Financial Interpolation – Yield curve and forward curve

On the move

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Presentation Background

- Presentation is the result of a validation we started in May 2010
- Goal of the validation was to ‘open’ black box that ALM vendor software was
- Recalculating in detail Regulatory Scenario’s for Income At Risk, Market Value and Future Market Value reports for EC for Interest Rate Risk
- Result is detailed insight in the implemented functionality and methodology in the software
- First part of the validation is recalculating the yield and forward curve, basis for this presentation

Agenda

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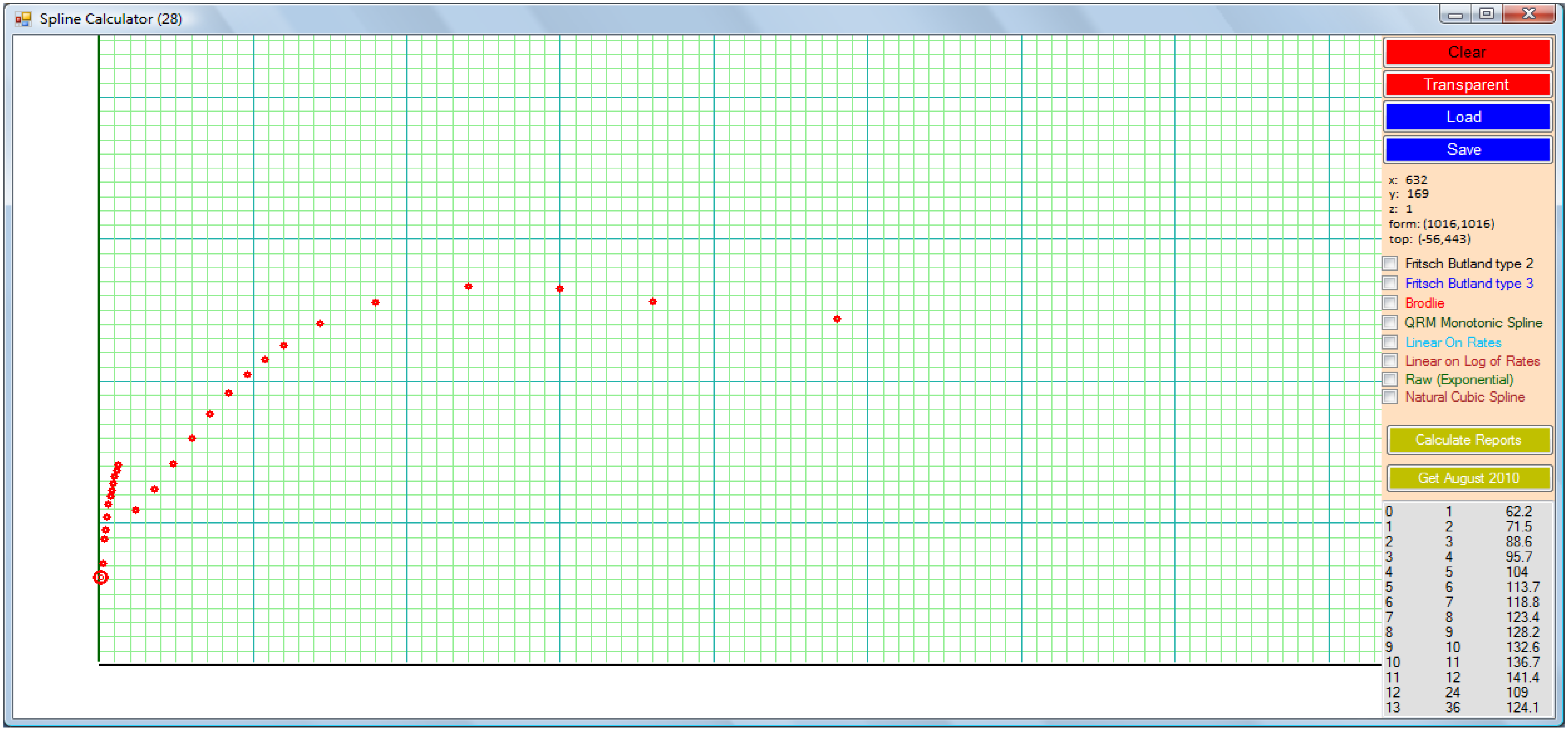
- Background information of interpolation techniques
- Interpolations with negative instantaneous forwards
- Monotonic cubic spline interpolation (MCS) on Capitalization Factor (CF) & MCS on term structure (TS)
- Arbitrage and correction mechanisms
- Questions and discussions

Interpolation

- Definition : a method of contructing new data points within a discrete set of points
- Special case of curve fitting (Nelson –Siegel) or regression analysis that goes trough all input points
- As validation we primarily looked at options available in software: *linear, exponential, cubic spline, monotonic cubic spline*
- Most rudement form is piecewise constant interpolation.
- We used Yield curve of August 2010 as basis

August 2010 Yield curve



Linear Interpolation

for t in [a,b], $f(t) = f(a) + (f(b) - f(a))\,(t - a)/(b - a)$

Error function $|R(t)| \leq \frac{1}{8}(b - a)^2 \max(g''(t))$
if g(t), the underlying function, is 2 times differentiable

- Interpolant is not differentiable, not smooth
- Instantantious forward curve is piecewise constant, not continuous
- McCulloch and Kochin [2000]: *this implies implausible expectations about future short-term interest rates or holding period returns. Should be avoided, especially when pricing derivatives whose value is dependent upon such forward values*.

Linear on Rates

Polynomial Interpolation

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- If we have n data points there is exactly one polynomial of degree at most n – 1 going through all the points (unisolvence theory)
- Non trivial to find a solution, requires solving system of linear equations with Vandermonde Matrix.

- Error term: $|R(t)| \leq \max(g^{(n+1)}(t)) \Pi^{n}_{i=0}(t-x_i) \;/\; (n + 1)!$

- Problem: Runge Phenomenon, i.e. Oscillation at the edges of an interval, shows that going to higher degrees does not improve accuracy

- Mitigation: choosing more interpolation points close to edge or spline interpolation

Spline Interpolation

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• Spline interpolation is a form of interpolation where the interpolant is a piecewise polynomial called a 'spline'.

