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Primer: Curve Stripping with Full Collateralisation

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PRIMER: CURVE STRIPPING WITH FULL COLLATERALISATION

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Abstract. In the past five years, it has become clear that there is no longer such a thing as a single "risk-free" interest rate term structure for each currency in the market, and proper pricing of cashflows must take into account basis spreads and collateralisation. An aspect of this issue is considered in this paper: Working in a cross-economy HJM type framework, the Fujii Shimada Takahashi (FST) theorem, specifying the present value of a fully collateralised derivative, is applied to stripping cross-currency swaps. The guiding principle in our approach is that it be based on an underlying arbitrage free interest rate model, in which values of Libors and FX forwards remain invariant under reasonable choices of collateral.

Contents

1. Introduction

The interest rate setting is standard cross-ecomony (XE) Heath Jarrow Morton (HJM) using accepted notation (such as in [7]) as much as possible. Thus in the ***domestic*** D-***economy***, $r(t)$ is the risk free or discount rate for discounting cashflows, $\mathbb{P}_0$ is the spot arbitrage free measure (expectation $\mathbf{E}_0$, multi-dimensional Brownian motion $W_0(t)$) with numeraire bank account β_t to accumulate $r(t)$

$$d\beta_t = \beta_t r(t)\,dt \quad \Rightarrow \quad \beta_t = exp\left\{\int_0^t r(s)\,ds\right\},$$

and $B(t,T)$ is the zero coupon bond paying 1-unit of D at T.

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Similar variables are used in the ***foreign*** F-***economy*** but with a superscript f attached, like $r^f(t)$, β^f_t, $\mathbb{P}^f_0$ ($\mathbf{E}^f_0$ and $W^f_0(t)$), $B^f(t,T)$ etc. The two economies are connected through the exchange rate $S(t)$ which is 1-unit of foreign F-currency in domestic-D units at time-t, i.e. $\mathsf{F}\,1 = \mathsf{D}\,S(t)$.

Familiarity with these and other standard HJM ingredients and concepts, such as instantaneous forwards $f(t,T)$, T-forward measures $\mathbb{P}_T$, changing between various equivalent measures $\mathbb{P}_0 \to \mathbb{P}_T$, $\mathbb{P}_0 \to \mathbb{P}^f_0$ or $\mathbb{P}_T \to \mathbb{P}^f_T$ etc. is assumed; for a review see [7]. Also no constraints, such as requiring HJM volatility functions be deterministic, are imposed, so generality is maintained.

Let $c(t)$ and $c^f(t)$ be respectively the domestic and foreign ***collateral rates*** i.e. the overnight rates paid or received for funds deposited as collateral for trades, which can be taken to be the rates used in US overnight index swaps (OIS) or their equivalent in other currencies. The discount and collateral rates may differ, so denote the difference by the ***spreads***

$$y(t) = r(t) - c(t)\ \mathsf{D} \quad \textit{and} \quad y^f(t) = r^f(t) - c^f(t)\ \mathsf{F}. \tag{1.1}$$

Given market data in the form of OIS and Libor swaps in D, OIS and Libor swaps in F, and a cross-economy (XE) swap between both economies collateralized by either D or F, our general task in this primer is to strip these swap curves today for information about the initial term structures of the risk-free, collateral and Libor rates in both economies (i.e. the initial values of variables that would be needed to launch the model dynamically in a HJM framework).

A ***basic principle*** in our approach[1] is that today's values of Libors and foreign exchange (FX) forwards are invariant under choice of collateral, i.e. they will be the same whichever one of the domestic D or foreign F currencies is chosen to collateralize a trade. To make numbers tally according to this principle therefore requires adjustments to discount curves. That makes sense because on the one hand Libors and FX forwards are fundamental market quoted numbers around which trades are constructed and specified (so one wants an objective not subjective number to work with), and on the other hand banks now employ a multitude of subjective discount curves, so adding a few more will make little difference.

