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Primer: The FST Theorem for Pricing with Foreign Collateral

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PRIMER: THE FST THEOREM FOR PRICING WITH FOREIGN COLLATERAL

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ABSTRACT. The Fujii Shimada Takahashi theorem for pricing derivatives collateralized in a foreign currency is reviewed.

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1. INTRODUCTION

The basic interest rate setting is cross-ecomony HJM using accepted notation (such as in [7]) as much as possible. So starting in the *domestic* D-economy, $r(t)$ is the *risk-free rate* assumed equal to the funding or *repo rate* for uncollateralized assets (both equities and bonds), and $c(t)$ will be the *collateral rate* for funds deposited as collateral for trades. Let $r(t)$ accumulate in a *bank account* $\beta(t) = exp\left\{\int_0^t r(s)\,ds\right\}$, and $c(t)$ accumulate in a *collateral account* $C(t) = exp\left\{\int_0^t c(s)\,ds\right\}$. The *spot measure* $\mathbb{P}_0$ (expectation $\mathbf{E}_0$, multi-dimensional Brownian motion $W_0(t)$) is taken to be that measure under which uncollateralized assets grow at $r(t)$, and collateralized assets grow at $c(t)$. Both uncollateralized $B(t,T)$ and collateralized $D(t,T)$ zero coupon bonds pay 1-unit of D at T.

Similar variables are used in the *foreign* F-economy but with a superscript f attached, like $r^f(t)$, β_t^f, $\mathbb{P}_0^f$ ($\mathbf{E}_0^f$ and $W_0^f(t)$), $B^f(t,T)$, $D^f(t,T)$ etc. The two economies are connected through the *exchange rate* $S(t)$ which is 1-unit of foreign F-currency in domestic-D units at time-t, i.e. $\mathsf{F}\,1 = \mathsf{D}\,S(t)$.

Usually the domestic and foreign collateral rates are taken to be the rates used in US *overnight index swaps* (OIS) or their equivalent in other currencies. The risk-free and collateral rates are often different, so denote the difference by the *spreads*

$$y(t) = r(t) - c(t)\ \mathsf{D} \quad and \quad y^f(t) = r^f(t) - c^f(t)\ \mathsf{F}. \tag{1.1}$$

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The aim of this primer is to review and establish the Fujii Shimada Takahashi theorem (FST-theorem) for pricing fully collateralized instruments when the collateral can be posted in either the domestic D or a foreign F currency.

Theorem 1. *When collateral is posted in a foreign* F*-currency, the present value* h_t *in domestic* D*-currency of a fully collateralized derivative paying* h_T *at* T *is*

$$h_t = \mathbf{E}_0 \left\{ h_T \exp \int_t^T \left[-r(s) + y^f(s) \right] ds \middle| \mathfrak{F}_t \right\}, \quad \textit{or} \tag{1.2}$$

$$= \mathbf{E}_0 \left\{ e^{-\int_t^T c(s) ds} h_T \exp \left[-\int_t^T \left[y(s) - y^f(s) \right] ds \right] \middle| \mathfrak{F}_t \right\}.$$

and when collateralized in domestic$\mathsf{D}$*-currency it is*

$$h_t = \mathbf{E}_0 \left\{ e^{-\int_t^T c(s) ds} h_T \middle| \mathfrak{F}_t \right\}. \quad \blacksquare \tag{1.3}$$

2. Collateralized options in the domestic economy

Working in the domestic D economy, in this section we obtain a pricing formula for an option fully collateralized in the D-currency by tracking it with a self-funding portfolio of assets that draw funds from both collateral and bank accounts.

The ingredients are: a vector of assets $A_t = \left(A_t^R,\, A_t^C\right)$ comprising uncollateralized equities and bonds A_t^R funded at r_t, and collateralized bonds A_t^C funded at c_t; bank and collateral accounts β_t and C_t; and a fully collateralized derivative h_t on A_t paying h_T at T.

We now establish the following result, which is essentially the FST-theorem (1.3) when collateral is posted in the domestic D-currency.