• A 'Spline' is a classic tool to draw curves used for example in ship building. It is a long thin flexible strip of wood or plastic. The elasticity of the spline material combined with the knots (input points) cause the strip to take a form to minimize the energy, creating the smoothies possible shape

• Splines were used by the Wright brothers to design the first plain

• Splines were used by Citroen, Renault and General moters for modeling automobile bodies

•First mathematical reference was in 1946

Cubic Spline Interpolation

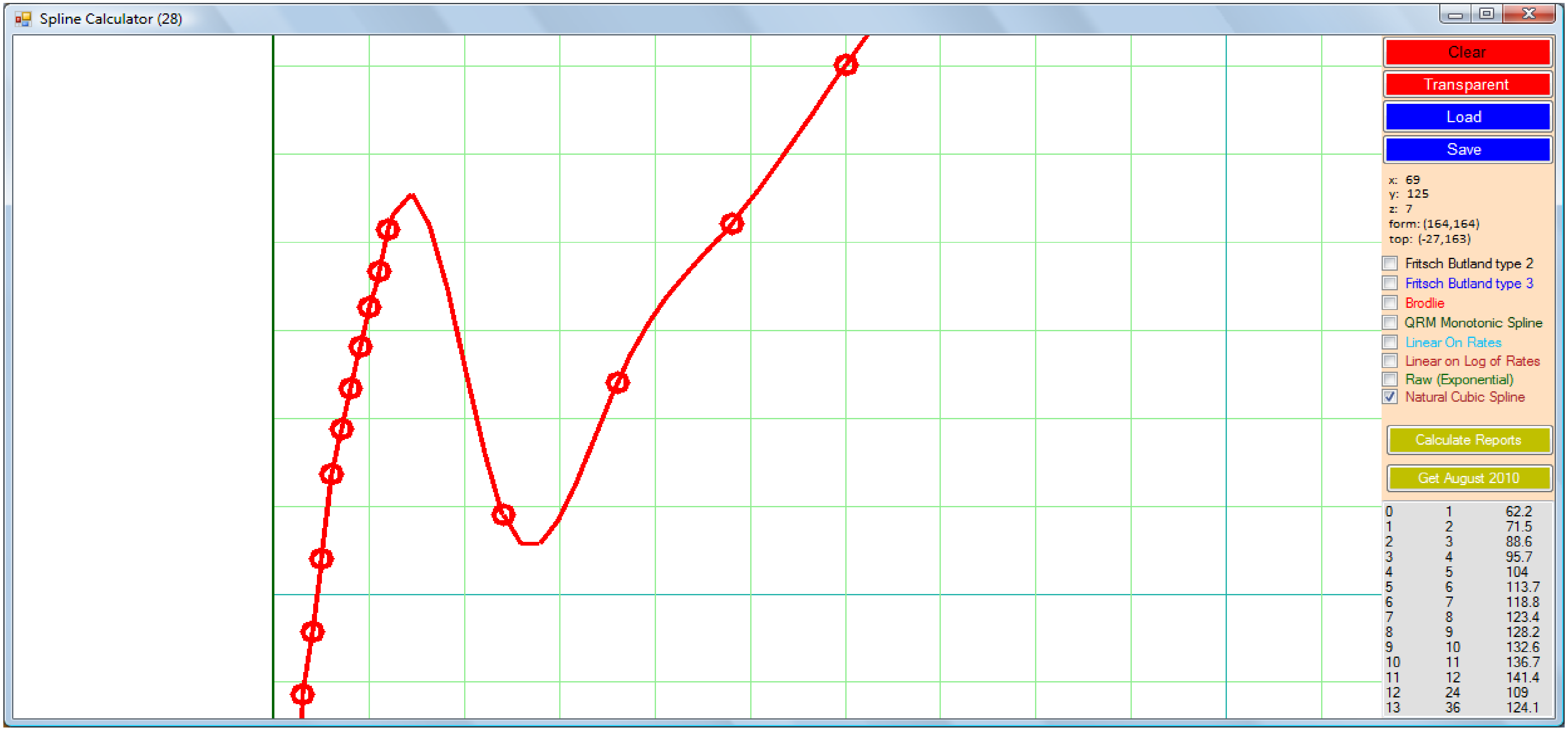
- 3^{rd} order polynomials are used to connect the input points.
- The resulting function is continuous and differentiable at the spline joints; this is known as the *Hermite* form.
- This is a minimum requirement for the *smoothness* of the curve. The curve is fully defined when values for the derivatives *d1,…,dn* at the input data points are defined.

Natural Cubic Spline

- 2 Times Continuous and Differentiable Cubic Spline
- Fully defined given conditions on derivative at start and endpoint
- Most smooth and elegant curve
- Best approach for mimicking a natural curving line between points
- Adventage: pleasing to the eye
- Adventage: energy minimizing
- Disadvantage: not monotonic
- Disadvantage: does not preserve local extremes
- Disadvantage: not local

Natural Cubic Spline

Not Monotonic and Extreme preserving



Monotonicity

- The interpolation is called ***monotonic*** if three points of monotonic increasing inputs lead to a monotonic increasing spline on the segment, same for decreasing.

- If three points of input data are not monotonic it means there is a local extreme. If interpolation maintains these extremes then the interpolation is called “preserving local extremes”.

- If a cubic spline is not *monotonic* it is possible to apply a filter to adjust the derivatives d1,…, dn such that the result is monotonic. This is called the ***Hyman*** filter.

Monotonicity

- In Fritsch and Carlson [3] exact conditions are defined on derivatives d1, …., dn such that third order polynomials exist that maintain the monotonicity. Derivatives have to lie within the so-called 'monotonicity area', an area within the 2 dimensional plane.

- In Fritsch and Butland [2] an algorithm is defined that generates derivatives *d1, …., dn* such that the interpolant is monotone. A version is implemented in Matlab, for example, under the name PCHIP for Piecewise Cubic Hermite Interpolating Polynomial.
- It calculates the derivatives at all points according to a formula G: (S is the delta function)

$$G_3(S_1, S_2) = \begin{cases} 0 & \text{if } S_1 S_2 \leqq 0, \\ \text{sign}\,(S_1)\dfrac{3|S_1||S_2|}{|S_1| + 2|S_2|} & \text{if } |S_2| \leqq |S_1|, \\ G(S_2, S_1) & \text{otherwise.} \end{cases}$$

