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Computing maximally smooth forward rate curves for coupon bonds

An iterative piecewise quartic polynomial interpolation method

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Abstract We present a simple and fast iterative, linear algorithm for simultaneously stripping the coupon payments from and smoothing the yield curve of the term structure of interest rates. The method minimizes pricing errors, constrains initial and terminal conditions of the curves and produces maximally smooth forward rate curves.

Keywords term structure of interest rates · yield curve · coupon stripping · curve interpolation

1 Introduction

In this paper we present an algorithm to construct maximally smooth forward rate and discount curves from the term structure of on-the-run U.S. Treasury bills and bonds. The maximum smoothness criterion produces more accurate prices for derivatives such as swaps (Adams 2001) and ensures that no artificial arbitrages will be introduced when using the constructed forward curve for pricing securities beyond those used to construct the curves.

Our method uses a piecewise, quartic polynomial interpolation of the forward curve based upon Adams and van Deventer (1994) as corrected by Lim and Xiao (2002). Their method, however, works only for zero coupon bonds so we extend the method to work for coupon bonds by simultaneously stripping the coupons and interpolating the spot curve. It is critical that these steps be done simultaneously in order to maintain consistency (Hagan and West 2006).

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The interpolation step alone can be accomplished with a linear algorithm but the inclusion of simultaneous stripping leads to a highly nonlinear problem. Our approach relies on an iterated linear algorithm that is simple to implement and very fast to compute while maintaining minimal pricing errors and maximum smoothness of the interpolated curves. We also correct some minor problems in Lim and Xiao (2002) related to the terminal conditions of spot and forward rate curves.

Our method computes the smoothest possible forward curve among the class of polynomials which is shown to be a, fourth-order polynomial constrained to have continuous second derivatives at all node points. Additional constraints fix the initial value $f(0)$ and impose the terminal slope of the forward curve to be zero. It is also possible to include other constraints such as non-negativity of the forward rates if necessary. The coefficients of the polynomials are chosen so as to minimize the pricing errors on the observed securities. The spot curve and discount functions are derived from the forward curve as described in Section 2. Details of the algorithm are discussed in Section 3.

The primary complication, and what makes modeling the term structure an interesting problem, is that we only observe prices and yields-to-maturity of a finite set of securities from several different markets including on-the-run and off-the-run treasuries, corporate bonds, LIBORs, SWAP rates, and various derivatives such as TIGR strips. Each of these markets has its own liquidity and risk characteristics that must be accounted for when constructing the discount function. Our choice is to use on-the-run treasuries because they are the most liquid and they have very low and uniform credit risk.

In the U.S. Treasury market we observe zero-coupon Treasury bills of maturities one, three, six and twelve months and semi-annual coupon paying Treasury bonds of maturities two, three, five, ten and thirty years. Generally this list of securities is supplemented with the Federal Funds rate or REPO rates of various maturities such as one day and one week in order to observe yields closer to zero maturity. We add the one-week LIBOR and then use this and the one month T-Bill rate to interpolate backward to the implied zero maturity rate $y_t(0)$. This approach provides us with a good estimate of the very short end of the spot curve without having to deal with the high daily volatility in overnight REPO rates.

Table 1 presents the yield curve data for July 10, 2008 for the on-the-run U.S. Treasuries. The first column of gives the maturity date of the bill or bond in decimals; the second column gives the annualized coupon rate (paid semi-annually) and is zero for the pure discount bills; the third column gives the quoted prices[1]; column four gives the yield-to-maturity using the appropriate day count conventions; and the final column reports the Macaulay duration of

[1] The quoted prices are *dirty prices* which is the actual cost of the bond and includes the accrued interest due to the current bond holder. Some sources quote the *clean prices* which do not include the accrued interest since the previous coupon payment.

Table 1 Observed yield curve data for on-the-run U.S. Treasuries on 7/10/2008: maturity date, Coupon rate, quoted price, yield-to-maturity, and Macaulay duration. The second row is the one week LIBOR and the first row is the $y(0)$ spot rate implied by backward linear interpolation.

Maturity	Coupon	Price	Yield	Duration
7/10/2008	0	100.00	1.426	0
7/17/2008	0	99.97	1.435	0.0192
8/7/2008	0	99.89	1.462	0.0767
10/9/2008	0	99.59	1.670	0.2492
1/8/2009	0	99.01	2.007	0.4983
7/2/2009	0	97.90	2.194	0.9774
6/30/2010	2.875	100.88	2.418	1.9315
6/30/2013	3.375	101.30	3.091	4.6271
5/15/2018	3.875	100.52	3.811	8.3114
2/15/2038	4.375	99.28	4.4186	17.0089

each security computed as

$$D = \frac{1}{P}\left(\sum_{i=1}^{n} \frac{t_i\, C/2}{(1+ytm)^{t_i}} + \frac{t_n\, 100}{(1+ytm)^{t_n}}\right) \tag{1}$$

where P is the bond price, $C/2$ is one half of the semi-annually paid coupon, ytm is the yield-to-maturity of the bond, and there are n coupon payments plus a face value of 100 paid at maturities $t_i,\ i = 1, \dots, n$.

From these ten observations we need to compute continuous discount, spot and forward rate functions. The first six securities are zero-coupons so their yield-to-maturities are already the appropriate spot rates and only interpolation between these points is required. The last four securities are coupon bonds so the coupon payments need to be stripped from the cash flows to compute the appropriate spot rates.

Figure 1 displays the spot curve and instantaneous forward rate curves computed by our algorithm for the the data given in Table 1. The spot curve is the solid black line, the instantaneous forward rate curve is the dashed line, and the observed yields are indicated by the small circles. The top figure shows all maturities and the bottom figure zooms in on the short maturities of the curves.

The figure reveals several important properties of the curves. First, it is clear that the forward curve is much more volatile than the spot curve. This is the reason that we focus on modeling the forward curve rather than the spot or discount curves. If we can produce a smooth forward curve then we are assured of producing smooth spot and discount curves. The particular term structure produces a maximum spot rate at about the twenty year maturity so our forward curve intersects the spot curve at that point. Note that the spot and forward curves begin at the same point at the zero maturity date and both curves are very stable at long maturities. This latter property is non-trivial and often not observed in polynomial interpolation methods.

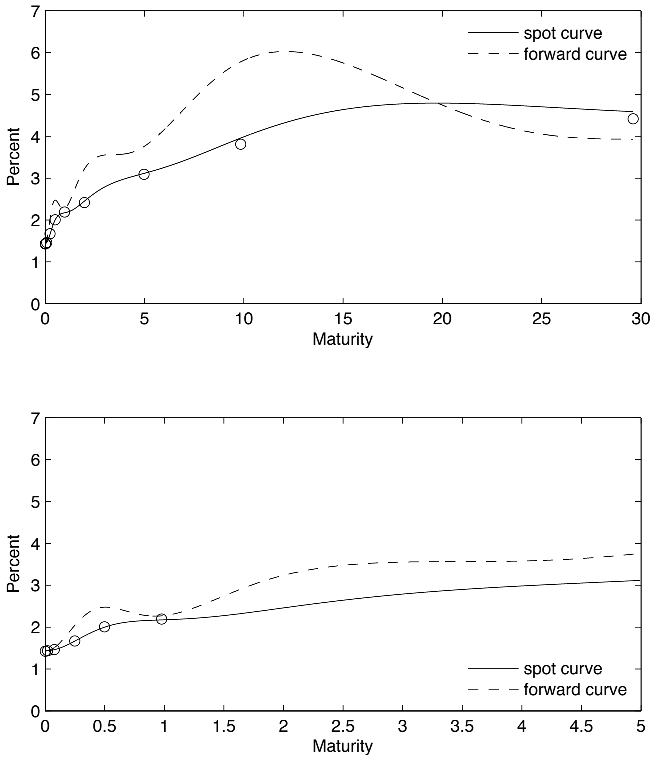

Fig. 1 Computed spot and forward rate curves for data in Table 1. Circles indicate the observed yields. The bottom figure shows the curves for short maturities.