Theorem 2. *The time-t price of the option fully collateralized in domestic* D*-currency is*

$$h_t = \mathbf{E}_0 \left\{ h_T \exp \left(-\int_t^T c(s)\, ds \right) \middle| \mathfrak{F}_t \right\} = h(t, A_t). \quad \blacksquare$$

Imagine that we (the bank) have sold the derivative at time t, put the proceeds in the collateral account, and will dynamically hedge it with a self-funding portfolio V_t containing assets and bank accounts. At time-t the collateral account should therefore contains ϕ_t^c units of C_t with $h_t = \phi_t^c C_t$, and for the next time step we set up a tracking portfolio

$$V_t = \phi_t C_t, \tag{2.1}$$

$$= \theta_t^R A_t^R + \theta_t^C A_t^C + \left(\phi_t^C + \phi_t\right) C_t + \psi_t \beta_t, \tag{2.2}$$

in which the asset components $\theta_t^R A^R$ and $\theta_t^C A^C$ are separately purchased via the transactions

$$\theta_t^R A_t^R + \psi_t \beta_t = 0, \qquad \theta_t^C A_t^C + \phi_t^C C_t = 0. \tag{2.3}$$

The self-funding condition for V_t is

$$dV_t = \theta_t^R dA_t^R + \theta_t^C dA_t^C + \left(\phi_t^C + \phi_t\right) dC_t + \psi_t d\beta_t. \tag{2.4}$$

Remark 3. Note that for this equation (2.4) to make sense there should be enough assets on the right-hand side of to eliminate all the components of the Brownian motion in the model.

Introduce the *starred* * discount variables comprising A_t^R discounted by β_t, and V_t and A_t^C both discounted by C_t

$$\begin{aligned} A_t^{R*} = \frac{A_t^R}{\beta_t} \qquad V = \frac{V_t}{C_t}, \qquad A_t^{C*} = \frac{A_t^C}{C_t} \qquad \Rightarrow \\ dA_t^R = dA_t^{R*}\beta_t + A_t^{R*}d\beta_t \qquad dV_t = dV_t^* C_t + V_t^* dC_t, \\ dA_t^C = dA_t^{C*}C_t + A_t^{C*}dC_t, \end{aligned} \tag{2.5}$$

substitute (2.5) into (2.4), and then gather terms to get

$$dV_t^* C_t + [V_t^* - \phi_t]\, dC_t = \theta_t^R dA_t^{R*}\beta_t + \theta_t^C dA_t^{C*}C_t + \left[\theta_t^R A_t^{R*} + \psi_t\right] d\beta_t + \left[\theta_t^C A_t^{C*} + \phi_t^C\right] dC_t.$$

Imposing the transaction conditions (2.3), and recalling that V_t is fully collateralized (2.1) gives

$$dV_t^* = \frac{\beta_t}{C_t}\theta_t^R dA_t^{R*} + \theta_t^C dA_t^{C*}. \tag{2.6}$$

Hence V_t discounted by the collateral account C_t is a martingale under the *spot measure* $\mathbb{P}_0$ which makes uncollateralized assets grow at r_t and collateralized assets grow at c_t; i.e. the measure under which they have SDEs

$$\frac{dA_t^R}{dA} = r_t dt + \sigma^R dW_0(t), \qquad \frac{dA_t^C}{dA} = c_t dt + \sigma^C dW_0(t). \tag{2.7}$$

So taking conditional expectations under $\mathbb{P}_0$, the value of the option h_t at time-t is, as quoted in Theorem-2,

$$h_t = \mathbf{E}_0 \left\{ h_T \exp\left(- \int_t^T c_s ds \right) \middle| \mathfrak{F}_t \right\} = h(t, A_t).$$

2.1. **Checking replication.** On the one hand, from (2.4), (2.3), and (2.1)

$$\begin{aligned} V_t + dV_t &= V_t + \theta_t^R dA_t^R + \theta_t^C dA_t^C + \left(\phi_t^C + \phi_t\right) dC_t + \psi_t d\beta_t, \\ &= V_t + \theta_t^R \left(dA_t^R - A_t^R r_t dt\right) + \theta_t^C \left(dA_t^C - A_t^C c_t dt\right) + V_t c_t dt. \end{aligned} \tag{2.8}$$

On the other hand, because h_t discounted by C_t is a $\mathbb{P}_0$-martingale, it must have SDE

$$dh_t = \Delta_t^R \left[dA_t^R - A_t^R r_t dt\right] + \Delta_t^C \left[dA_t^C - A_t^C c_t dt\right] + h_t c_t dt, \tag{2.9}$$

in terms of the *deltas* of the option

$$\Delta_t^R = \frac{\partial h_t}{\partial A_t^R} \quad \Delta_t^C = \frac{\partial h_t}{\partial A_t^C}.$$

Comparing (2.8) and (2.9) we see that if the amounts invested in the assets in the replicating portfolio (2.2) are the deltas of the option, i.e. $\theta_t^R = \Delta_t^R$ and $\theta_t^C = \Delta_t^C$, then we have both

$$V_t = h_t \quad \textit{and} \quad V_t + dV_t = h_t + dh_t.$$

That shows the portfolio V_t does indeed track the option value h_t.