Monotonic Cubic Spline

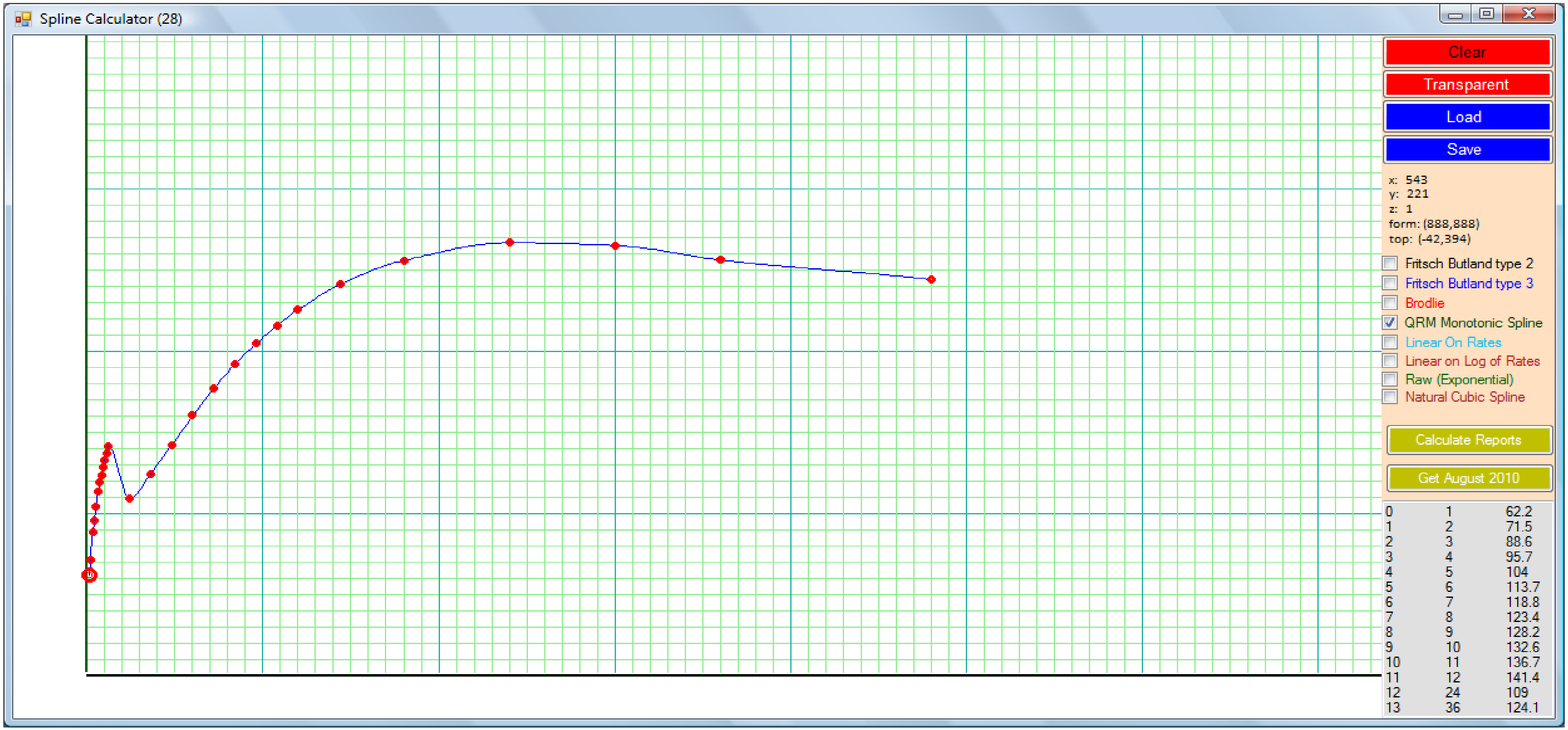
- A big advantage of this approach is that interpolation is strongly **local**, a small change in the input data only results in a small change in the interpolant, contrary to natural cubic spline

- The cause of flatness is that the formula for Fritsch-Butland has a tendency to calculate derivatives that are near the origin of the monotonicity area. The *curvature* implied by the input data is not well maintained. The curvature is the shape that is intuitively implied by the input data.

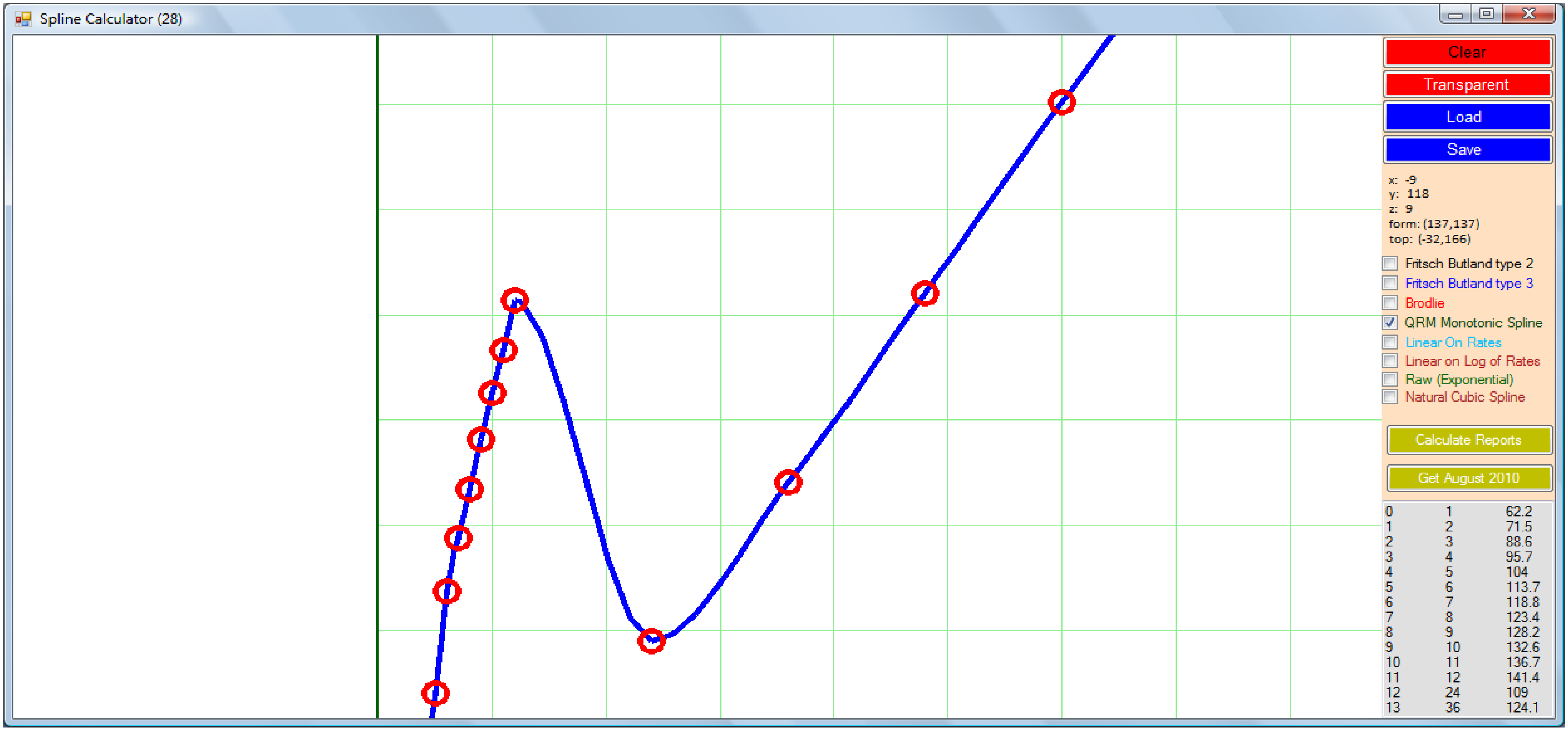
- This means that the relative error O, the mathematical measure for smoothness, at the input points is relatively big. The result is far from C^2 in other words.

- Another implication of the flatness is that implied forward curves becomes very jumpy.