2 Background

Before describing our algorithm it will be useful to first review some of the analytics of the term structure of interest rates so that we can establish notation and better understand the constraints that we need to impose on our curve stripping and interpolation problem.

Let $P_t(T)$ be the period t price of a bond that matures at time $t+T$ with a face value of \$1 and pays a coupon of C at dates t_i, $t \leq t_1 < \cdots < t_n \leq t+T$. The continuous compounding yield-to-maturity of this bond is the rate ytm which solves

$$P_t(T) = 1 \cdot \exp(-ytmT) + \sum_{i=1}^{n} C_i \exp(-ytm\, t_i). \qquad (2)$$

The coupon payments, since they may come relatively soon, have a large impact on the bond price and the computed yield-to-maturity.

In the special case where the bond is a pure discount bond which pays no coupons and \$1 at time $t + T$, we use the notation

$$\delta_t(T) = \exp(-y_t(T)\,T) \tag{3}$$

or alternatively,

$$y_t(T) = -\frac{1}{T}\ln(\delta_t(T)), \forall\, T \geq 0, \tag{4}$$

where $\delta_t(T) \in (0, 1]$ is the discount rate at time t for a \$1 payoff at time $t + T$, and $y_t(T)$ is the time t spot rate for maturity date $t + T$. In this case, since there are no coupon payments before maturity, the yield-to-maturity is identical to the spot rate and the discount rate is the price of the zero-coupon bond. In contrast, the yield-to-maturity of a coupon paying bond is not necessarily equal to the spot rate $y_t(T)$ and the bond price will not equal the discount rate for maturity T. Thus, as we see in Figure 1 for the ten-year bond, for example, the spot curve may not go through the yield-to-maturity's of the bonds.

Define $F_t(n, T)$ to be the time t price of a forward contract to deliver a $T - n$ maturity pure discount bond at date $t + n$ that matures at date $t + T$. To avoid arbitrage, it must be the case that

$$F_t(n, T) = \frac{\delta_t(T)}{\delta_t(n)}. \tag{5}$$

Let $f_t(n, T)$ be the continuously compounded yield to maturity of this forward contract where

$$F_t(n, T) = \exp\left(-f_t(n, T)(T - n)\right). \tag{6}$$

Usng (3), (4) and (5) implies

$$f_t(n, T) = \frac{1}{T - n}\left(y_t(T)T - y_t(n)n\right). \tag{7}$$

In the limit, as $n \longrightarrow T$, we get the *instantaneous* forward rate at time t for maturity T as

$$f_t(T) = \frac{d}{dT}\left(y_t(T)T\right) = y_t(T) + T\,\frac{d}{dT}\,y_t(T). \tag{8}$$

The forward curve is above the spot curve when spot rates are increasing with maturity (the normal yield curve) and below the spot curve when spot rates are decreasing with maturity (an inverted yield curve). Note that the forward curve intersects the spot curve at any maxima or minima of the spot curve. If the yield curve is flat then the forward rate equals the spot rate for all maturities. Also, note that $f_t(0) = y_t(0)$ so that the spot curve and the instantaneous forward curve start at the same point.

The spot rate $y_t(T)$ is essentially an average of the forward rates up to maturity T. To see this note that

$$\int_0^T f_t(\tau)\,d\tau = \int_0^T d\,(\tau\, y_t(\tau)) = T\, y_t(T) \tag{9}$$

or

$$y_t(T) = \frac{1}{T}\int_0^T f_t(\tau)\,d\tau. \tag{10}$$

Thus, the forward curve is more volatile than the spot curve, which is evident in our computed curves in Figure 1.

Our task is to find the discount function $\delta_t(T)$, the spot function $y_t(T)$ and the instantaneous forward function $f_t(T)$ for all $T \geq 0$ such that they are consistent with observed market data and satisfy the relationships

$$\delta_t(T) = \exp(-y_t(T)T) = \exp\left(-\int_0^T f_t(\tau)\,d\tau\right) \tag{11a}$$

$$f_t(T) = y_t(T) + T\,\frac{d}{dT}y_t(T) = -\frac{\frac{d}{dT}\delta_t(T)}{\delta_t(T)} \tag{11b}$$

$$y_t(T) = \frac{1}{T}\int_0^T f_t(\tau)\,d\tau = -\frac{1}{T}\ln\delta_t(T) \tag{11c}$$

$$y_t(0) = f_t(0). \tag{11d}$$

It is also reasonable to expect that the spot and forward curves should be well-behaved as maturities increase beyond that of the longest observed security. This is particularly important if we need to price, say, a forward contract on a thirty-year bond five or ten years hence. Thus, we assume that the spot rate, and therefore the forward rate, approaches a positive asymptote:

$$\lim_{T\to\infty} y_t(T) = \lim_{T\to\infty} f_t(T) = y_t^{\infty}. \tag{12}$$

Again, this property is evident in our curves in Figure 1.

Finally, when we model the term structure we assume that current prices should not permit any arbitrage opportunities among the securities. The no-arbitrage condition implies that the discount function must be monotonically decreasing which, in turn, implies that the forward curve must be everywhere positive. We see from Figure 1 that our forward curves satisfy the no-arbitrage condition.

3 Iterative Piecewise Quartic Polynomial Interpolation Algorithm

Our goal is a simple and fast algorithm to compute the forward, spot and discount rates associated with the current on-the-run Treasury yield curve. The method of Adams and van Deventer (1994) and Lim and Xiao (2002) is linear in the unknown coefficients but only when all securities are zero coupon bonds.

As we will see, when coupon bonds are introduced the algorithm becomes nonlinear. Our approach is to modify the Lim and Xiao (2002) algorithm and deal with the nonlinearity by iterating over a sequence of linear problems. Compared to other nonlinear algorithms, such as Manzano and Blomvall (2004) and Hagan and West (2006), our approach is simple to code, fast and very stable.

To make the description of the algorithm as concrete as possible we will refer to the specific data from Table 1 where there are nine observed securities in the yield curve plus the initial condition on the settlement date. The first five securities are zero coupon bonds with maturities from seven days to about one year. The last four securities are the coupon paying bonds with maturities between two and thirty years.

We will denote the maturities of the securities as $T_i,\ i = 1, \ldots, m$ and the price of the bond with maturity T_i as $P(T_i)$. The yield-to-maturities are computed according to the appropriate day count conventions for the specific bills and bonds. To simplify our presentation we will use yields based upon continuous compounding with actual day counts.

A bond is just a sequence of cash flows on specific days. In our example there are 10,812 days between the settlement date of July 10, 2008 and the maturity of the thirty year bond on February 15, 2038. Let $Z(T_i)$ denote the 10,812 element cash flow vector for the bond maturing at date T_i. For our first five securities, the zero coupons, $Z(T_i)$ will contain all zeros except for the value of \$100 at day T_i. For example, $Z(T_3)$ contains \$100 in the 91^{st} element—the days to maturity of the three month T-Bill maturing on October 9, 2008. The cash flow vector for bonds includes the semi-annual coupon payments as well as the final face value payment. For example, the two year 2.875 coupon bond maturing on June 30, 2010 has cash flows of $\$2.875/2 = \$1.4375$ on days $\{174, 355, 539\}$ and a cash flow of \$101.4375 on maturity day 720.

Using the notation from Section 2, let δ, y and f be the discount rate, spot rate and instantaneous forward rate functions to be computed. Our method computes the forward curve as a , piecewise, quartic polynomial function and then derives the spot curve and discount function from it. Given estimates of these functions, we could compute the price for each security as

$$\widehat{P(T_i)} = \hat{\delta}'\, Z(T_i),\ i = 1, \ldots, m, \tag{13}$$

where $\hat{\delta}$ is the 10,812 element vector of discount rates computed from the forward function $\hat{f}$.