A ***basic tool*** with be the Fujii Shimada Takahashi (FST) theorem for pricing derivatives collateralized in domestic and foreign currencies (see [4, 5, 6]), which was reviewed in the Numerix primer [2]. We state it in the next section, along with its many corollaries.

For a simple exposition, but without losing much generality, we will assume throughout this note that all swaps forward start at T_0, complete at T_n, and reset quarterly at T_j $(j = 0, 1, .., n-1)$ with *coverage* $\delta_j = T_{j+1} - T_j$.

A simple ***XE basis swap*** exchanges principal $P = S(0)\,P^f$ at its start T_0, then swaps Libor plus ***basis spread*** b_n through the life of the swap (fixing at T_{j-1} and swapping at T_j for $j = 1, .., n$), and finally at completion T_n returns the principal. The cashflows are shown in the following diagram:

(1.2)

Time	*FX*	T_0	T_1	$\cdots$	T_{j+1}	$\cdots$	T_n
F	1	P^f	$P^f\delta^f_0\left[\ell^f(T_0) + b_n\right]$	$\cdots$	$P^f\delta^f_j\left[\ell^f(T_j) + b_n\right]$	$\cdots$	$-P^f + P^f\delta^f_{n-1}\left[\ell^f(T_{n-1}) + b_n\right]$
D	$S(0)$	P	$P\delta_0\ell(T_0)$	$\cdots$	$P\delta_j\ell(T_j)$	$\cdots$	$-P + P\delta_{n-1}\ell(T_{n-1})$

[1] First explicitly stated to the author as crucial by Michael Nealon

The ***XE stripping equation*** arises when these cash flows are present valued and summed to zero.

One should bear in mind that swap stripping produces ***initial values only*** of variables like discount functions or Libors. Thus the dynamics and mathematical structure of zero coupons $B(t,T)$, Libors $L(t,T)$ and associated measures like $\mathbb{P}_T$ remain unchanged if we juggle the initial values $B(0,T)$, $D(0,T)$, $L(0,T)$ and $B^f(0,T)$, $D^f(0,T)$, $L^f(0,T)$ to satisfy stripping requirements.

America is the world's biggest economy, and the USD is the reserve currency underpinning much world trade. The consequent large number of cross-economy (XE) swaps between the US and other countries therefore form a convenient benchmark for analyzing other XE swaps. In this note, the uniqueness of the US economy is encapsulated in the assumption that US spreads are zero, and we seek to explain and fit XE basis spreads, by letting spreads be non-zero in other economies. Hence a ***constant assumption about USD*** throughout this note will be

$$\boxed{\text{For USD} \quad r(t) = c(t) \quad \text{and} \quad \mathrm{y(t) = r(t) - c(t) = 0} \quad \text{always.}} \tag{1.3}$$

The US is our basic reference economy underlying all cross-economy swaps.

2. The Fujii Shimada Takahashi (FST) theorem

Theorem 1. *When collateral is posted in a foreign* F*-currency, the present value* h_t *in domestic* D*-currency of a fully collateralized derivative paying* h_T *at* T *is*

$$\begin{aligned} h_t &= \mathbf{E}_0\left\{ h_T \exp\int_t^T \left[-r(s) + y^f(s)\right] ds \middle| \mathfrak{F}_t \right\}, \quad \textit{or} \\ &= \mathbf{E}_0\left\{ e^{-\int_t^T c(s)ds} h_T \exp\left[-\int_t^T \left[y(s) - y^f(s)\right] ds\right] \middle| \mathfrak{F}_t \right\}, \end{aligned} \tag{2.1}$$

and when collateralized in domestic D*-currency it is*

$$h_t = \mathbf{E}_0\left\{ e^{-\int_t^T c(s)ds} h_T \middle| \mathfrak{F}_t \right\}. \quad \blacksquare \tag{2.2}$$

Some useful simplifications in the FST Theorem-1 occur when some of the collateral and risk-free rates coincide:

