the uncollateralized derivative (9) with the collateralized product (8). Calculating the sensitivity with respect to S_0 in (9), (8) we find that the hedge ratio (delta) is $\frac{\exp(-sT)}{1}$, i.e. we need $\frac{\exp(-sT)}{1}$ units of the collateralized product.

	in $t = 0$		in $t = T$	
Collateral Contract:				
			$+S(T)$	receive underlying
	$S(0)$	receive collateral	$-S(0)\exp(rT)$	pay collateral
Net:	$S(0)$	(10)	0	
Claim (without funding):				
			$+S(T)$	receive underlying
Net:	0		$S(T)$	(11)
Claim (with funding / funded replication):				
			$+S(T)$	receive underlying
	$S(0)\exp(-sT)$	funding (bond)	$-S(0)\exp(rT)$	funding (bond).
Net:	$S(0)\exp(-sT)$	(12)	0	

Table 3: Flows of a single claim in $t = T$ generated by collateralization and funding upon maturity $t = T$ and inception $t = 0$. Note: A change in $S(0)$ will generate a flow in $t = 0$ corresponding to the sensitivity of the shown flow, even if inception was prior in $t < 0$.

Let us analyze the transactions behind (8) and (9): From Table 3 we see why we are not hedged with a hedge ratio of 1 to 1: while the collateralized contract creates cash flows in $t = 0$ if $S(0)$ changes, see (10), the claim has a cash flow $S(T)$ in $t = T$, see (11). But something is missing: our liquidity management will match the $+S(T)$ in T to a corresponding bond, e.g. to finance a P&L take out. If we include the funding, i.e. the transactions which have to be performed by liquidity management, we see that the correct hedge ratio is $\exp(-sT)$ to 1, (10) and (12).

We can interpret the result in alternative way: If $S(0)$ changes there will be a margin call on the collateral and this margin call has to be funded. Hence, funding costs apply for the change in $S(0)$. From Table 3 we also see the convexity effect described in Section 5.2. A margin call of $\Delta S(t)$ in t on the collateralized product is accompanied by a re-hedge given by a funding zero bond with notional $\Delta S(t)\exp(r(T-t))\exp(s(T-t))$ (which then generates the cash flow offsetting the margin call). The correlation term here is the multiplication of $\Delta S(t)$ by $\exp(s(T-t))$.

Table 4 summarized the sensitivities.

6.2.2 Example: Structured Bond and Collateralized Structured Swap

The following example is more related to the setup found in interest rate products. Again, we consider a very simple example, namely one interest rate period of a structured coupon $C(T_1)$. Consider a bond paying $C(T_1) + 1$ in T_1. Discounting with funding the time $t = 0$ value is

$$V_{\text{bond,fd}}(0) = \mathrm{E}\left(\frac{(C(T_1)+1)\exp(-sT_1)}{N(T_1)} \mid \mathcal{F}_0 \right).$$

In addition, consider a collateralized hedge swap where collateral is accrued/discounted according to the numéraire N. Let the hedge swap exchange $C(T_1)$ with $L(T_1) + s_0$.

Product	Sensitivities (Valuing under Funding)		
	Collateralized		Uncollateralized
Claim S(T)	$\exp(rT)\exp(-r^{\mathrm{fd}}T)$	$\not\longleftrightarrow$	$\exp(rT)\exp(-r^{\mathrm{fd}}T)$
Funding	$1-\exp(rT)\exp(-r^{\mathrm{fd}}T)$	$\not\longleftrightarrow$	0
Total	1	$\longleftrightarrow$	$\exp(-sT)$

Table 4: Sensitivities of the components of a collateralized product and a funded (uncollateralized) product. When we value under funding, the funding or an uncollateralized claim has zero sensitivity to the underlying: the funding costs are part of the valuation. The collateralization can be seen as contract specific funding, and when it its valued under funding the collateral will have a non-zero sensitivity to the underlying. In sum, the sensitivities of a collateralized and uncollateralized claim differ. Note that for $s = 0$ the mismatch disappears.

