Primer: The FST Theorem for Pricing with Foreign Collateral

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Abstract. The Fujii Shimada Takahashi theorem for pricing derivatives collateralized in a foreign currency is reviewed.

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1. Introduction

The basic interest rate setting is cross-economy HJM using accepted notation (such as in [7]) as much as possible. So starting in the domestic D-economy, \( r(t) \) is the risk-free rate assumed equal to the funding or repo rate for uncollateralized assets (both equities and bonds), and \( c(t) \) will be the collateral rate for funds deposited as collateral for trades. Let \( r(t) \) accumulate in a bank account \( \beta(t) = \exp\left\{ \int_0^t r(s) \, ds \right\} \), and \( c(t) \) accumulate in a collateral account \( C(t) = \exp\left\{ \int_0^t c(s) \, ds \right\} \).

The spot measure \( P_0 \) (expectation \( E_0 \), multi-dimensional Brownian motion \( W_0(t) \)) is taken to be that measure under which uncollateralized assets grow at \( r(t) \), and collateralized assets grow at \( c(t) \). Both uncollateralized \( B(t,T) \) and collateralized \( D(t,T) \) zero coupon bonds pay 1-unit of D at T.

Similar variables are used in the foreign F-economy but with a superscript \( f \) attached, like \( r^f(t), \beta^f(t), P^f_0, (E^f_0 \text{ and } W^f_0(t)), B^f(t,T), D^f(t,T) \) etc. The two economies are connected through the exchange rate \( S(t) \) which is 1-unit of foreign F-currency in domestic-D units at time-\( t \), i.e. \( F \ D = D \ S(t) \).

Usually the domestic and foreign collateral rates are taken to be the rates used in US overnight index swaps (OIS) or their equivalent in other currencies. The risk-free and collateral rates are often different, so denote the difference by the spreads

\[
y(t) = r(t) - c(t) \ D \quad \text{and} \quad y^f(t) = r^f(t) - c^f(t) \ F.
\]

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The aim of this primer is to review and establish the Fujii Shimada Takahashi theorem (FST-theorem) for pricing fully collateralized instruments when the collateral can be posted in either the domestic $D$ or a foreign $F$ currency.

**Theorem 1.** When collateral is posted in a foreign $F$-currency, the present value $h_t$ in domestic $D$-currency of a fully collateralized derivative paying $h_T$ at $T$ is

$$h_t = E_0 \left\{ h_T \exp \left[ - \int_t^T \left[ -r(s) + y^f(s) \right] ds \right] \bigg| \mathcal{F}_t \right\},$$

or

$$h_t = E_0 \left\{ e^{-\int_t^T c(s) ds} h_T \exp \left[ - \int_t^T \left[ y(s) - y^f(s) \right] ds \right] \bigg| \mathcal{F}_t \right\}.$$

and when collateralized in domestic $D$-currency it is

$$h_t = E_0 \left\{ e^{-\int_t^T c(s) ds} h_T \bigg| \mathcal{F}_t \right\}.$$

1. **Collateralized options in the domestic economy**

Working in the domestic $D$ economy, in this section we obtain a pricing formula for an option fully collateralized in the $D$-currency by tracking it with a self-funding portfolio of assets that draw funds from both collateral and bank accounts.

The ingredients are: a vector of assets $A_t = (A_t^R, A_t^C)$ comprising uncollateralized equities and bonds $A_t^R$ funded at $r_t$, and collateralized bonds $A_t^C$ funded at $c_t$; bank and collateral accounts $\beta_t$ and $C_t$; and a fully collateralized derivative $h_t$ on $A_t$ paying $h_T$ at $T$.

We now establish the following result, which is essentially the FST-theorem (1.3) when collateral is posted in the domestic $D$-currency.