Corollary 2. *When the foreign risk-free rate is the foreign collateral rate, i.e. when* $r^f(t) = c^f(t)$ and *collateral is posted in foreign-*F*, the present value of* $h(t)$ *becomes*

$$h(t) = \mathbf{E}_0\left\{ e^{-\int_t^T r(s)ds} h(T) \middle| \mathfrak{F}_t \right\} = B(t,T)\,\mathbf{E}_T\left\{ h(T) \middle| \mathfrak{F}_t \right\},$$

where $\mathbf{E}_T$ *is expectation under the domestic forward measure* $\mathbb{P}_\mathbb{T}$ $\blacksquare$.

Corollary 3. *When the domestic risk-free rate is the domestic collateral rate* $r(t) = c(t)$ *and collateral is posted in domestic-*D *again*

$$h(t) = \mathbf{E}_0\left\{ e^{-\int_t^T r(s)ds} h(T) \middle| \mathfrak{F}_t \right\} = B(t,T)\,\mathbf{E}_T\left\{ h(T) \middle| \mathfrak{F}_t \right\}. \quad \blacksquare$$

Corollary 4. *If foreign and domestic economies are interchanged so that:* $h(\cdot)$ *is in foreign-*F *written* $h^f(\cdot)$*, the domestic risk-free rate is the domestic collateral rate, i.e.* $r(t) = c(t)$*, and collateral is posted in domestic-*D*, then Corollary-2 changes to*

$$h^f(t) = \mathbf{E}_0^f\left\{ e^{-\int_t^T r^f(s)ds} h^f(T) \middle| \mathfrak{F}_t \right\} = B^f(t,T)\,\mathbf{E}_T^f\left\{ h^f(T) \middle| \mathfrak{F}_t \right\}. \quad \blacksquare$$

It is also possible to construct a maturity T dependent measure associated with the ***collateralized zero coupon bond***

$$D(t,T) = \mathbf{E}_0\left\{ e^{-\int_t^T c(s)ds} \middle| \mathcal{F}_t \right\} = B(t,T)\,\mathbf{E}_T\left\{ e^{\int_t^T y(s)ds} \middle| \mathcal{F}_t \right\}, \tag{2.3}$$

that is similar in some ways to forward measures. Define $\overline{\mathbb{P}}_T$ (expectation $\overline{\mathbf{E}}_T$, Brownian motion $\overline{W}_T(t)$) by

$$\overline{\mathbb{P}}_T = Z_T\,\mathbb{P}_0 \quad \textit{where} \quad Z(T) = \frac{e^{-\int_0^T c(s)ds}}{D(0,T)}, \tag{2.4}$$

$$\Rightarrow \quad \overline{\mathbf{E}}_T\left\{ X(T) | \mathcal{F}_t \right\} = \frac{\mathbf{E}_0\left\{ e^{-\int_t^T c(s)ds} X(T) \middle| \mathcal{F}_t \right\}}{D(t,T)}.$$

If the spread $y(t)$ is deterministic (in particular $y(t) = 0$), then $\overline{\mathbb{P}}_T$ becomes the standard T-forward measure $\mathbb{P}_T$ because then $Z(T) = \frac{1}{\beta(T)B(0,T)}$, which is the Radon-Nikodym derivative for $\mathbb{P}_T$.

A helpful result for modeling SDEs for $c(t)$ and $D(t,T)$ is:

Corollary 5. *The value of the collateralized zero coupon bond discounted by the collateral rate is a $\mathbb{P}_0$-martingale*

$$\mathbf{E}_0\left\{ e^{-\int_0^t c(u)du} D(t,T) \middle| \mathfrak{F}_s \right\} = e^{-\int_0^s c(u)du} D(s,T)$$

for any $0 \le s \le t \le T$. Hence the drift under $\mathbb{P}_0$ of $D(t,T)$ is $c(t)$. ■

In the literature $\overline{\mathbb{P}}_T$ is often referred to as the T-forward measure induced by $D(t,T)$ as numeraire, although that may not accord with the strict definition of a T-forward measure.