Monotonic Cubic Spline



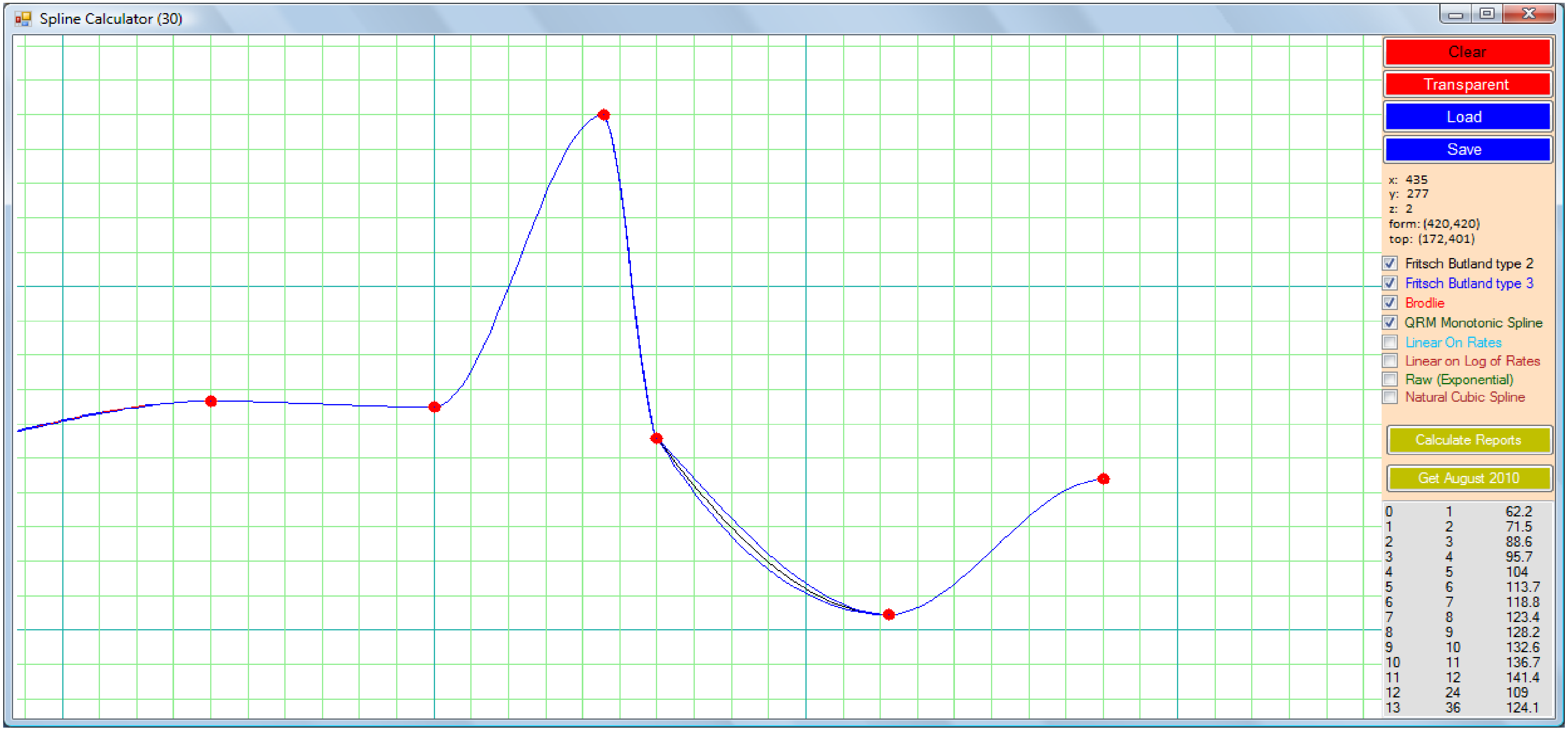
Monotonic and Extreme Preserving



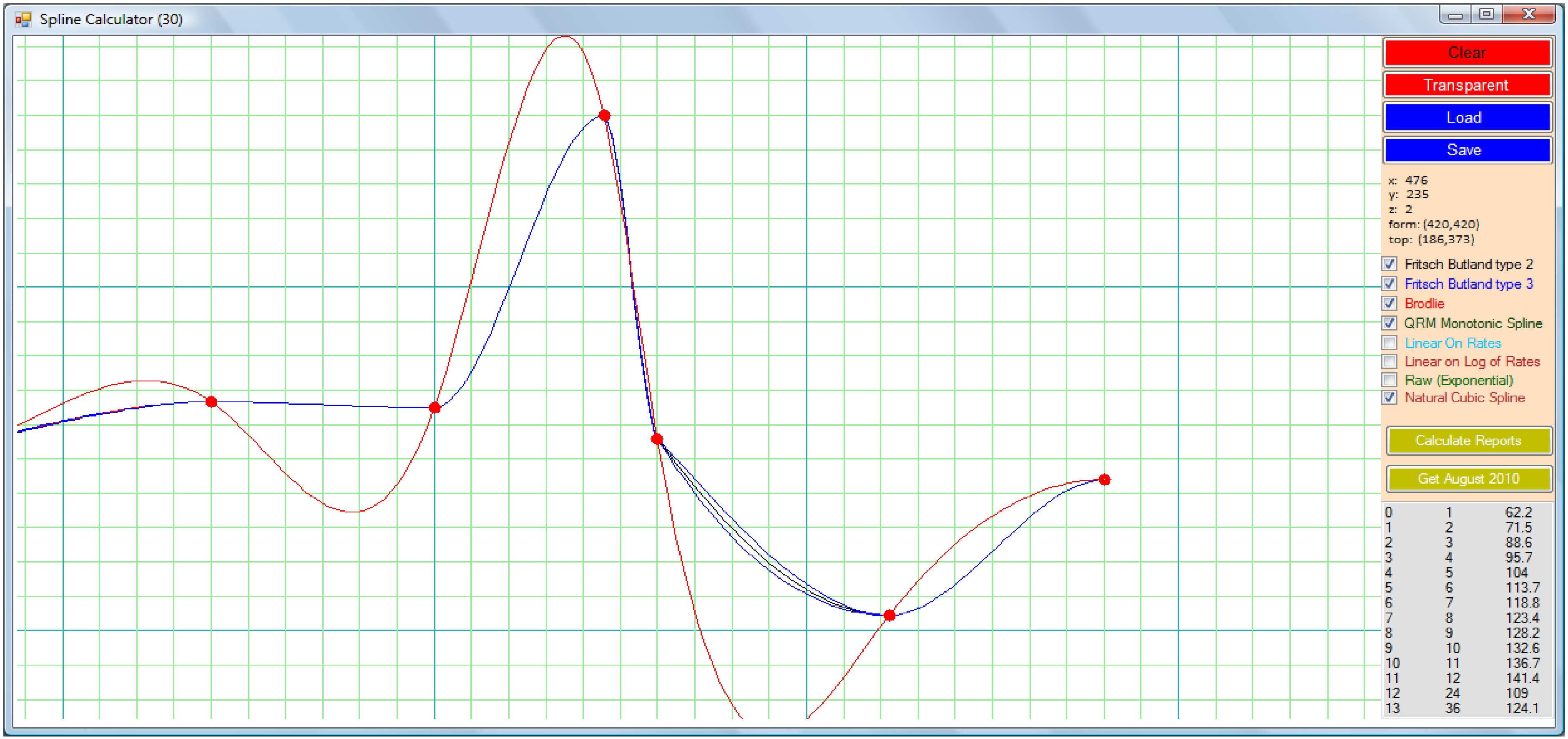
Monotonic vs Natural

Add two local extremes

Different types of Monotonic



Difference with Natural Spline



Not Local

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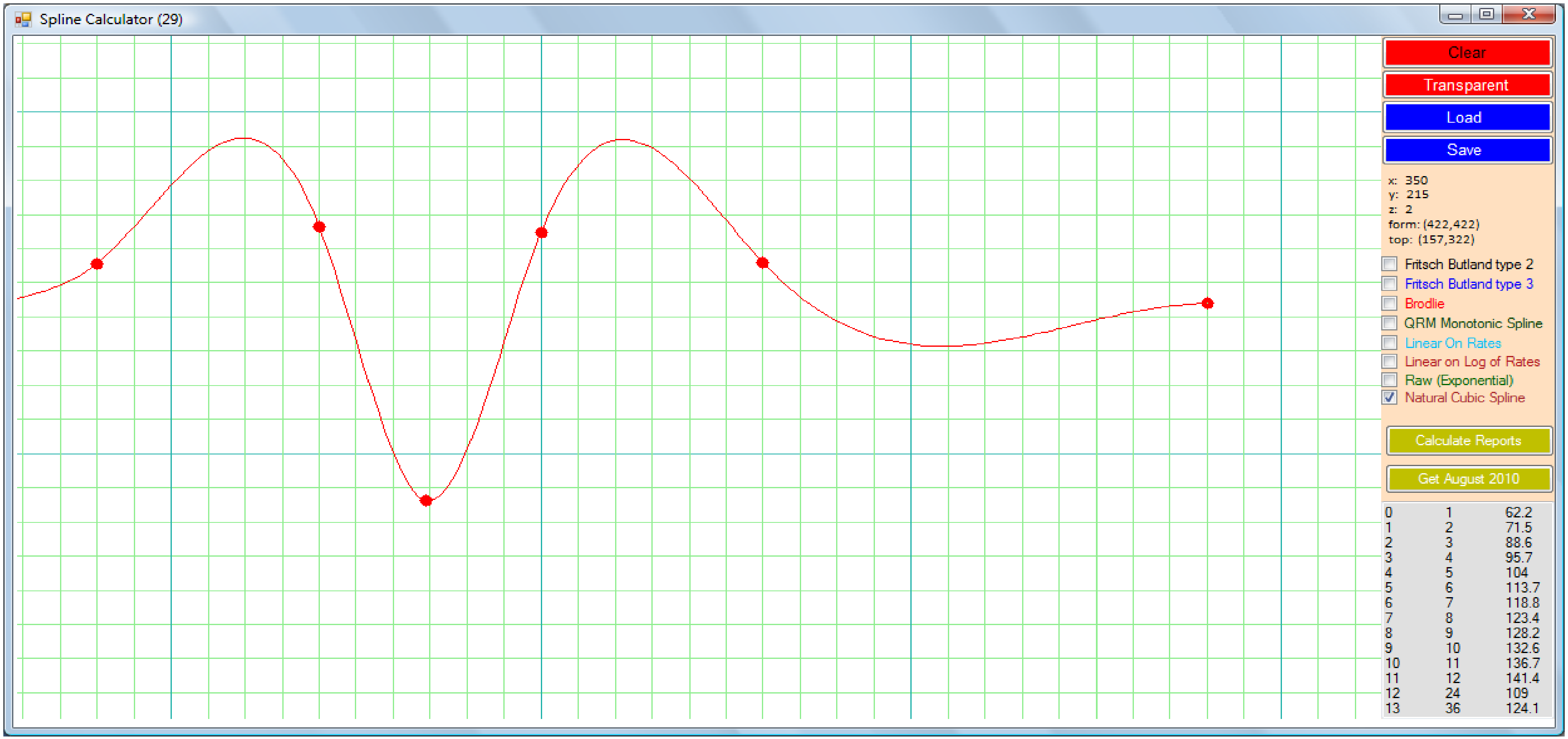


Features of different interpolations

METHODS FOR CONSTRUCTING A YIELD CURVE

Yield curve type	Forwards positive?	Forward smoothness	Method local?	Forwards stable?	Bump hedges local?
Linear on discount	no	not continuous	excellent	excellent	very good
Linear on rates	no	not continuous	excellent	excellent	very good
Raw (linear on log of discount)	yes	not continuous	excellent	excellent	very good
Linear on the log of rates	no	not continuous	excellent	excellent	very good
Piecewise linear forward	no	continuous	poor	very poor	very poor
Quadratic	no	continuous	poor	very poor	very poor
Natural cubic	no	smooth	poor	good	poor