Since it is collateralized the $t = 0$ collateral is

$$V_{\text{swap,col}}(0) \;=\; \mathrm{E}\left(\frac{C(T_1) - (L(T_1) + s_0)}{N(T_1)} \mid \mathcal{F}_0 \right).$$

Let us assume that the structured coupon $C(T_1)$ is such that the bond is at par, i.e. $V_{\text{bond,fd}}(0) = 1$. The swap margin is usually chosen such that the corresponding swap is at par, i.e. $V_{\text{swap,col}}(0) = 0$. Calculating the sensitivity with respect to the structured coupon (or any of its components) we see that the correct hedge ratio is $1 : \exp(-\bar{s}T_1)$, with $\bar{s} = s$.[6] The hedge portfolio with 1 unit of the bond and $-\exp(-\bar{s}T_1)$ unit of the swap then is

$$\begin{aligned} V_{\text{floater,fd}}(0) \;=\; \mathrm{E}\Bigg(& \frac{((L(T_1) + s_0) + 1)\exp(-sT_1)}{N(T_1)} \\ & + \frac{(C(T_1) - (L(T_1) + s_0))\exp(-(s-\bar{s})T_1)}{N(T_1)} \mid \mathcal{F}_0 \Bigg) \end{aligned}$$

and assuming that s is not stochastic with $s = \bar{s}$ we find

$$= \mathrm{E}\left(\frac{((L(T_1) + s_0) + 1)\exp(-sT_1)}{N(T_1)} \mid \mathcal{F}_0 \right) \;=\; 1,$$

which is the floating rate bond (valued under funding). From this we see that the swap margin might have been determined from the floater. We also see the different roles of the swap margin and hedge ratio: The swap margin compensates for the higher discounting of the notional, while the hedge ratio compensates for the different exposure to the structure coupon. Again we see that the hedge ratio applied to the collateralized swap depends on the funding costs used in the uncollateralized bond.

6.3 Theta and the Cash Account

We saw that the sensitivity with respect to the underlying depends on the funding, i.e., it is different for a collateralized and non-collateralized trade. Nevertheless the

[6] We use the symbol $\bar{s}$ to distinguish it from the possibly stochastic value s, but assuming s is not stochastic we have $s = \bar{s}$.

sensitivities do result in the correct hedge ratio. We now take a look at the theta of the two trades, i.e., the derivative with respect to time, $\frac{\partial}{\partial t}$.

6.3.1 Example

We consider a deterministic cash flow $C(T)$ in T and the deterministic numeraires

$$N(t) \;=\; \exp(r\,t), \qquad N^{\mathrm{fd}}(t) \;=\; \exp((r+s)\,t).$$

Under funded replication the cash flow is valued as

$$V^{\mathrm{fd}}(0) \;=\; N^{\mathrm{fd}}(0)\mathrm{E}\left(\frac{C(T)}{N^{\mathrm{fd}}(T)} \mid \mathcal{F}_0\right) \;=\; C(T)\exp(-(r+s)T),$$

while the collaterlized trade is valued as

$$V(0) \;=\; N(0)\mathrm{E}\left(\frac{C(T)}{N^{\mathrm{fd}}(T)} \mid \mathcal{F}_0\right) \;=\; C(T)\exp(-rT).$$

With funded replication the cash flow has a theta of

$$\frac{\partial}{\partial t}V^{\mathrm{fd}}(t) \;=\; (r+s)\,C(T)\exp(-(r+s)(T-t))$$

while the collateralized product has a theta of

$$\frac{\partial}{\partial t}V(t) \;=\; r\,C(T)\exp(-r(T-t)).$$

Take into account the hedge ratio we have, assuming a static hedge we have

$$\exp(-s(T-t))\frac{\partial}{\partial t}V(t) \;=\; r\,C(T)\exp(-(r+s)(T-t)).$$

Assuming a dynamic hedge

$$\frac{\partial}{\partial t}(\exp(-s(T-t))V(t)) \;=\; (r+s)\,C(T)\exp(-(r+s)(T-t)).$$

We see that the theta match if we take the dynamic hedging (induced by funding) into account, but otherwise they do not agree.

The reason, why the thetas of the two products do not match is simple, but maybe not obvious: In the context on funded replication, $V(t)$ is not the value process of the collateralized deal!