Nevertheless, $D(t,T)$ as numeraire does make collateralized trades $\overline{\mathbb{P}}_T$-martingales, because changing measures from $\mathbb{P}_0$ to $\overline{\mathbb{P}}_T$ allows Theorem-1 to be restated as:

Corollary 6. *When payment and pricing currencies are different*

$$\overline{\mathbf{E}}_T\left\{ h(T)\exp\left[-\int_0^T \left[y(s) - y^f(s) \right] ds \right] \middle| \mathfrak{F}_t \right\} = \frac{h(t)\exp\left[-\int_0^t \left[y(s) - y^f(s) \right] ds \right]}{D(t,T)}, \tag{2.5}$$

and when payment and pricing currencies are the same.

$$\overline{\mathbf{E}}_T\left\{ h(T) | \mathfrak{F}_t \right\} = \frac{h(t)}{D(t,T)} \quad ■ \tag{2.6}$$

3. Modeling the collateral rate

Model the dynamics of the collateralized zero $D(t,T)$ similarly to HJM by introducing the *collateralized forward rate* $c(t,T)$ defined by

$$D(t,T) = \exp\left\{ -\int_t^T c(t,u)\,du \right\} \qquad c(t) = c(t,t),$$
$$\textit{with} \quad dc(t,T) = \alpha(t,T)\,dt + \eta(t,T)\,dW_0(t).$$

Differentiating and applying Ito, the SDE for $D(t,T)$ is given by

$$d\int_t^T c(t,u)\,du = -c(t)\,dt + \left[\int_t^T \alpha(t,u)\,du\right]dt + \left[\int_t^T \eta(t,u)\,du\right]dW_0(t) \qquad \Rightarrow$$

$$\frac{dD(t,T)}{D(t,T)} = c(t)\,dt - \left[\int_t^T \alpha(t,u)\,du\right]dt - \left[\int_t^T \eta(t,u)\,du\right]dW_0(t) + \frac{1}{2}\left|\int_t^T \eta(t,u)\,du\right|^2 dt.$$

But from Corollary-5, the drift of $D(t,T)$ is $c(t)$, so

$$\int_t^T \alpha(t,u)\,du = \frac{1}{2}\left|\int_t^T \eta(t,u)\,du\right|^2 \quad \Rightarrow \quad \alpha(t,T) = \eta(t,T)\int_t^T \eta(t,u)\,du.$$

Hence SDEs for $c(t,T)$ and $D(t,T)$ are respectively

$$dc(t,T) = \eta(t,T)\left[\int_t^T \eta(t,u)\,du\,dt + dW_0(t)\right]dt, \tag{3.1}$$

$$\frac{dD(t,T)}{D(t,T)} = c(t)\,dt - \left[\int_t^T \eta(t,u)\,du\right]dW_0(t), \tag{3.2}$$

which have exactly the same form as the SDEs for the HJM instantaneous forward $f(t,T)$and zero coupon bond $B(t,T)$.

Integrating (3.2) over the interval $[t,T]$ identifies the Radon-Nikodym (2.4)

$$D(T,T) = 1 = D(t,T)\,e^{\int_t^T c(s)ds}\mathcal{E}\left\{-\int_t^T\int_s^T \eta(s,u)\,du\,dW_0(t)\right\} \tag{3.3}$$

$$\Rightarrow \quad Z(T) = \frac{e^{-\int_0^T c(s)ds}}{D(0,T)} = \mathcal{E}\left\{-\int_0^T\int_s^T \eta(s,u)\,du\,dW_0(t)\right\},$$

showing, from the Girsanov theorem, that $\overline{W}_T(t)$ given by

$$d\overline{W}_T(t) = dW_0(t) + \int_t^T \eta(t,u)\,du\,dt$$

is Brownian motion under the measure $\overline{\mathbb{P}}_T$. Hence the forward collateral rate $c(t,T)$ is a $\overline{\mathbb{P}}_T$ martingale.