The smoothest function is the one that has the minimum acceleration or smallest absolute second derivatives over its range. A straight line is very smooth but would not be a good choice for the forward curve because it would create large pricing errors. The optimization problem is

$$\min_f \int_0^{T_m} \left(f''(t)\right)^2\, dt \tag{14}$$

subject to the pricing constraints $P(T_i) = \widehat{P(T_i)},\ i = 1, \ldots, m$, the initial condition $f(0) = y(0)$ ($= 1.426$ in our example) and the terminal condition

$f'(T_m) = 0$. The last condition ensures that the forward and spot curves are well-behaved beyond the maximum observed maturity.

Adams and van Deventer (1994) use variational calculus to show that the solution to this optimization problem is a quartic polynomial of the form

$$f(t) = at^4 + bt + c \tag{15}$$

where the coefficients on the cubic and quadratic terms are constrained to be zero. However, Lim and Xiao (2002) point out that the derivation of Adams and van Deventer (1994) neglected to take into account that the pricing constraint also involves the forward function f. With this correction, they find the solution to be an unconstrained quartic function

$$f(t) = at^4 + bt^3 + ct^2 + dt + e. \tag{16}$$

The implications of this error are substantial because there are now two additional parameters to estimate for each piecewise quartic polynomial. The exact identification of parameters that Adams and van Deventer (1994) used will no longer be possible so we may not be able to exactly price all bonds on our yield curve.

Lim and Xiao (2002) introduce a minor error into their algorithm with an incorrectly constrained terminal condition of the forward curve. Assume that there are m bonds in our yield curve with maturities $\{T_1, \ldots, T_m\}$. We impose the constraint $f'(\infty) = 0$ by adding an $(m+1)^{st}$ spline so that $f(t) = \textit{constant}$ for all $t > T_m$. We may think of the the maturity of this additional segment as being $T_{m+1} = \infty$. We can see how this terminal constraint works by examining the top graph in Figure 1. In this example, at the maturity of thirty years, the longest maturity bond in our sample, the spot rate has a slightly negative slope and the forward rate has smoothly approached a constant value of 3.93 percent. From the identity $y_t(T) = f_t(T) + T\, y_t'(T)$, the spot curve will now fall monotonically toward the constant forward curve for $T > T_m$.

Since we are using piecewise polynomials, it is useful to define the indicator function

$$I_{[T_i, T_{i-1}]}(t) = \begin{cases} 1 & \text{if } T_{i-1} \le t \le T_i \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

so that we may write the forward curve polynomial as

$$f(t) = \sum_{i=1}^{m+1} I_{[T_i, T_{i-1}]}(t)\, f_i(t) \tag{18}$$

where $f_i(t) = a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i$. In our algorithm, if we have m bonds and $m+1$ quartic polynomial splines, there will be $5(m+1)$ coefficients to estimate.

Since $f_i(t)$ is linear in the coefficients, $\int f''(t)\, dt$ will be quadratic in the coefficients so the objective function (14) may be written as a quadratic form $X'HX$ where X is a $5(m+1)$ vector of coefficients and H is an $5(m+1) \times 5(m+1)$ known matrix that is specified in Appendix A. Thus, the first-order conditions with respect to the coefficient vector X will be linear in the unknowns.

To ensure maximum smoothness of the forward curve, (14) is optimized subject to $4m+5$ constraints:

(i) m pricing constraints

$$P(T_i) = \widehat{P(T_i)} = \hat{\delta}'\, Z(T_i),\ i = 1, \ldots, m$$

(ii) m continuity conditions at the nodes

$$f_{i+1}(T_i) = f_i(T_i),\ i = 1, \ldots, m;$$

(iii) m differentiability conditions at the nodes

$$f'_{i+1}(T_i) = f'_i(T_i),\ i = 1, \ldots, m;$$

(iv) m twice differentiability conditions at the nodes

$$f''_{i+1}(T_i) = f''_i(T_i),\ i = 1, \ldots, m,$$

(v) an initial boundary condition $f(0) = y_0$; and
(vi) four restrictions on the terminal polynomial so that $f'(\infty) = 0$.

Since there are $5(m+1)$ coefficients to estimate, we will have m more coefficients than constraints.

Recall that Adams and van Deventer (1994) incorrectly have two additional restrictions per polynomial so they have only $3(m+1)$ coefficients to estimate. They drop the twice-differentiability condition (iv) and there are only two terminal polynomial restrictions for them so that they have $3m+3$ restrictions and their model is exactly identified. This explains how they are able to produce zero pricing errors on all of their m zero coupon bonds. Note, however, that their incorrectly restricted polynomials will produce a forward curve that does not have the maximum smoothness property that they desire.

Introducing coupon bonds into the Adams and van Deventer (1994) algorithm also introduces a nonlinearity into the algorithm. To see this consider the pricing constraints (i). From (13) we see that the coupon paying bond price is linear in the discount function δ but from (11a) δ is nonlinear in the forward function f and therefore in the coefficients of the polynomials. For zero coupon bonds we observe the spot rate and the discount rate at that maturity so we can avoid this nonlinearity by using the observed values. When we add coupon bonds, however, the spot and discount rates are no longer observed so we cannot avoid this nonlinearity. As it turns out, this nonlinearity is nontrivial and often leads to unstable algorithms. One solution is to strip the coupons from the coupon bonds in a prior step and then apply a linear algorithm to the stripped securities. Although simple, we will demonstrate below that this approach leads to inconsistent spot and forward curves.

Our approach is to interpolate and to strip simultaneously but to maintain stability by using an iterative linear algorithm. A critical step involves *approximating* the spot rate $y(T_i)$ for the bond maturing at T_i, stripping the value of the coupon payments from the bond and then pricing the bond face value using the zero-coupon bond price given by (11a) so that the log of the zero coupon bond price is linear in the polynomial coefficients. We may then solve our nonlinear optimization problem using a sequence of linear steps. The details of the linear steps of the piecewise quartic polynomial interpolation (PQPI) algorithm are similar to those of Lim and Xiao (2002) and are described in the Appendix A along with our corrections for the terminal conditions. In the next section we describe the iterative algorithm.

3.1 The Iterative Algorithm

Pseudocode for the main iterative algorithm is shown in Figure 3.1. After reading the yield curve data from Table 1 and storing the vector of node points, we first compute the forward curve over the observed zero coupon bills. In our example this includes the first five securities up to maturity July 2, 2009. Since these are all zero coupon bills we can do this with a slightly modified version of the linear Lim and Xiao (2002) PQPI algorithm.

Next we add the first coupon bond, the sixth security or the two year 2.875 coupon bond maturing on June 30, 2010, to our list of securities. The yield of 2.418% for this bond includes the coupon payments so we must strip the coupons and compute the spot rate, $\tilde{y}(T_6)$ at this maturity date T_6. We do this by first getting an initial estimate of $\tilde{y}(T_6)$ using a simple linear bootstrap method. Using this estimated value, we use PQPI to compute an estimate of the forward curve $\tilde{f}$ up through maturity T_6. Using $\tilde{f}$ we compute the estimated price $\tilde{P}(T_6)$ of the two year bond. This method will not produce an estimate that is consistent with the interpolation up to T_{i-1} from the previous step since some of the coupon payments of the two year bond will occur before T_{i-1}, T_5, in our example.

To determine which direction to adjust $\tilde{y}(T_6)$ we perturbe $\tilde{y}(T_6)$ up and down by $\tilde{y}(T_6)/100$ to get $\tilde{y}_u$ and $\tilde{y}_d$. From these perturbations we compute $\tilde{f}_u$ and $\tilde{f}_d$ and then $\tilde{P}_u$ and $\tilde{P}_d$. We then approximate the derivative $d\tilde{P}(T_6)/d\tilde{y}(T_6)$ using the centered difference method

$$\frac{d\tilde{P}(T_6)}{d\tilde{y}(T_6)} \approx \frac{\tilde{P}_u - \tilde{P}_d}{\tilde{y}_u - \tilde{y}_d} = \frac{\tilde{P}_u - \tilde{P}_d}{\tilde{y}(T_6)/50}. \tag{19}$$

We now update our estimate of the spot rate at T_6 using

$$\tilde{\tilde{y}}(T_6) = \tilde{y}(T_6) + \frac{d\tilde{y}(T_6)}{d\tilde{P}(T_6)} \left(P(T_6) - \tilde{P}(T_6) \right). \tag{20}$$

We repeat this gradient updating of $\tilde{y}(T_6)$ until the estimated spot rate changes by less than 10^{-9}.