**Theorem 2.** The time-$t$ price of the option fully collateralized in domestic $D$-currency is

$$h_t = E_0 \left\{ h_T \exp \left( - \int_t^T c(s) ds \right) \bigg| \mathcal{F}_t \right\} = h(t, A_t).$$

Imagine that we (the bank) have sold the derivative at time $t$, put the proceeds in the collateral account, and will dynamically hedge it with a self-funding portfolio $V_t$ containing assets and bank accounts. At time-$t$ the collateral account should therefore contain $\phi_t$ units of $C_t$ with $h_t = \phi_t C_t$, and for the next time step we set up a tracking portfolio

$$V_t = \phi_t C_t,$$

$$= \theta_t^R A_t^R + \theta_t^C A_t^C + (\phi_t^C + \phi_t) C_t + \psi_t \beta_t,$$

in which the asset components $\theta_t^R A_t^R$ and $\theta_t^C A_t^C$ are separately purchased via the transactions

$$\theta_t^R A_t^R + \psi_t \beta_t = 0, \quad \theta_t^C A_t^C + \phi_t^C C_t = 0.$$

The self-funding condition for $V_t$ is

$$dV_t = \theta_t^R dA_t^R + \theta_t^C dA_t^C + (\phi_t^C + \phi_t) dC_t + \psi_t d\beta_t.$$

**Remark 3.** Note that for this equation (2.4) to make sense there should be enough assets on the right-hand side of to eliminate all the components of the Brownian motion in the model.
Introduce the starred * discount variables comprising $A_t^R$ discounted by $\beta_t$, and $V_t$ and $A_t^C$ both discounted by $C_t$

$$A_t^{R*} = \frac{A_t^R}{\beta_t}, \quad V = \frac{V_t}{C_t}, \quad A_t^{C*} = \frac{A_t^C}{C_t} \quad \Rightarrow$$

(2.5) 

$$dA_t^R = dA_t^{R*} \beta_t + A_t^{R*} d\beta_t \quad dV_t = dV_t^{*} C_t + V_t^{*} dC_t,$$

$$dA_t^C = dA_t^{C*} C_t + A_t^{C*} dC_t,$$

substitute (2.5) into (2.4), and then gather terms to get

$$dV_t^{*} C_t + [V_t^{*} - \phi_t] dC_t = \theta_t^R dA_t^{R*} \beta_t + \theta_t^C dA_t^{C*} C_t + [\theta_t^R A_t^{R*} + \psi_t] d\beta_t + [\theta_t^C A_t^{C*} + \phi_t^C] dC_t.$$

Imposing the transaction conditions (2.3), and recalling that $V_t$ is fully collateralized (2.1) gives

(2.6) 

$$dV_t^{*} = \frac{\beta_t}{C_t} \theta_t^R dA_t^{R*} + \theta_t^C dA_t^{C*}.$$

Hence $V_t$ discounted by the collateral account $C_t$ is a martingale under the spot measure $\mathbb{P}_0$ which makes uncollateralized assets grow at $r_t$ and collateralized assets grow at $c_t$; i.e. the measure under which they have SDEs

(2.7) 

$$\frac{dA_t^R}{dA} = r_t dt + \sigma_t^R dW_t(0) , \quad \frac{dA_t^C}{dA} = c_t dt + \sigma_t^C dW_t(0) .$$

So taking conditional expectations under $\mathbb{P}_0$, the value of the option $h_t$ at time-$t$ is, as quoted in Theorem-2,

$$h_t = \mathbb{E}_0 \left\{ h_T \exp \left( - \int_t^T c_s ds \right) \bigg| \mathcal{F}_t \right\} = h(t, A_t) .$$

2.1. Checking replication. On the one hand, from (2.4), (2.3), and (2.1)

$$V_t + dV_t = V_t + \phi_t^R dA_t^{R*} + \phi_t^C dA_t^{C*} + (\phi_t^C + \phi_t) dC_t + \psi_t d\beta_t,$$

(2.8) 

$$= V_t + \theta_t^R (dA_t^R - A_t^{R*} r_t dt) + \theta_t^C (dA_t^C - A_t^{C*} c_t dt) + V_t c_t dt.$$

On the other hand, because $h_t$ discounted by $C_t$ is a $\mathbb{P}_0$-martingale, it must have SDE

(2.9) 

$$dh_t = \Delta_t^R [dA_t^R - A_t^{R*} r_t dt] + \Delta_t^C [dA_t^C - A_t^{C*} c_t dt] + h_t c_t dt,$$

in terms of the deltas of the option

$$\Delta_t^R = \frac{\partial h_t}{\partial A_t^R} \quad \Delta_t^C = \frac{\partial h_t}{\partial A_t^C} .$$