If $\sigma(t,T)$ is the volatility of the HJM instantaneous forward rate $f(t,T)$, then similarly to (3.3) we have

$$B(T,T) = 1 = B(t,T)\,e^{\int_t^T r(s)ds}\mathcal{E}\left\{-\int_t^T\int_s^T \sigma(s,u)\,du\,dW_0(t)\right\} \tag{3.4}$$

Dividing (3.3) by (3.4) gives the following expression for the spread $y(t)$

$$\exp\int_t^T y(s)\,ds = \frac{D(t,T)}{B(t,T)}\frac{\mathcal{E}\left\{-\int_t^T\int_s^T \eta(s,u)\,du\,dW_0(t)\right\}}{\mathcal{E}\left\{-\int_t^T\int_s^T \sigma(s,u)\,du\,dW_0(t)\right\}},$$

$$= \frac{D(t,T)}{B(t,T)}\mathcal{E}\left\{\int_t^T\int_s^T [\sigma(s,u) - \eta(s,u)]\,du\,dW_T(t)\right\}.$$

So an ***alternative*** to modeling the collateral rate $c(t)$ directly, is to ***model the spread*** $y(t) = r(t) - c(t)$ via

$$\exp \int_t^T y(s)\, ds = \frac{D(t,T)}{B(t,T)} \mathcal{E}\left[\int_t^T \phi(s,T)\, dW_T(s)\right] \tag{3.5}$$

where $\phi(t,T)$ can be regarded as the ***instantaneous volatility of the spread***. Alternatively, inverting this result

$$\exp\left[-\int_t^T y(s)\, ds\right] = \frac{B(t,T)}{D(t,T)} \mathcal{E}\left[-\int_t^T \phi(s,T)\, \overline{W}_T(s)\right].$$

Furthermore, the change of measure directly from $\overline{\mathbb{P}}_T$ to $\mathbb{P}_T$ is

$$\overline{\mathbb{P}}_T = \frac{B(0,T)}{D(0,T)} e^{\int_0^T y(s)ds}\, \mathbb{P}_T = \mathcal{E}\left[\int_0^T \phi(s,T)\, dW_T(s)\right] \mathbb{P}_T \qquad \Rightarrow \tag{3.6}$$

$$\overline{\mathbf{E}}_T\left\{ X(T) \middle| \mathcal{F}_t \right\} = \frac{B(t,T)}{D(t,T)} \mathbf{E}_T\left\{ \exp \int_t^T y(s)\, ds\, X(T) \middle| \mathcal{F}_t \right\} = \mathbf{E}_T\left\{ \mathcal{E}\left[\int_t^T \phi(s,T)\, dW_T(s)\right] X(T) \middle| \mathcal{F}_t \right\}.$$

Or, inverting again, the change of measure from $\mathbb{P}_T$ to $\overline{\mathbb{P}}_T$ will be

$$\mathbb{P}_T = \frac{D(0,T)}{B(0,T)} e^{-\int_0^T y(s)ds}\, \overline{\mathbb{P}}_T = \mathcal{E}\left[-\int_0^T \phi(s,T)\, d\overline{W}_T(s)\right] \overline{\mathbb{P}}_T. \tag{3.7}$$

4. Uncollateralized cross-economy swaps

To set the scene for dealing with collateralized domestic, foreign and cross-economy (XE) basis swaps, recall how such swaps were previously stripped when uncollateralized, and Libor was the discount curve.

Taking the Libor curve to also be the discount curve, domestic-D swap equations starting at T_0 and completing at T_n with swaprate ω_n

$$\omega_n \sum_{j=0}^{n-1} \delta_j B(0,T_{j+1}) = \sum_{j=0}^{n-1} \delta_j B(0,T_{j+1})\, L(0,T_j) = B(0,T_0) - B(0,T_n), \tag{4.1}$$

were locally stripped in the domestic-D economy to yield both domestic zeros $B(0,T_{j+1})$ and domestic Libors $L(0,T_j)$.