At the conclusion of this iterative process we have updated our forward curve through the maturity of the first coupon bond. We now add the next coupon bond and repeat this process until we have computed the forward curve

Algorithm 3.1: IPQPI($datafile$)

comment: Input data from Table 1.

$$\begin{cases} \{P, T, C, \dots\} \leftarrow input(datafile) \\ \text{Prices, Maturities, Coupons, etc.} \\ \text{There are } m_z \text{ zero coupon bills and } m_c \\ \quad \text{coupon bonds for } m \text{ total securities.} \end{cases}$$

comment: Construct nodes.

$$\begin{cases} \text{The polynomial nodes are at the} \\ \quad \text{security maturity dates:} \\ \quad \{T_1, \dots, T_{m_z}, T_{m_z+1}, \dots, T_m\} \end{cases}$$

comment: Compute f for bills.

$$\begin{cases} \text{Compute forward curve over the } m_z \\ \quad \text{zero coupon bills.} \\ f \leftarrow PQPI\big(y(T_1), \dots, y(T_{m_z})\big) \end{cases}$$

comment: Loop over coupon bonds.

for $i \leftarrow m_z + 1$ **to** m

$$\textbf{do} \begin{cases} \tilde{y}(T_i) \leftarrow bootstrap(f, T_i) \\ \text{Add } \tilde{y}(T_i) \text{ to list of zero yields} \\ \textbf{repeat} \\ \begin{cases} \text{Perturb } \tilde{y}(T_i) \text{ and recompute } f \\ f \leftarrow PQPI\big(y(T_1), \dots, \tilde{y}(T_i)\big) \\ \text{Compute price sensitivity } dP/d\tilde{y} \\ \text{Adjust } \tilde{y}(T_i) \end{cases} \\ \textbf{until } f \text{ correctly prices bonds 1 to i} \end{cases}$$

comment: Compute final curves.

$$\begin{cases} f \leftarrow PQPI\big(y(T_1), \dots, y(T_{m_z}), \\ \qquad \tilde{y}(T_{m_z+1}), \dots, \tilde{y}(T_m)\big) \\ y(T) \leftarrow \frac{1}{T} \int_0^T f(t)\, dt \\ \delta(T) \leftarrow \exp(-y(T)\, T) \end{cases}$$

comment: Compute final pricing errors.

$\text{error}_i \leftarrow (P_i - \delta' \,\text{CashFlows}_i),\ i = 1, \dots, m.$

Fig. 2 Pseudocode for the Iterated Piecewise Quartic Polynomial Interpolation Algorithm

over all securities in our sample. After the final forward curve is computed we calculate the pricing errors and other statistics useful for evaluating the estimated curves.

It is important to note that, although the forward curve is computed many times, each computation is the solution of a linear system of equations as detailed in the PQPI step in Appendix A.

4 Results

The results of our algorithm applied to the yield curve data of Table 1 are show in Figure 1 and Table 2. As expected, the greatest volatility in the forward curve is at the short maturities. Although there are several zero coupon bonds at the shorter maturities, the longer bonds have coupon payments during the

Table 2 Actual and estimated bond prices for the yield curve on date 7/10/2008.

Maturity	Actual Price	Est. Price
7/10/2008	100.0000	100.0000
7/17/2008	99.9725	99.9725
8/7/2008	99.8880	99.8880
10/9/2008	99.5854	99.5854
1/8/2009	99.0092	99.0092
7/2/2009	97.8992	97.8992
6/30/2010	100.8800	100.8805
6/30/2013	101.3000	100.3037
5/15/2018	100.5200	100.5484
2/15/2038	99.2800	99.2800

first year that influence the shape of the forward curve in this region. The bottom graph in Figure 1 shows the forward and spot curves for maturities up to five years. As required by theory, the forward and spot curves start at the same point and the movement of the forward curve reflects the subtle variations required in the spot curve in order to capture the influence of the intervening coupon payments. Most critically, the computed forward curve never goes negative so no false arbitrage signals are produced by the algorithm.

As shown inTable 2, the computed forward curve prices the observed bonds quite accurately. There are zero pricing errors for the zero coupon bonds. While one might expect this since the spot rates for these bonds are directly observed, recall that the forward curve is also influenced by the coupon payments from longer maturity coupon bonds intermingled amongst these bills. Thus, it is a nontrivial result to produce zero pricing errors for the bills when bonds are included in the set of securities. The largest error is 2.84 cents on the ten year bond which amounts to a 0.0285% error. In this particular example, the ten year bond is the most difficult to price because the spot curve becomes inverted between the ten year and thirty year bonds.

5 Evaluation and Comparison

Adams (2001) aptly observes, “Most practitioners judge the quality of a zero curve not by the quality of the underlying mathematics, but by the quality of the curve itself” (p. 19). Evaluating the forward curve, however, is somewhat subjective because there is no unique solution to the curve stripping and interpolation problem. We have chosen to impose maximum smoothness and a terminal zero slope condition on our forward curve because from our experience we believe these to be important criteria in our applications.

There are many other approaches to this problem. To aid in the comparison of our algorithm to others we give a very brief overview of the most critical of these alternative algorithms below. For a more detailed discussion see Fisher (2004) and Jordan and Mansi (2003).

5.1 Alternative Algorithms

Early approaches (McCulloch 1971, 1975; Vasicek and Fong 1982) used polynomial or exponential splines to fit the discount function directly. One advantage of these approachs is that the polynomials offer enough degrees of freedom so that the security prices can be fit with minimal error. The problems are two-fold. First, you must already know the discount rate for long maturity bonds and these are not directly observed. Second, as can be seen from (11b), even a very smooth appearing discount function may create erratic forward rate curves. In the extreme case, if the splines in the discount function are not differentiable at the nodes then the forward rate curve will have discontinuities at those points implying arbitrage opportunities for derivative securities that are, in fact, merely an artifact of the interpolation algorithm.

Probably the most widely used yield curve model at present is that of Nelson and Siegel (1987) along with the extension of Svensson (1995). Jordan and Mansi (2003) conclude that these are the best models and Gimeno and Nave (2006) report that nine out of the thirteen central banks that report their estimation methods to the Bank of International Settlements, use the Svensson augmented version of the Nelson-Siegel model. Consequently, we will review these methods in detail and use them as benchmarks against which we can compare our model.

Nelson and Siegel (1987) observe that the yield curve generally follows rather simple monotone increasing concave or occasionally humped or even S-shapes curves. To match this fact they propose a global approach that fits a specific functional form for the forward rate curve

$$f_t(T) = \beta_0 + \beta_1 \exp\left(-\frac{T}{\tau}\right) + \beta_2 \left[\frac{T}{\tau}\exp\left(\frac{-T}{\tau}\right)\right] \tag{21}$$

which, using (11c) gives the spot curve

$$\begin{aligned} y_t(T) =& \beta_0 + \beta_1 \left[\frac{1-\exp\left(-\frac{T}{\tau}\right)}{\frac{T}{\tau}}\right] \\ &+ \beta_2 \left(\left[\frac{1-\exp\left(-\frac{T}{\tau}\right)}{\frac{T}{\tau}}\right] - \exp\left(-\frac{T}{\tau}\right)\right). \end{aligned} \tag{22}$$

The first term, the constant β_0, is interpreted as the long run level of interest rates, the second term is interpreted as a short-term component that determines the slope of the spot curve for short maturities, and the third term is interpreted as a medium term component that can capture a hump or dip in the spot curve. The coefficients are estimated by minimizing the pricing errors for the observed securities and we expect $\beta_0 > 0$, $\beta_0 + \beta_1 > 0$ and $\tau > 0$.