3. Collateralization in a foreign currency

For simplicity, we focused on pricing a derivative on a single uncollateralized asset A_t in the domestic-D economy, when that derivative is collateralized in foreign-F currency. Let SDEs for A_t under $\mathbb{P}_0$, and for a similary uncollateralized foreign asset A_t^f under $\mathbb{P}_0^f$ be respectively:

$$\frac{dA_t}{A_t} = r_t dt + a_t dW_0(t), \qquad \frac{dA_{f\,t}}{A_t^f} = r_t^f dt + a_t^f dW_0^f(t).$$

Change of measure between $\mathbb{P}_0$ and $\mathbb{P}_0^f$ gets defined by the *parity pricing principle*: present valuing a domestic-D cashflow $X(T)$ at T through both economies gives

$$S(t)\,\beta^f(t)\,\mathbf{E}_0^f\left\{\left.\frac{X(T)}{S(T)\,\beta^f(T)}\right|\mathfrak{F}_t\right\} = \beta(t)\,\mathbf{E}_0\left\{\left.\frac{X(T)}{\beta(T)}\right|\mathfrak{F}_t\right\} \tag{3.1}$$

which defines the change of measure as

$$\mathbb{P}_0^f = \frac{S(T)\,\beta^f(T)}{S(0)\,\beta(T)}\mathbb{P}_0.$$

We now establish the foreign F-currency version of Theorem-2, i.e the FST result (1.2).

Theorem 4. *The time-t price of the option fully collateralized in foreign F-currency is*

$$h_t = \mathbf{E}_0\left\{\left. h(T)\exp\int_t^T \left[-r(s) + y^f(s)\right] ds\right|\mathfrak{F}_t\right\},$$

where $y^f(t) = r^f(t) - c^f(t)$ is the difference between the foreign risk free and collateral rates. ■

The option values h_t and h_T in domestic D-currency translate into values $h_t/S(t)$ and $h_T/S(T)$ in foreign F-currency. So applying Theorem-2 in the foreign economy with collateral in foreign-F we get

$$\frac{h_t}{S(t)} = \mathbf{E}_0^f\left\{\left.\frac{h_T}{S(T)}\exp\left(-\int_t^T c^f(s)\,ds\right)\right|\mathfrak{F}_t\right\}.$$

Rewriting this equation and changing measures using (3.1) then gives

$$\begin{aligned} h_t &= \beta^f(t)\,S(t)\,\mathbf{E}_0^f\left\{\left.\frac{h_T}{\beta^f(T)\,S(T)}\exp\int_t^T y^f(s)\,ds\right|\mathfrak{F}_t\right\}, \\ &= \beta(t)\,\mathbf{E}_0\left\{\left.\frac{h_T}{\beta(T)}\exp\int_t^T y^f(s)\,ds\right|\mathfrak{F}_t\right\}, \end{aligned}$$

which is Theorem-4.

4. Articulating the FST theorem

Some useful simplifications in Theorem-1 occur when some of the collateral and risk-free rates coincide:

Corollary 5. *When the foreign risk-free rate is the foreign collateral rate, i.e. when $r^f(t) = c^f(t)$* and *collateral posted in foreign-F, the present value of $h(t)$ becomes*

$$h(t) = \mathbf{E}_0\left\{\left. e^{-\int_t^T r(s)ds}h(T)\right|\mathfrak{F}_t\right\} = B(t,T)\,\mathbf{E}_T\left\{\left. h(T)\right|\mathfrak{F}_t\right\},$$

where $\mathbf{E}_T$ *is expectation under the domestic forward measure* $\mathbb{P}_{\mathbb{T}}$ $\blacksquare$.