Comparing (2.8) and (2.9) we see that if the amounts invested in the assets in the replicating portfolio (2.2) are the deltas of the option, i.e. $\theta_t^R = \Delta_t^R$ and $\theta_t^C = \Delta_t^C$, then we have both

$$V_t = h_t \quad \text{and} \quad V_t + dV_t = h_t + dh_t.$$

That shows the portfolio $V_t$ does indeed track the option value $h_t$. 
3. Collateralization in a foreign currency

For simplicity, we focused on pricing a derivative on a single uncollateralized asset \( A_t \) in the domestic-D economy, when that derivative is collateralized in foreign-F currency. Let SDEs for \( A_t \) under \( P_0 \), and for a similarly uncollateralized foreign asset \( A_f^t \) under \( P_f^0 \) be respectively:

\[
\frac{dA_t}{A_t} = r_t dt + a_t dW_0(t), \quad \frac{dA_f^t}{A_f^t} = r_f^t dt + a_f^t dW_f^0(t).
\]

Change of measure between \( P_0 \) and \( P_f^0 \) gets defined by the parity pricing principle: present valuing a domestic-D cashflow \( X(T) \) at \( T \) through both economies gives

\[
S(t) \beta_f(t) E_f^0 \left\{ \frac{X(T)}{S(T) \beta_f(T)} \middle| \mathfrak{F}_t \right\} = \beta(t) E_0 \left\{ \frac{X(T)}{\beta(T)} \middle| \mathfrak{F}_t \right\},
\]

which defines the change of measure as

\[
P_f^0 = \frac{S(T) \beta_f(T)}{S(0) \beta(T)} P_0.
\]

We now establish the foreign F-currency version of Theorem-2, i.e the FST result (1.2).

**Theorem 4.** The time-\( t \) price of the option fully collateralized in foreign F-currency is

\[
h_t = E_0 \left\{ h(T) \exp \int_t^T [-r(s) + y_f(s)] ds \middle| \mathfrak{F}_t \right\},
\]

where \( y_f(t) = r_f(t) - c_f(t) \) is the difference between the foreign risk free and collateral rates. ■

The option values \( h_t \) and \( h_T \) in domestic D-currency translate into values \( h_t/S(t) \) and \( h_T/S(T) \) in foreign F-currency. So applying Theorem-2 in the foreign economy with collateral in foreign-F we get

\[
\frac{h_t}{S(t)} = E_f^0 \left\{ \frac{h_T}{S(T)} \exp \left( - \int_t^T c_f(s) ds \right) \middle| \mathfrak{F}_t \right\}.
\]

Rewriting this equation and changing measures using (3.1) then gives

\[
h_t = \beta_f(t) S(t) E_f^0 \left\{ \frac{h_T}{\beta_f(T) S(T)} \exp \int_t^T y_f(s) ds \middle| \mathfrak{F}_t \right\},
\]

\[
= \beta(t) E_0 \left\{ \frac{h_T}{\beta(T)} \exp \int_t^T y_f(s) ds \middle| \mathfrak{F}_t \right\},
\]

which is Theorem-4.

4. Articulating the FST theorem

Some useful simplifications in Theorem-1 occur when some of the collateral and risk-free rates coincide:

**Corollary 5.** When the foreign risk-free rate is the foreign collateral rate, i.e. when \( r_f(t) = c_f(t) \) and collateral posted in foreign-F, the present value of \( h(t) \) becomes

\[
h(t) = E_0 \left\{ e^{-\int_t^T r(s)ds} h(T) \middle| \mathfrak{F}_t \right\} = B(t, T) E_T \left\{ h(T) \middle| \mathfrak{F}_t \right\},
\]
where $E_T$ is expectation under the domestic forward measure $\mathbb{P}_T$. ■

**Corollary 6.** When the domestic risk-free rate is the domestic collateral rate $r(t) = c(t)$ and collateral is posted in domestic-$D$ again

$$h(t) = E_0 \left\{ e^{-\int_0^T r(s) ds} h(T) \bigg| \mathcal{F}_t \right\} = B(t, T) E_T \left\{ h(T) \bigg| \mathcal{F}_t \right\}. \quad \Box$$