Hermite/Bessel	no	smooth	very good	good	poor
Financial	no	smooth	poor	good	poor
Quadratic natural	no	smooth	poor	good	poor
Hermite/Bessel on rt function	no	smooth	very good	good	poor
Monotone piecewise cubic	no	continuous	very good	good	good
Quartic	no	smooth	poor	very poor	very poor
Monotone convex (unameliorated)	yes	continuous	very good	good	good
Monotone convex (ameliorated)	yes	continuous	good	good	good
Minimal	no	continuous	poor	good	very poor

TABLE 1. A synopsis of the comparison between methods

Linear interpolation on interest rates with negative forwards

Known rates:

- $R_1 = 8\%$ & $R_2 = 5\%$

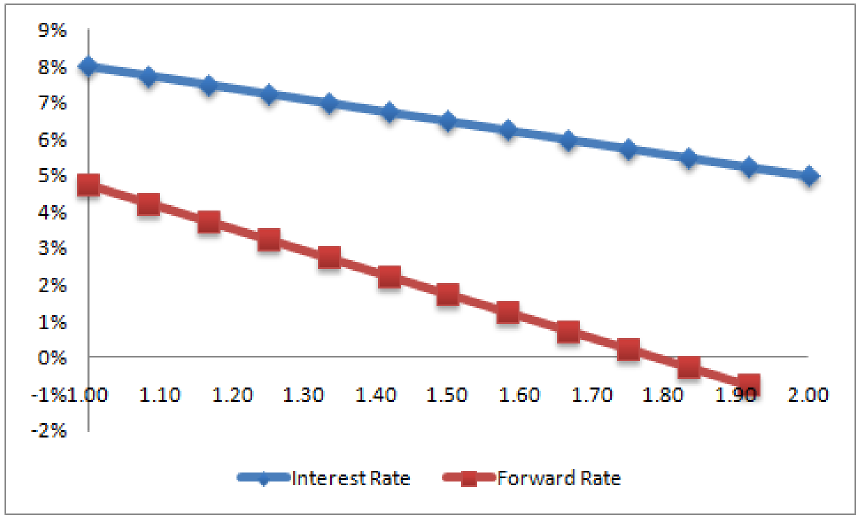
Linear interpolated rates:

- $R_{1.08} = 7.75\%$ … $R_{1.92} = 5.25\%$

Forward rates derived from interest rates as below:

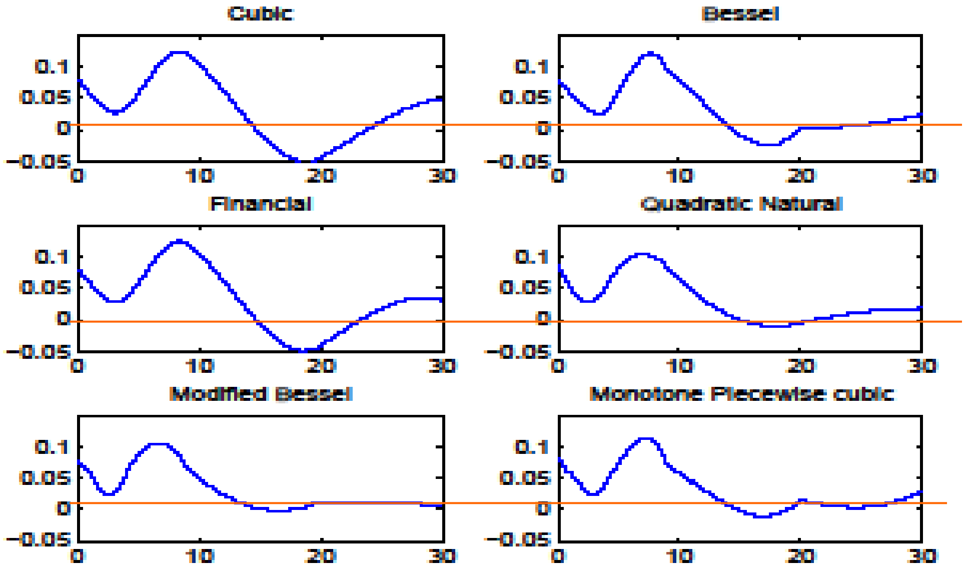
$$f(t_1, t_2) = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$$

However, the forward rates after 1.8 year are negative!