Corollary 6. *When the domestic risk-free rate is the domestic collateral rate* $r(t) = c(t)$ *and collateral is posted in domestic-*D *again*

$$h(t) = \mathbf{E}_0 \left\{ e^{-\int_t^T r(s)ds} h(T) \middle| \mathfrak{F}_t \right\} = B(t,T)\, \mathbf{E}_T \left\{ h(T) | \mathfrak{F}_t \right\}. \quad \blacksquare \tag{4.1}$$

Corollary 7. *If foreign and domestic economies are interchanged so that:* $h(\cdot)$ *is in foreign-*F *written* $h^f(\cdot)$*, the domestic risk-free rate is the domestic collateral rate, i.e.* $r(t) = c(t)$*, and collateral is posted in domestic-*D*, then equation (4.1) transforms to*

$$h^f(t) = B^f(t,T)\, \mathbf{E}_T^f \left\{ h^f(T) \middle| \mathfrak{F}_t \right\}. \quad \blacksquare \tag{4.2}$$

It is also possible to construct a maturity T dependent measure associated with the *collateralized zero coupon bond*

$$D(t,T) = \mathbf{E}_0 \left\{ e^{-\int_t^T c(s)ds} \middle| \mathcal{F}_t \right\}, \tag{4.3}$$

that is similar in some ways to forward measures. Define $\overline{\mathbb{P}}_T$ (expectation $\overline{\mathbf{E}}_T$) by

$$\overline{\mathbb{P}}_T = Z_T\, \mathbb{P}_0 \quad \textit{where} \quad Z(T) = \frac{e^{-\int_0^T c(s)ds}}{D(0,T)}, \tag{4.4}$$

$$\Rightarrow \quad \overline{\mathbf{E}}_T \left\{ X(T) | \mathcal{F}_t \right\} = \frac{\mathbf{E}_0 \left\{ e^{-\int_0^T c(s)ds} X(T) \middle| \mathcal{F}_t \right\}}{\mathbf{E}_0 \left\{ e^{-\int_0^T c(s)ds} \middle| \mathcal{F}_t \right\}} = \frac{\mathbf{E}_0 \left\{ e^{-\int_t^T c(s)ds} X(T) \middle| \mathcal{F}_t \right\}}{D(t,T)}.$$

If the spread $y(t)$is deterministic, then $\overline{\mathbb{P}}_T$ becomes the standard T-forward measure $\mathbb{P}_T$ because

$$Z(T) = \frac{\exp\left[-\int_0^T \left[r(s) - y(s) \right] ds \right]}{\mathbf{E}_0 \exp\left\{ -\int_0^T \left[r(s) - y(s) \right] ds \right\}} = \frac{1}{\beta(T)\, B(0,T)}, \tag{4.5}$$

which is the Radon-Nikodym derivative for $\mathbb{P}_T$. In particular, $\overline{\mathbb{P}}_T = \mathbb{P}_T$ when $y(t) = 0$ and the collateral rate is the overnight rate i.e. $c(t) = r(t)$.

From the definition (4.3) of $D(t,T)$, for any $0 \le s \le t \le T$

$$\begin{aligned} \mathbf{E}_0 \left\{ e^{-\int_0^t c(u)du} D(t,T) \middle| \mathfrak{F}_s \right\} &= \mathbf{E}_0 \left\{ \mathbf{E}_0 \left\{ e^{-\int_0^T c(u)du} \middle| \mathcal{F}_t \right\} \middle| \mathfrak{F}_s \right\} \\ &= \mathbf{E}_0 \left\{ e^{-\int_0^T c(u)du} \middle| \mathcal{F}_s \right\} = e^{-\int_0^s c(u)du} D(s,T), \end{aligned}$$

giving a result that will prove useful, for example, in modeling $c(t)$ and $D(t,T)$ as SDEs:

Corollary 8. *The value of the collateralized zero coupon bond discounted by the collateral rate*

$$e^{-\int_0^t c(s)ds} D(t,T)$$

is a $\mathbb{P}_0$*-martingale, confirming (see 2.7) the drift under* $\mathbb{P}_0$ *of* $D(t,T)$ *is* $c(t)$. $\blacksquare$

In the literature $\overline{\mathbb{P}}_T$ is often referred to as the T-forward measure induced by $D(t,T)$ as numeraire because it makes collateralized trades $\overline{\mathbb{P}}_T$-martingales; that is, changing measures from $\mathbb{P}_0$ to $\overline{\mathbb{P}}_T$ allows Theorem-1 to be restated as:

Corollary 9. *When payment and pricing currencies are different*

$$\overline{\mathbf{E}}_T\left\{ h(T)\exp\left[-\int_0^T\left[y(s)-y^f(s)\right]ds\right]\middle|\,\mathfrak{F}_t\right\}=\frac{h(t)\exp\left[-\int_0^t\left[y(s)-y^f(s)\right]ds\right]}{D(t,T)},\tag{4.6}$$

and when payment and pricing currencies are the same.

$$\overline{\mathbf{E}}_T\left\{ h(T)|\,\mathfrak{F}_t\right\}=\frac{h(t)}{D(t,T)}\quad\blacksquare\tag{4.7}$$

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