The Nelson-Siegel functional form has been widely adopted because it has several appealing features. The spot and forward curves are always smooth and continuously differentiable. As the maturity increases we get

$$\lim_{T\to\infty} y_t(T) = \lim_{T\to\infty} f_t(T) = \beta_0 \tag{23}$$

so that the curves are well-behaved in the limit—a property not always observed in polynomial models of the term structure. Note that, in the limit, the spot curve becomes a flat function so the forward curve will asymptotically approach the fixed limiting spot rate.

At the short-end of the term structure we get

$$\lim_{T \to 0} y_t(T) = \lim_{T \to 0} f_t(T) = \beta_0 + \beta_1 \tag{24}$$

so the constraint that the forward and spot curves begin at the same point is satisfied. For a normal yield curve that is an upward sloping concave function we would expect to find $\beta_1 < 0$.

One short-coming of the Nelson-Siegel model is that, with only four free parameters, the model cannot be calibrated to fit all current security prices and in some cases the pricing errors can be substantial. To improve this aspect of the model, Svensson (1995) added an additional term to the forward curve to get

$$\begin{aligned} f_t(T) = \beta_0 + \beta_1 \exp\left(-\frac{T}{\tau_1}\right) + \beta_2 \left[\frac{T}{\tau_1} \exp\left(\frac{-T}{\tau_1}\right)\right] \\ + \beta_3 \left[\frac{T}{\tau_2} \exp\left(\frac{-T}{\tau_2}\right)\right] \end{aligned} \tag{25}$$

which yields the spot curve

$$\begin{aligned} y_t(T) =& \beta_0 + \beta_1 \left[\frac{1 - \exp\left(-\frac{T}{\tau_1}\right)}{\frac{T}{\tau_1}}\right] \\ &+ \beta_2 \left(\left[\frac{1 - \exp\left(-\frac{T}{\tau_1}\right)}{\frac{T}{\tau_1}}\right] - \exp\left(-\frac{T}{\tau_1}\right)\right) \\ &+ \beta_3 \left(\left[\frac{1 - \exp\left(-\frac{T}{\tau_2}\right)}{\frac{T}{\tau_2}}\right] - \exp\left(-\frac{T}{\tau_2}\right)\right). \end{aligned} \tag{26}$$

The additional term allows more flexibility in the shapes of the forward and spot curves and the model is easier to calibrate to securities prices since it has six free parameters.

There are many variations of the Nelson and Siegel (1987) model, including Diebold and Li (2006) who make a natural and powerful extension by allowing the parameters to vary over time. The additional degrees of freedom improve the calibration properties of their model at the cost of additional complexity in modeling the factors that drive the coefficient movements.

Other sophisticated nonlinear approaches include the monotone convex spline method of Hagan and West (2006) and the nonlinear dynamic programming method of Manzano and Blomvall (2004). These methods, while very promising, are highly nonlinear and difficult to implement. Our goal is to provide an easily implemented algorithm that is superior to the most commonly used algorithms in practice.

5.2 Comparison of Algorithms

We will compare our IPQPI method to four alternative methods:

LBLI	A simple linear bootstrap and linear interpolation method applied to the spot curve.
LBPQPI	A linear bootstrap method on the spot rates of the coupons with a piecewise quartic polynomial interpolation through these rates.
NS	The Nelson-Siegel method described in 5.1.
SV	The Svensson extension of Nelson-Siegel described in 5.1.

The linear bootstrap with linear interpolation (LBLI) method is simply a piecewise linear interpolation of the spot curve using a bootstrap method to compute the spot rates at each node. This is straightforward for zero coupon bonds when the the spot rates at those maturities are directly observed. For bonds, we add one bond at a time and adjust the estimated spot rate up or down until that bond is correctly priced. Note that the linear segments determined in prior steps are not adjusted during this process. The resulting spot curve will be continuous but not differentiable at the node points. This method is admittedly a "straw man" but it is useful for illustrating some important features of the IPQPI method.

The LBPQPI method computes the spot rates at the maturities of the bonds in the same way as LBLI except, in the final step, we use a piecewise quartic polynomial interpolation of the computed spot rates to estimate the forward curve. This approach separates the interpolation and stripping steps and is the most straightforward extension of the various methods, including the Lim and Xiao (2002) method, applied to coupon bonds but that require spot rates as inputs.

The Nelson-Siegel (NS) and Svensson (SV) methods are described in 5.1 and, due to their widespread use, are the most serious contenders for our IPQPI method.

Table 3 reports the pricing errors of the five algorithms for each of securities as well as some summary statistics to evaluate the methods. The first two columns give the maturities and actual prices of the on-the-run treasuries for the yield curve on July 10, 2008. The first six securities (including the settlement date in the first row) are zero coupon bills and the last four securities are the coupon bonds. Columns three through seven in the top portion of the table report the pricing error in cents $\left[100 * \left(P(T_i) - \hat{P}(T_i)\right)\right]$ of each security for each method.

The bottom portion of the table reports three useful summary statistics. "MDwError" is the root mean square Macaulay duration weighted percentage pricing errors

$$\text{MDwError} = \sqrt{\sum_{i=1}^{m} \frac{1}{MD_i} \left(100 \frac{P(T_i) - \hat{P}(T_i)}{P(T_i)}\right)^2}. \tag{27}$$

The Macaulay durations are defined in equation (1) and reported in Table 1. The idea underlying this measure is to give increased weight to the short-term securities because it is more difficult to fit the short end of the yield curve than the long end. The seven day spot yield can move considerably without effecting bond prices by much but even a small movement in the thirty year rate has a large impact on other bond prices.

The "Ave Abs Error" row reports the average of the absolute pricing errors in cents:

$$\text{Ave Abs Error} = \frac{1}{m}\sum_{i=1}^{m}\left|100\left(P(T_i) - \hat{P}(T_i)\right)\right|. \tag{28}$$

"Smoothness' reports the smoothness of the instantaneous forward curve as measured by the integral of the squared second derivatives

$$\text{Smoothness} = \left(\sqrt{\int_0^{T_m} \left(f''(t)\right)^2\, dt}\right)^{-1} \tag{29}$$

$$\approx \left(\sqrt{\sum_{t=2}^{T_m-1} \left(f(t+1) - 2f(t) + f(t-1)\right)^2}\right)^{-1} \tag{30}$$

where the second equation is the discrete approximation. The smoothness values in Table 3 are computed with f measured in percentages rather than decimals. Taking the square root of the integral converts the units back to percents for easier interpretation. Taking the inverse of the measure means that the less "jerk" there is in the forward curve, the larger the Smoothness value will be. A straight line would have a smoothness score of infinity.

Figures 3 through 6 compare the discount functions, spot curves and instantaneous forward curves generated by each of these methods. Note that the discount functions in Figure 3 and spot curves in Figure 5 for all the methods look "reasonably" similar and it would be difficult to choose among them on the basis of these curves alone. The changes in the discount function shown in Figure 4 and the forward curves shown in Figure 6, however, differ dramatically. This is a visual indication of why modeling the forward rate curve directly is critical in fitting the yield curve.

Table 3 and the associated Figures 3 through 6 below illustrate the key features of the various methods and what we believe are the advantages of the IPQPI method. Note first that the LBLI method produces zero pricing errors for both the zero coupon bills and the coupon bearing bonds. This is precisely what the LBLI algorithm is designed to do. The spot yields at each node are chosen to exactly price the security maturing at that node. Since there is no feedback from one node to previous nodes during the computations, zero pricing errors is an easy criteria to satisfy. The problem with this method is that, because the spot curve is piecewise linear, it produces discontinuities in the spot and forward curves at the nodes. The reason for the discontinuities is clear from equations (11a) and (11b). The very low smoothness statistic

Table 3 Pricing errors and summary statistics of the five algorithms for the yield curve on 7/10/2008. The first column gives the maturity dates of the securities. The first six securities with maturities through 7/2/2009 are zero coupon bills. The pricing errors are reported in cents so an error of 2.0814 means that the bond was underpriced by \$0.020814. MDwError is the root mean square Macaulay duration weighted percentage pricing error. Ave Abs Error is the average absolute pricing error in cents. Smoothness is the inverse of the square root of the integral of the squared second derivative of the instantaneous forward curve.