Cubic interpolations with negative forwards

- Arbitrage free environment



The forward curves under various cubic interpolation methods for the given rates

- Reference: Hagen and West, interpolation methods for curve construction, 2006.

Instantaneous forwards vs Monthly fowards

Where are instantaneous forward rates used in practice?

- Commonly used forward rates: 1month, 3month, 1year…

Method:

Change continuous compounding to monthly compounding

E.g. 1 month spot forward rates, R(t)

$$R(t) = 100 * \frac{\left(\frac{DF_{0,t}}{DF_{0,t+1}} - 1\right)}{\tau(t, t+1)}$$

- If the DFs are monotonically decreasing, R(t) is positive! (Monotonic piecewise cubic interpolation on DFs, linear on DFs…)

Application of Monotonic cubic spline interpolation

- **Two options of applying Monotonic cubic spline (MCS)**
- *Option 1. MCS on term structure (yields)*
- *Option 2. MCS on capitalization factor*

What is capitalization factor(CF)?

$$CF(t) = -\ln\left(DF(t)\right)$$

- If it is continuous compounding,
- So, $DF(t) = e^{-rt}$ $\quad f(t) = \frac{\partial}{\partial t} r(t) t$

$$CF(t) = rt$$

- If CF(t) is twice differentiable, once differentiable function of CF(t) is continuous. Therefore, forward rate f(t) is continuous.

MCS on TS vs MCS on CF (1)

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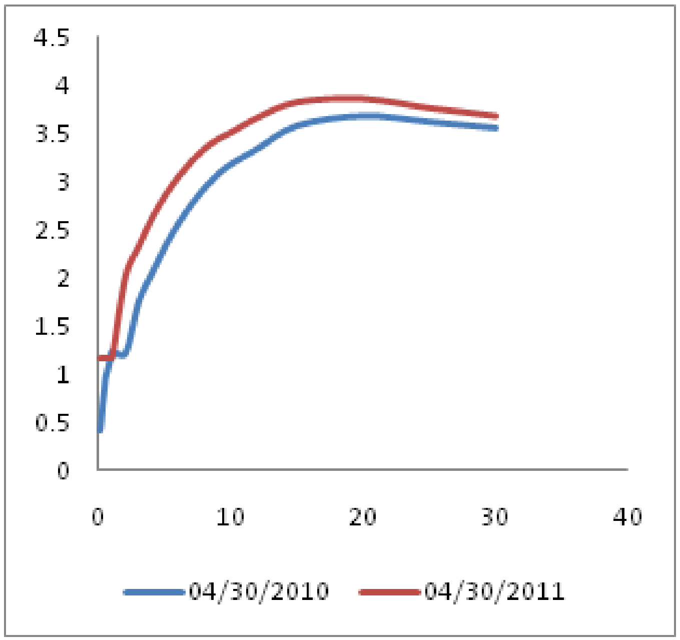
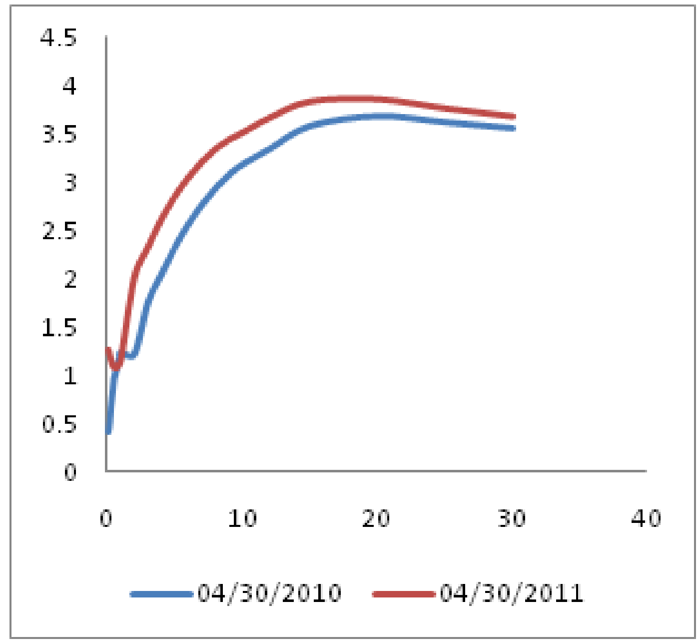
Forward rates in 12 month (MCS on TS)



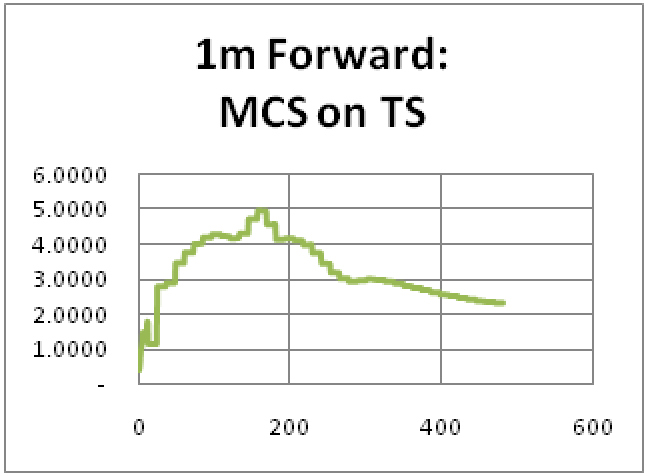
Forward rates in 12 month (MCS on CF)



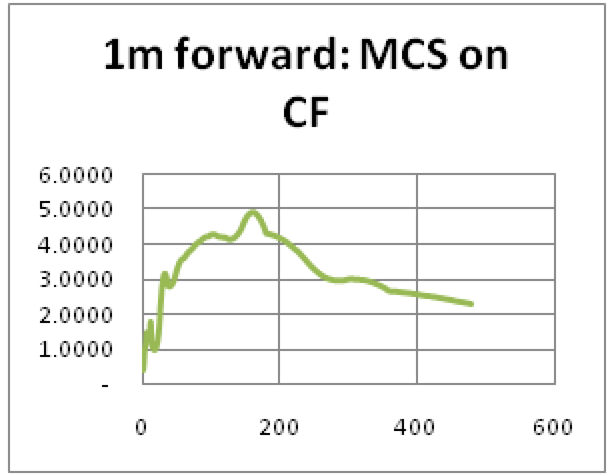
Forward rates in 12m horizon based on MCS on TS have a relatively Flat curve at the first year due to linear interpolation on non-payment date discount factors

MCS on TS vs MCS on CF (2)

1m Forward(MCS on TS)



1m Forward (MCS on CF)



The method MCS on CF gives smoother 1 month forward curve than that on TS. 1 month forward rates can be used for variable rate mortgage calculation.