6.3.2 Value Process of a Collateralized Product

Let $V(t)$ denote the time-t value of a collaterlized product, i.e., $V(t)$ is the amount of collateral held. As before, we assume that $V(t)$ is obtained by N-relative expectation, where N is the collateral numéraire.

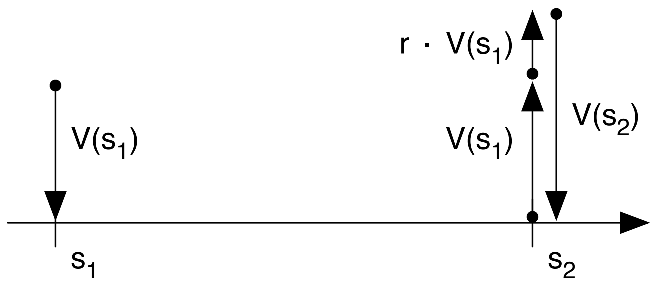

Figure 1: Cash flows of a collateralized product with margin call date s_1, s_2, assuming no market movement, the theta on the margin calls cancels the interest payed on

At the next margin call date, say $t + \mathrm{d}t$, apart from a convexity effect the average net cash flow is zero: we receive interest on the collateral of $V(t)r(t)\mathrm{d}t$ and exchange the collateral through the margin call $V(t + \mathrm{d}t) - V(t)$ (which is $\mathrm{theta} \mathrm{d}t$), see Figure 1.[7]

While there is no future net cash flow (expected), we hold a cash (collateral) position in t, which accrues according to our funding rate.

Put simple, $V(t)$ is the amount of cash held in time-t and the value process of a cash position is

$$T \mapsto V(t) \exp(\int_t^T r(\tau) + s(\tau) \mathrm{d}\tau).$$

6.4 Interpretation of the Sensitivities

While the theta calculated from the collateral amount $V(t)$ is not correct, the market sensitivities calculated from $V(t)$ do give the correct hedge ratios. The reason behind this becomes clear if we go back to the cross-currency interpretation of funded replication: the value $V(t)$ is in "collateral currency" while with funded replication $V^{\mathrm{fd}}(t)$ is in "funded currency". If we want to compare values and sensitivities we have to convert to a common currency first, using our conversion factor $FX(t)$, i.e., the collateralized value in "funded currency" is $FX(t)V(t)$.

Since $FX(0) = 1$ we find that the conversion is without effect if we consider present ($t = 0$) values. The same applies to market sensitivities with respect to some risk factor x: Since $FX(0) = 1$ independent of all market parameters x, we have $\frac{\partial}{\partial x} FX(0) = 0$. Thus

$$\frac{\partial}{\partial x} \left(FX(t)V(t)\right)|_{t=0} \ = \ FX(0) \frac{\partial V}{\partial x}(0) \ = \ \frac{\partial}{\partial x} V(0).$$

On the other hand we obtain for the theta of the collateralized deal (in "funded cur-

[7] Here $r(t)$ is the collateral short rate, i.e., $\mathrm{d}N(t) = r(t)N(t)\mathrm{d}t$ where N is the collateral numeraire and V/N is an $\mathbb{Q}^N$-martingale.

rency")

$$\frac{\partial}{\partial t}\left(FX(t)V(t)\right)|_{t=0} = FX(0)\frac{\partial V}{\partial t}(0) + V(0)\frac{\partial FX}{\partial t}(0)$$
$$= \frac{\partial V}{\partial t}(0) + V(0)\frac{\partial FX}{\partial t}(0).$$

Under the model assumption

$$\mathrm{d}FX(t) = s(t)FX(t)\mathrm{d}t, \qquad FX(0) = 1,$$

we find

$$\frac{\partial}{\partial t}\left(FX(t)V(t)\right)|_{t=0} = \frac{\partial V}{\partial t}(0) + s(0)V(0).$$

We see that in funded currency the theta gets the additional term which correspond to the spread earned on the cash/collateral position - as in the example above.

Note that this also shows that there is a link between the cash process, collateralized products and non-collateralized products.

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Notes

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