Maturity	Actual Price	LBLI	LBPQPI	NS	SV	IPQPI
7/10/2008	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7/17/2008	99.9725	0.0000	0.0000	0.2365	-0.0704	0.0000
8/7/2008	99.8880	0.0000	0.0000	0.9745	0.2143	0.0000
10/9/2008	99.5854	0.0000	0.0000	0.2747	1.1759	0.0000
1/8/2009	99.0092	0.0000	0.0000	-9.6634	-2.7669	0.0000
7/2/2009	97.8992	0.0000	0.0000	-14.1885	1.8282	0.0000
6/30/2010	100.8800	0.0000	-0.1384	-5.4618	-0.8438	-0.0480
6/30/2013	101.3000	0.0000	2.0716	60.1537	4.8588	-0.3701
5/15/2018	100.5200	0.0000	-0.3559	4.1946	-11.1614	-2.8419
2/15/2038	99.2800	0.0000	223.8056	-60.1885	10.1220	0.0000
MDwError		0.0000	0.5467	0.3764	0.0721	0.0100
Ave Abs Error		0.0000	22.6372	15.5336	3.3042	0.3260
Smoothness		0.5046	644.24	16625.22	16.54	644.08

of 0.5046 reflects these discontinuities in the forward curve that are clearly visible in Figure 6 and in the changes in the discount rates shown in Figure 4. Even though all of the securities used to construct the curves are priced exactly, the discontinuities in the discount and forward curves would create large pricing errors in derivatives and other out-of-sample securities priced from these curves.

The LBPQPI method applies a PQPI to the spot rates $\{y(T_1), \ldots, y(T_m)\}$ computed by LBLI rather than using the linear segments between the nodes of the spot curve in the LBLI method. As expected, this produces a very smooth forward curve. This process maintains the zero pricing errors for the zero coupon bills but introduces pricing errors for the coupon bonds. The thirty year bond has a very large \$2.24 pricing error. This occurs because the linear segment between the ten year and thirty year node points used to compute the thirty year spot rate $y(T_{30})$ contains forty coupon payments for the thirty year bond. When PQPI is used to smooth the forward curve the value of these coupon payments can change dramatically and create large pricing errors. This method clearly illustrates the importance of simultaneously stripping the coupon bonds and interpolating the forward curve. Doing these steps sequentially introduces inconsistencies between the spot rates and the intervening forward and spot curves.

The very high smoothness score of the Nelson-Siegel method illustrates why this approach remains so popular. The primary disadvantage of the NS method is that the possible shapes of the spot and forward curves are quite limited and may produce substantial pricing errors. The average pricing error of fifteen cents with maximum errors of sixty cents are unacceptably large for most applications. The Svensson extension of NS allows for more flexible

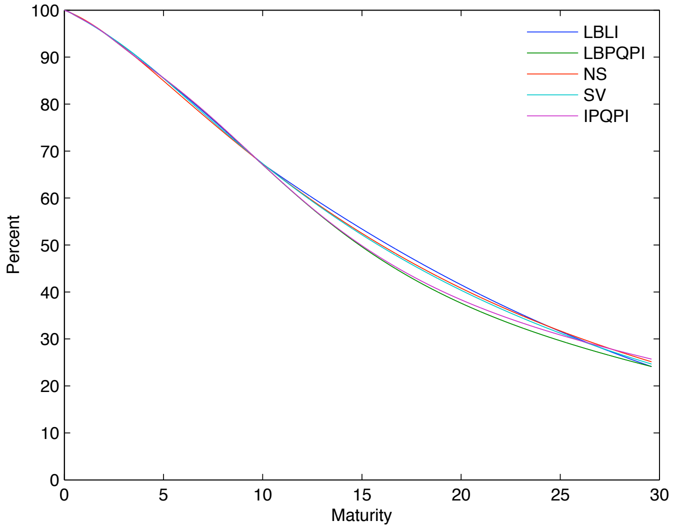

Fig. 3 Discount functions of the five algorithms.

curves and more accurate pricing with the average pricing error dropping to just over three cents with a maximum error of eleven cents. The cost of this increased pricing accuracy is a less smooth forward curve. The additional term in the SV method introduces a dip in the forward curve around maturities of one year that does not occur in the NS forward curve. These two methods nicely illustrate the trade-off between smoothness of the forward curve and accuracy in pricing.

The IPQPI method has low pricing errors while maintaining the smoothest possible forward curve among the class of piecewise polynomials. The largest pricing error is 2.84 cents on the ten year bond. We often find our largest pricing errors for the ten year bond and this is likely due to the special role that the ten year plays as a hedging instrument in mortgage backed securities.

5.3 Discussion

In this paper we have described our Iterated Piecewise Quartic Polynomial Interpolation algorithm for simultaneously interpolating and stripping a yield curve consisting of coupon paying bonds. Our algorithm is accurate, flexible and relies only upon the solution of linear systems of equations and produces maximally smooth forward curves.

Overall, our IPQPI method performs very well. Although the Nelson-Siegel algorithm produces smoother forward curves it does so at the cost of much larger pricing errors. IPQPI produces pricing errors an order of magni-

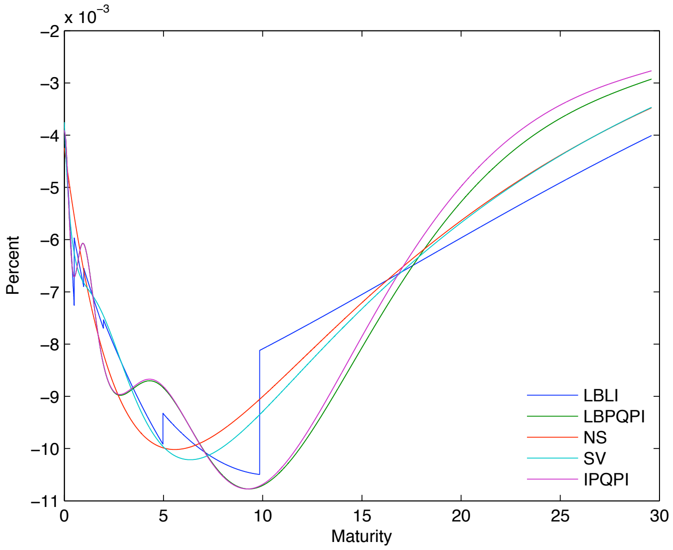

Fig. 4 Changes in the discount functions of the five algorithms. Note the discontinuities in the LBLI algorithm.

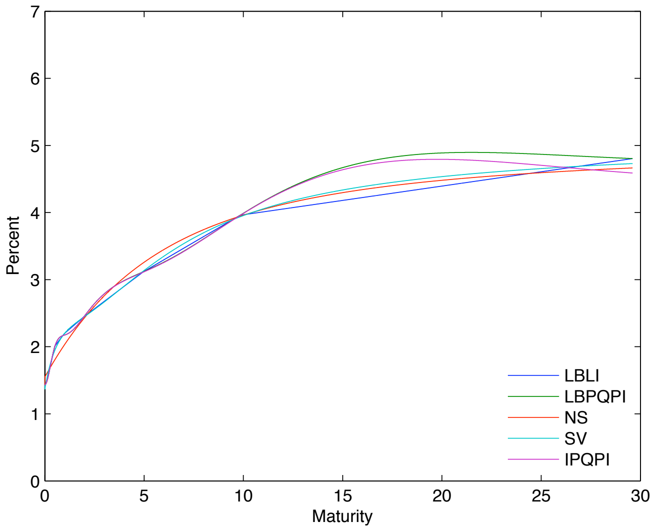

Fig. 5 Spot functions of the five algorithms.