**Corollary 7.** If foreign and domestic economies are interchanged so that: $h(\cdot)$ is in foreign-$F$ written $h^f(\cdot)$, the domestic risk-free rate is the domestic collateral rate, i.e. $r(t) = c(t)$, and collateral is posted in domestic-$D$, then equation (4.1) transforms to

$$h^f(t) = B^f(t, T) E^f_T \left\{ h^f(T) \bigg| \mathcal{F}_t \right\}. \quad \Box$$

It is also possible to construct a maturity $T$ dependent measure associated with the collateralized zero coupon bond

$$D(t, T) = E_0 \left\{ e^{-\int_0^T c(s) ds} \bigg| \mathcal{F}_t \right\},$$

that is similar in some ways to forward measures. Define $\bar{\mathbb{P}}_T$ (expectation $\bar{E}_T$) by

$$\bar{\mathbb{P}}_T = Z_T \mathbb{P}_0 \quad \text{where} \quad Z(T) = \frac{e^{-\int_0^T c(s) ds}}{D(0, T)},$$

$$\Rightarrow \quad \bar{E}_T \left\{ X(T) \bigg| \mathcal{F}_t \right\} = \frac{E_0 \left\{ e^{-\int_0^T c(s) ds} X(T) \bigg| \mathcal{F}_t \right\}}{E_0 \left\{ e^{-\int_0^T c(s) ds} \bigg| \mathcal{F}_t \right\}} = \frac{E_0 \left\{ e^{-\int_0^T c(s) ds} X(T) \bigg| \mathcal{F}_t \right\}}{D(t, T)}.$$

If the spread $y(t)$ is deterministic, then $\bar{\mathbb{P}}_T$ becomes the standard $T$-forward measure $\mathbb{P}_T$ because

$$Z(T) = \frac{\exp \left\{ - \int_0^T [r(s) - y(s)] ds \right\}}{E_0 \exp \left\{ - \int_0^T [r(s) - y(s)] ds \right\}} = \frac{1}{\beta(T) B(0, T)},$$

which is the Radon-Nikodym derivative for $\mathbb{P}_T$. In particular, $\bar{\mathbb{P}}_T = \mathbb{P}_T$ when $y(t) = 0$ and the collateral rate is the overnight rate i.e. $c(t) = r(t)$.

From the definition (4.3) of $D(t, T)$, for any $0 \leq s \leq t \leq T$

$$E_0 \left\{ e^{-\int_0^T c(u) du} D(t, T) \bigg| \mathcal{F}_s \right\} = E_0 \left\{ E_0 \left\{ e^{-\int_0^T c(u) du} \bigg| \mathcal{F}_t \right\} \bigg| \mathcal{F}_s \right\} = E_0 \left\{ e^{-\int_0^T c(u) du} \bigg| \mathcal{F}_s \right\} = e^{-\int_0^s c(u) du} D(s, T),$$

giving a result that will prove useful, for example, in modeling $c(t)$ and $D(t, T)$ as SDEs:

**Corollary 8.** The value of the collateralized zero coupon bond discounted by the collateral rate

$$e^{-\int_0^T c(s) ds} D(t, T)$$

is a $\mathbb{P}_0$-martingale, confirming (see 2.7) the drift under $\mathbb{P}_0$ of $D(t, T)$ is $c(t)$. ■

In the literature $\bar{\mathbb{P}}_T$ is often referred to as the $T$-forward measure induced by $D(t, T)$ as numeraire because it makes collateralized trades $\bar{\mathbb{P}}_T$-martingales; that is, changing measures from $\mathbb{P}_0$ to $\bar{\mathbb{P}}_T$ allows Theorem-1 to be restated as:
Corollary 9. When payment and pricing currencies are different

\[
E_T \left\{ h(T) \exp \left[ -\int_0^T [y(s) - y'(s)] \, ds \right] \right\} = \frac{h(t) \exp \left[ -\int_0^t [y(s) - y'(s)] \, ds \right]}{D(t,T)},
\]

and when payment and pricing currencies are the same.

\[
E_T \{ h(T) \mid F \} = \frac{h(t)}{D(t,T)} \quad \blacksquare
\]

REFERENCES