MCS on TS vs MCS on CF (3)

- **Conclusion**

- MCS on TS is easier and more straightforward than MCS on CF.

- But MCS on CF generates better forward curves than MCS on TS.

No arbitrage equilibrium vs Limited arbitrage

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- Arbitrage: Profit without risk ("free lunch")
- But equilibrium prices offer no arbitrage opportunities

Underlying hypothesis:

- No transaction cost;
- Full information;
- Demand and supply can achieve equilibrium

However, there are limits:

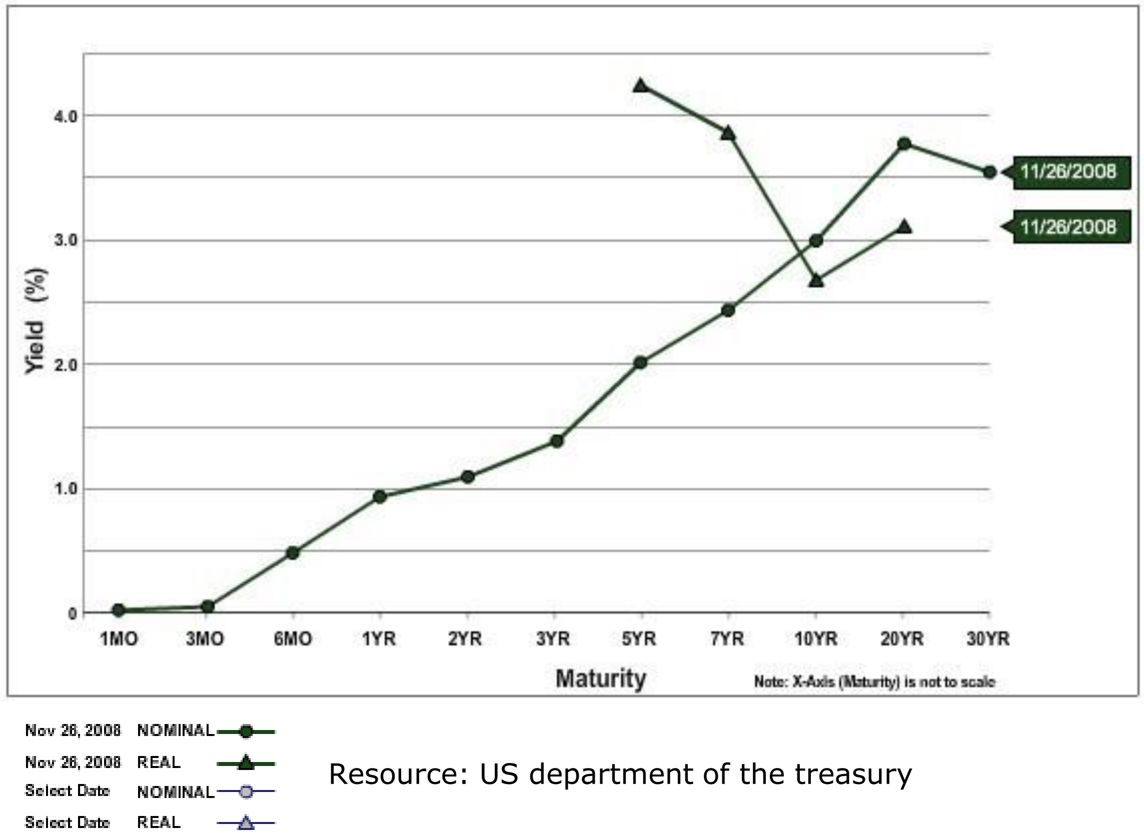
- High transaction cost;
- Lack of information;
- Limited assets(UK)

→ Limited arbitrage

Limited arbitrage example: US treasury yield curve

Strongly inverted yield leads to non-monotonic discount factors

Term	Discount Factors
5Y	0.812508637
7Y	0.767117291
10Y	0.767611372
20Y	0.736908129



Resource: US department of the treasury

Negative forwards undesirable

- ***Correction Mechanism found during validation***
- simply to keep the discount factors non-increasing and make sure that these forward rates are greater than a certain level, e.g. 0.02bp

Pros vs Cons (1)

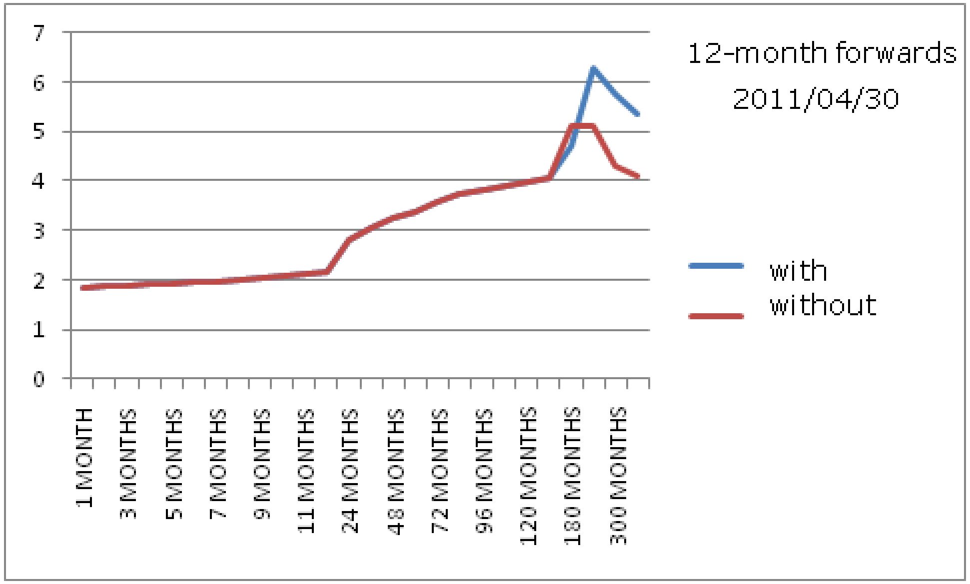
- ***Correction mechanism***
- **Pros**
- Simple and no negative forwards after correction

- **Cons**
- Change the shape of the yield curve
- Unnecessary adjustments are possible, e.g. 1m negative forwards happen after 20 years, but 1m forwards after 20 years won't be used if you have a variable rate mortgage of 10 years
- Shape of other positive forward curves are changed (only the 1 month spot forward rates get the most impact by the non-monotonic discount factors and other forward curves may not receive negative forward rates at all)

Pros vs Cons (2)

Correction mechanism 1

e.g. Forward rates in 12 month horizon



No negative forwards without correction!

But shape of forwards are changed after correction!