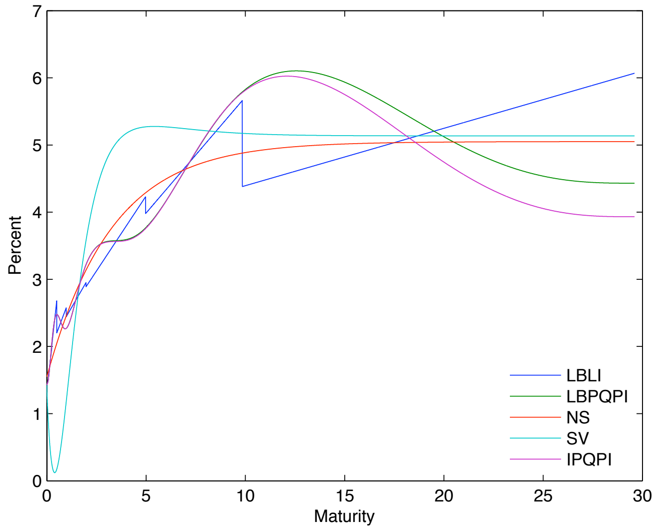

Fig. 6 Forward functions of the five algorithms. Note the discontinuities in the LBLI algorithm.

tude smaller than the Svensson algorithm while at the same time producing smoother forward curves. Computationally, IPQPI is very stable and an order of magnitude faster than the highly nonlinear Svensson approach. The method is also easy to modify by adding additional constraints for specific pricing applications.

As noted, we usually observe our largest pricing error for the ten year bond and we speculate that this is because the ten year is heavily used for hedging mortgage-backed securities. Consequently, the price of the on-the-run ten year bond is often on "special" depending upon the current books of bond brokers. It is possible to add an additional constraint to IPQPI to reflect these "specials" and to reduce the pricing error at the ten year. We do not go into this extension here because it is often a case-by-case issue.

No doubt our method is not the final word on yield curve stripping and interpolation algorithms. As Hagan and West (2006) point out, quartic-based methods such as ours can be sensitive to perturbations in bond prices in the sense that a small change in the bond price at a specific maturity can have substantial influence on the forward curve several nodes away. Our simulation experiments have confirmed this. One implication of this is that hedging portfolios based upon quartic methods can be relatively expensive to maintain. Of course, it depends upon what you are hedging and how, but one potential way to deal with this in IPQPI would be to add an additional constraint with increasing penalties for movements away from an initial forward curve at a specific maturity. Our method is flexible enough to easily add such constraints.

Another issue worth exploring is the dependency of IPQPI on polynomials. One of the reasons that the Nelson and Siegel and the Svensson methods work so well is that they use exponential functions that are essentially infinite order polynomials with highly constrained coefficients. It is likely that a more general class of basis functions would produce better results albeit at a much higher computational cost.

A Appendix: Piecewise Quartic Polynomial Interpolation (PQPI)

The procedure for interpolating the zero coupon portion of the yield curve is similar to that described in Lim and Xiao (2002) except that we handle the terminal condition on the forward curve differently. Our approach adds one additional segment to the piecewise spline function with some additional coefficient restrictions on the terminal spline. Since the matrix sizes differ to reflect these changes, we provide the details of this step here.

In the first subsection of the appendix we show how to construct the objective function as a quadratic expression. In the second subsection we construct the constraints as linear equations and in the final subsection we show how to solve the PQPI system.

A.1 Specifying the Objective Function

Noting that the objective function is piecewise with nodes at the maturity of each security. Let $f_i(t) = a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i$ denote the quartic polynomial between node T_{i-1} to node T_i so that $f_i''(t) = 12a_i t^3 + 6b_i t^2 + 2c_i$. Then the objective function (14) may be written as

$$
\begin{aligned}
&\int_0^{T_{m+1}} \left(f''(t)\right)^2 dt \\
&\quad = \sum_{i=1}^{m+1} \int_{T_{i-1}}^{T_i} f_i''(t)\, dt \\
&\quad = \sum_{i=1}^{m+1} \int_{T_{i-1}}^{T_i} \left(12a_i t^2 + 6b_i t + 2c_i\right)^2 \, dt \\
&\quad = \sum_{i=1}^{m+1} \Big(\frac{144}{5} \Delta_i^5 a_i^2 + 36\Delta_i^4 a_i b_i + 12\Delta_i^3 b_i^2 \\
&\qquad + 16\Delta_i^3 a_i c_i + 12\Delta_i^2 b_i c_i + 4\Delta_i c_i^2 \Big) \\
&\quad = \sum_{i=1}^{m+1} x_i' h_i x_i \\
&\quad = \mathbf{X'HX}
\end{aligned}
\tag{A.1}
$$

where $\Delta_i^n = T_i^n - T_{i-1}^n$,

$$
x_i = \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \\ e_i \end{pmatrix}, \quad h_i = \begin{pmatrix} \frac{144}{5}\Delta_i^5 & 18\Delta_i^4 & 8\Delta_i^3 & 0 & 0 \\ 18\Delta_i^4 & 12\Delta_i^3 & 6\Delta_i^2 & 0 & 0 \\ 8\Delta_i^3 & 6\Delta_i^2 & 4\Delta_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.2}
$$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \end{pmatrix}_{5(m+1)\times 1}, \tag{A.3}$$

$$\mathbf{H} = \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{m+1} \end{pmatrix}_{5(m+1)\times 5(m+1)}, \tag{A.4}$$

and each of the 0's as well as h_{m+1} is a 5×5 matrix of zeros.

Recall that we have m securities with maturities $T_1, \ldots, T_m$ where m is the number of securities. We do not consider the settlement date in this list so T_1 is the maturity of the first *real* security—the one-week LIBOR in our example. We define $T_0 = 0$. Note that we have added an $m+1$ node in (A.1). The reason for this additional node is to impose the terminal condition on the forward curve which is incorrectly handled inLim and Xiao (2002). This turns out to be more complicated than expected and we will discuss this correction in more detail below.

The important observation to make is that the objective function is quadratic so that the gradient is linear in the coefficients. We will now show that the constraint functions are also linear functions of the coefficients.

A.2 Specifying the Constraint Functions

The zero-coupon bond price given by (11a) may be written as

$$\begin{aligned} -\ln\delta(T_j) &= \int_0^{T_j} f(t)\,dt \\ &= \sum_{j=1}^{i} \int_{T_{i-1}}^{T_i} f_j(t)\,dt \\ &= \sum_{j=1}^{i} \int_{T_{j-1}}^{T_j} \left(a_j t^4 + b_j t^3 + c_j t^2 + d_j t + e_j\right) dt \\ &= \sum_{j=1}^{i} \left(\frac{1}{5} a_j T_j^5 + \frac{1}{4} b_j T_j^4 + \frac{1}{3} c_j T_j^3 \right. \\ &\qquad \left. + \frac{1}{2} d_j T_j^2 + e_j T_j\right) \end{aligned} \tag{A.5}$$

so that the log of the zero-coupon bond price is linear in the coefficients x.

Recalling our previous notation $\Delta_j^n = T_j^n - T_{j-1}^n$, we can write the difference of the log prices of two consecutive zero coupon bonds as

$$\begin{aligned} -\ln\left(\frac{\delta(T_j)}{\delta(T_{j-1})}\right) &= \frac{1}{5}\Delta_j^5 a_j + \frac{1}{4}\Delta_j^4 b_j + \frac{1}{3}\Delta_j^3 c_j \\ &\quad + \frac{1}{2}\Delta_j^2 d_j + \Delta_j e_j. \end{aligned} \tag{A.6}$$

The full set of pricing constraints for all m bonds may then be written in matrix form as

$$\mathbf{A}_1\mathbf{X} = \mathbf{B}_1 \tag{A.7}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} DT_1 & 0_{1\times5} & \cdots & 0_{1\times5} \\ 0_{1\times5} & DT_2 & \cdots & 0_{1\times5} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1\times5} & 0_{1\times5} & \cdots & DT_m \end{pmatrix}_{m\times5(m+1)}, \tag{A.8}$$

$$DT_j = \left(\tfrac{1}{5}\Delta_j^5,\ \tfrac{1}{4}\Delta_j^4,\ \tfrac{1}{3}\Delta_j^3,\ \tfrac{1}{2}\Delta_j^2,\ \Delta_j\right)_{1\times5}, \tag{A.9}$$

and

$$\mathbf{B}_1 = \begin{pmatrix} \ln\big(\delta(T_1)/\delta(T_0)\big) \\ \ln\big(\delta(T_2)/\delta(T_1)\big) \\ \vdots \\ \ln\big(\delta(T_m)/\delta(T_{m-1})\big) \end{pmatrix}_{m\times1}. \tag{A.10}$$