No difference till 120 term point

Reason:

the negative 1 month spot forward rates are prevented, because the correction mechanism1 corrects the discount factors; This then also impacts the 12-month forward curve

Alternative (1)

- ***Alternative correction mechanism:***
- Set a floor on the forward rates, e.g. 0.01bp. If negative forwards occur, these negative forwards will be automatically changed into positive values (the floor)

Alternative (2)

- **Pros**
- Simple and keep the shape of the yield

- **Cons**
- Identify negative forwards on the individual product level

Advices from validation

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- RMVM advises to apply MCS based on the theoretic research
- RMVM advises ALM to base MCS on the Capitalizations Factors, not on the Term Structure as this is considered a better option.
- RMVM and ALM advise the software vendor to give a clearer signal when the correction mechanism is used to prevent negative forward rates.
- RMVM suggests that vender software offer the option for users to choose not to apply correction mechanism after the warning signal shows up.

Questions

- Thank you!

Appendix- formulas (1)

- Periodic compounding:

$$FV\ of\ 1euro = (1euro * \frac{annual\ rate}{n})^{n*T}$$

- Continuous compounding:

$$FV\ of\ 1euro$$
$$= 1euro * \lim_{n \to \infty} \left(1 + \frac{annual\ rate}{n}\right)^{n*T}$$
$$= 1euro * e^{annual\ rate*T}$$

Appendix- formulas (2)

$$DF(0,t_1) * DF(t_1,t_2) = DF(0,t_2)$$

$$e^{-r_1 * t_1} * e^{-f_{t_1,t_2} * (t_2 - t_1)} = e^{-r_2 * t_2}$$

$$f_{t_1,t_2} = \frac{r_2 * t_2 - r_1 * t_1}{t_2 - t_1}$$

- Continuous compounding:

$$f_t = \lim_{t_2 - t_1 \to 0} f_{t_1,t_2} = \lim_{t_2 - t_1 \to 0} \frac{r_2 * t_2 - r_1 * t_1}{t_2 - t_1}$$

$$= \partial r(t) t / \partial t$$

Monotonic cubic spline on term structure (1)

- There are 4 steps to obtain discount factors(DF):
- *Step 1* For spot rates with the structure term of one year or less(i<=12)

$$DF_i = (1 + r_{s,i} * \frac{actual}{360})^{-1} \quad (1)$$

- *Step 2* The unknown coupon rates at the payment date(e.g. i=132) are interpolated by MCS using the same formulas which are used for MCS on CF method.
- *Step 3* After step 2, all the par coupon rates at the payment date (i=24, 36…) are available for calculating discount factors. The discount factor is calculated by the formula (2).

$$DF_i = \frac{1 - r_{c,i} * \sum_{n=12,24\ldots}^{i-12} DF_n}{1 + r_{c,i}} \quad (2)$$

Monotonic cubic spline on term structure (2)

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- *Step 4* For the par coupon rates at the non-payment date (e.g. 13month), discount factors are calculated based on the previous end year's discount factor (e.g. DF_{13}) and the coming end year's discount factor (e.g. DF_{24}) by the formula (3).

$$DF_i = DF_s * \left(\frac{DF_u}{DF_s}\right)^{act(s,i)/act(s,u)} \qquad (3)$$

  where s=12, 24… , u=24, 36, and s< i <u.

- MCS on Term Structure uses linear interpolation to obtain the non-payment date discount factors.

Monotonic cubic spline on capitalization factor(1)

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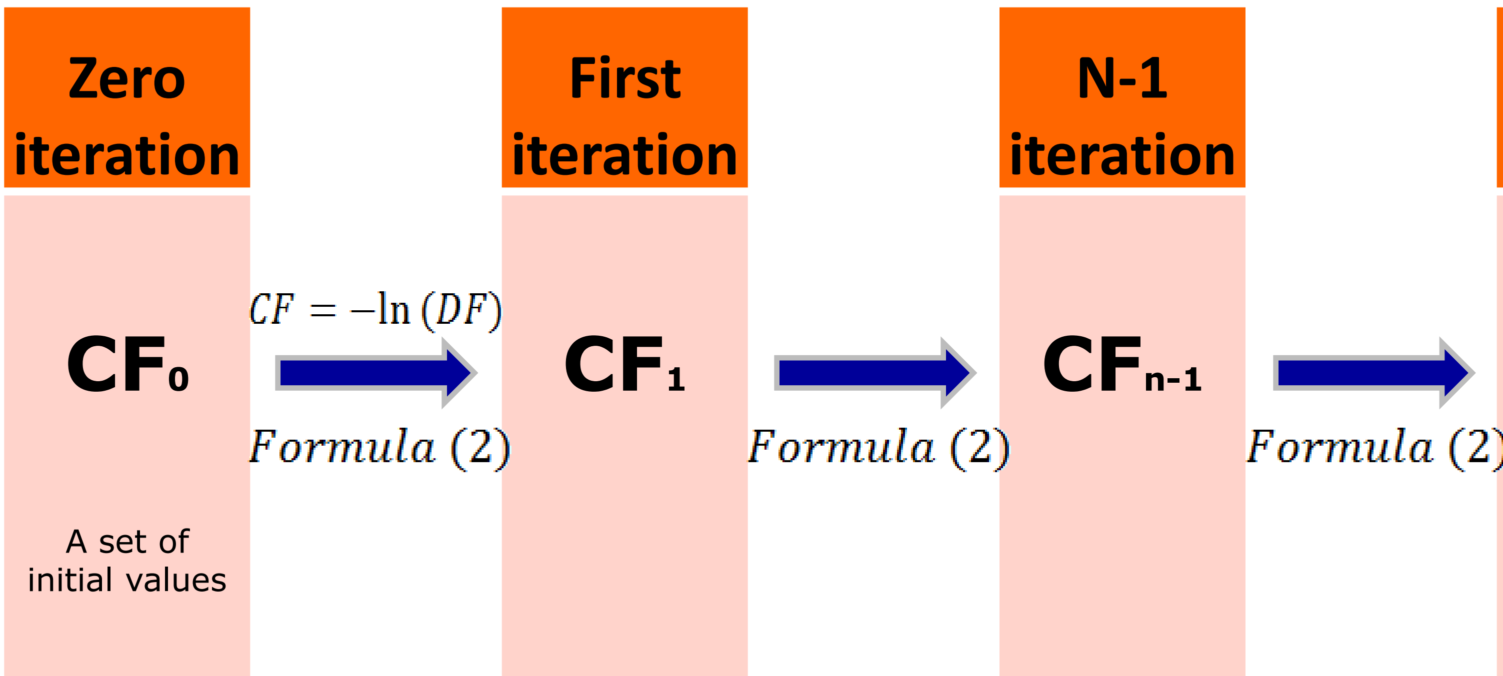
- Discount factors within 1 year: Same method with MCS on TS
- Normally, the discount factors after 1 year at the payment date (i=24, 36, …) is calculated by the formula (2) based on the corresponding coupon rate and the previous discount factors at the end of each year

$$DF_i = \frac{1 - r_{c,i} * \sum_{n=12,24\ldots}^{i-12} DF_n}{1 + r_{c,i}} \quad (2)$$

- However, not every end year's coupon rate is available, for example 132 month' coupon rate. Therefore, the discount factors at payment dates are found though rounds of iterations using the formula above and interpolation

Monotonic cubic spline on capitalization factor(2)



$$DF_i = \frac{1 - r_{c,i} * \sum_{n=12,24\ldots}^{i-12} DF_n}{1 + r_{c,i}} \quad (2)$$