Next we impose constraints to ensure that the forward curve remains smooth as it transitions through node points in the piecewise polynomial approximation. To ensure continuity at the node points we require that the forward rate at node T_i has the same value whether computed using the left-side polynomial or the right-side polynomial. Thus, we impose

$$f_{i+1}(T_i) = f_i(T_i),\ \ i = 1, \ldots, m, \tag{A.11}$$

or

$$\begin{aligned}(a_{i+1} - a_i)T_i^4 + (b_{i+1} - b_i)T_i^3 + (c_{i+1} - c_i)T_i^2 \\ + (d_{i+1} - d_i)T_i + (e_{i+1} - e_i) = 0,\ \ i = 1, \ldots, m.\end{aligned} \tag{A.12}$$

Define $T4_i = (T_i^4, T_i^3, T_i^2, T_i, 1)_{1\times5}$ and write all m of these constraints in matrix form as

$$\mathbf{A}_2 x = \mathbf{B}_2 \tag{A.13}$$

where

$$\mathbf{A}_2 = \begin{pmatrix} -T4_1 & T4_1 & 0_{1\times5} & 0_{1\times5} & \cdots & 0_{1\times5} & 0_{1\times5} \\ 0_{1\times5} & -T4_2 & T4_2 & 0_{1\times5} & \cdots & 0_{1\times5} & 0_{1\times5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{1\times5} & 0_{1\times5} & 0_{1\times5} & 0_{1\times5} & \cdots & -T4_m & T4_m \end{pmatrix} \tag{A.14}$$

is an $m \times 5(m+1)$ matrix and $\mathbf{B}_2$ is a $m \times 1$ vector of zeros.

To impose differentiability at the nodes we require

$$f'_{i+1}(T_i) = f'_i(T_i),\ \ i = 1, \ldots, m \tag{A.15}$$

or

$$\begin{aligned}4(a_{i+1} - a_i)T_i^3 + 3(b_{i+1} - b_i)T_i^2 + 2(c_{i+1} - c_i)T_i \\ + (d_{i+1} - d_i) = 0,\ \ i = 1, \ldots, m.\end{aligned} \tag{A.16}$$

Define $T3_i = (4T_i^3, 3T_i^2, 2T_i, 1, 0)_{1\times5}$ and write all m of these constraints in matrix form as

$$\mathbf{A}_3\mathbf{X} = \mathbf{B}_3 0_{m\times1} \tag{A.17}$$

where

$$\mathbf{A}_3 = \begin{pmatrix} -T3_1 & T3_1 & 0_{1\times5} & 0_{1\times5} & \cdots & 0_{1\times5} & 0_{1\times5} \\ 0_{1\times5} & -T3_2 & T3_2 & 0_{1\times5} & \cdots & 0_{1\times5} & 0_{1\times5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{1\times5} & 0_{1\times5} & 0_{1\times5} & 0_{1\times5} & \cdots & -T3_m & T3_m \end{pmatrix} \tag{A.18}$$

is an $m \times 5(m+1)$ matrix and $\mathbf{B}_3$ is a $m \times 1$ vector of zeros.

To ensure that the first derivatives of the forward curve are smooth at the nodes we impose

$$f''_{i+1}(T_i) = f''_i(T_i),\ i = 1, \ldots, m, \tag{A.19}$$

or

$$12(a_{i+1} - a_i)T_i^2 + 6(b_{i+1} - b_i)T_i \\ + 2(c_{i+1} - c_i) = 0,\ i = 1, \ldots, m. \tag{A.20}$$

Define $T2_i = (12T_i^2, 6T_i, 2, 1, 0, 0)_{1\times 5}$ and write all m of these constraints in matrix form as

$$\mathbf{A}_4\mathbf{X} = \mathbf{B}_4 \tag{A.21}$$

where

$$\mathbf{A}_4 = \begin{pmatrix} -T2_1 & T2_1 & 0_{1\times 5} & 0_{1\times 5} & \cdots & 0_{1\times 5} & 0_{1\times 5} \\ 0_{1\times 5} & -T2_2 & T2_2 & 0_{1\times 5} & \cdots & 0_{1\times 5} & 0_{1\times 5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{1\times 5} & 0_{1\times 5} & 0_{1\times 5} & 0_{1\times 5} & \cdots & -T2_m & T2_m \end{pmatrix} \tag{A.22}$$

is an $m \times 5(m+1)$ matrix and $\mathbf{B}_4$ is a $m \times 1$ vector of zeros.

To ensure the boundary condition $f(0) = y_0$, we simply impose $e_1 = y_0$. The terminal boundary condition $f'(T_m) = 0$ is more difficult to impose. Lim and Xiao (2002) use the condition $d_1 = 0$ which is clearly incorrect. We impose the terminal condition by adding an additional $(m+1)^{st}$ segment to the piecewise polynomial with the coefficient restrictions $a_{m+1} = b_{m+1} = c_{m+1} = d_{m+1} = 0$ so that $f(t) = e_{m+1}$ for all $t > T_m$. The terminal height of the forward function is left unconstrained and the continuity and smoothness constraints described above will ensure a smooth transition to the zero slope of the forward curve at node T_m.

These five boundary conditions may be written in matrix notation as

$$\mathbf{A}_5\mathbf{X} = \mathbf{B}_5 \tag{A.23}$$

where

$$\mathbf{A}_5 = \begin{pmatrix} 0\ 0\ 0\ 0\ 1\ 0 \cdots 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \cdots 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \cdots 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \cdots 0\ 0\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \cdots 0\ 0\ 0\ 1\ 0 \end{pmatrix}_{5\times 5(m+1)} \tag{A.24}$$

and

$$\mathbf{B}_5 = \begin{pmatrix} y_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{A.25}$$

Stacking all of these linear constraints gives

$$\mathbf{A}\mathbf{X} = \mathbf{B} \tag{A.26}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \\ \mathbf{A}_4 \\ \mathbf{A}_5 \end{pmatrix}_{(4m+5)\times 5(m+1)} \quad \text{and} \quad \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \\ \mathbf{B}_5 \end{pmatrix}_{(4m+5)\times 1}. \tag{A.27}$$

A.3 Solving the PQPI System

The constrained optimization problem may now be written in matrix notation as

$$\min_{\mathbf{X},\boldsymbol{\lambda}} Z(\mathbf{X},\boldsymbol{\lambda}) = \mathbf{X}'\mathbf{H}\mathbf{X} + \boldsymbol{\lambda}'(\mathbf{A}\mathbf{X} - \mathbf{B}) \tag{A.28}$$

where $\boldsymbol{\lambda}$ is the $4m+5$ vector of Lagrange multipliers corresponding to the constraints.

The first-order conditions are

$$\frac{\partial}{\partial \mathbf{X}} Z(\mathbf{X},\boldsymbol{\lambda}) = 2\mathbf{H}\mathbf{X} + \mathbf{A}'\boldsymbol{\lambda} = 0 \tag{A.29}$$

and

$$\frac{\partial}{\partial \boldsymbol{\lambda}} Z(\mathbf{X},\boldsymbol{\lambda}) = \mathbf{A}\mathbf{H}\mathbf{X} - \mathbf{B} = 0, \tag{A.30}$$

or

$$\begin{pmatrix} 2\mathbf{H} & \mathbf{A}' \\ \mathbf{A} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{B} \end{pmatrix} \tag{A.31}$$

from which we find the explicit solution

$$\begin{pmatrix} \mathbf{X}^* \\ \boldsymbol{\lambda}^* \end{pmatrix} = \begin{pmatrix} 2\mathbf{H} & \mathbf{A}' \\ \mathbf{A} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mathbf{B} \end{pmatrix}. \tag{A.32}$$

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