Similarly, foreign-F swap equations

$$\omega_n^f \sum_{j=0}^{n-1} \delta_j^f B^f(0,T_{j+1}) = \sum_{j=0}^{n-1} \delta_j^f B^f(0,T_{j+1})\, L^f(0,T_j) = B^f(0,T_0) - B^f(0,T_n), \tag{4.2}$$

were locally stripped in the foreign-F economy to yield both foreign zeros $B^f(0,T_{j+1})$ and foreign Libors $L^f(0,T_j)$.

At this point the foreign-F zeros $B^f(0,T_{j+1})$ were made disjoint from the foreign-F Libors $L^f(0,T_j)$, with the Libors $L^f(0,T_j)$ retained and the zeros $B^f(0,T_{j+1})$ discarded. The point of this process is to construct a new foreign-F discount function $B^f(0,T_{j+1})$ defined from the XE swap yet consistent with the retained foreign-F Libors $L^f(0,T_j)$ redefined in terms of the settlement process $\ell^f(t)$ as

$$L^f(0,T_j) = \mathbf{E}^f_{T_{j+1}} \ell^f(T_j) \tag{4.3}$$

under the forward measure $\mathbb{P}^f_{T_{j+1}}$ specified by the new $B^f(0,T_{j+1})$ (see the next Section-5 for amplification).

As described in (1.2), an XE basis swap exchanges Libors at basis spread b_n, with principal exchange $P = S(0)P^f$ at start T_0 and return at completion T_n. So present valuing (1.2) the XE stripping equation in this case is

$$\begin{aligned} P^f \left[B^f(0,T_0) - \sum_{j=0}^{n-1} \delta^f_j B^f(0,T_{j+1}) \left[L^f(0,T_j) + b_n \right] - B^f(0,T_n) \right] S(0) \\ = P \left[B(0,T_0) - \sum_{j=0}^{n-1} \delta_j B(0,T_{j+1}) L(0,T_j) - B(0,T_n) \right], \end{aligned} \tag{4.4}$$

which simplifies to zero on the domestics side, giving equations

$$\sum_{j=0}^{n-1} \delta^f_j B^f(0,T_{j+1}) \left[L^f(0,T_j) + b_n \right] = B^f(0,T_0) - B^f(0,T_n)$$

to be stripped for the new foreign-F discount function $B^f(0,T_{j+1})$.

To sum up, the stripping process has several steps:

(1) Locally strip swaps to find the local values of Libor which must be retained in a XE setting.
(2) Discard the foreign discount function.
(3) Define a new foreign discount function to fit the the XE swap.
(4) Define foreign Libor vis-a-vis the new discount function, so the resultant structure is mathematically consistent.

5. Defining Libor

When Libor is the discount curve it is simple a function two zero coupons, one maturing at T and the other three months later at $T_1 = T + \delta$

$$L(t,T) = \frac{B(t,T)}{B(t,T_1)} - 1.$$

But when Libor is disjoint from the discount curve, it needs to be defined in terms of its ***settlement process*** $\ell(t)$ as in (4.3). The basic approach is to define ***forward Libor*** $L(t,T)$ as that number which present values to zero a swaplet setting to $\ell(T)$ at time T and paying at T_1.

In the uncollateralized case, as in Section-4, that simply produces

$$B(t,T_1)\,\mathbf{E}_1 \left\{ \delta \left[L(t,T) - \ell(T) \right] \middle| \mathfrak{F}_t \right\} = 0 \quad \Rightarrow \quad L(t,T) = \mathbf{E}_1 \left\{ \ell(T) \middle| \mathfrak{F}_t \right\}.$$

A similar sort of result holds in the USD economy because of the assumption (1.3) that US spreads are zero making $\overline{\mathbb{P}}_T = \mathbb{P}_T$. From Theorem-1 applied to a domestic-D USD swaplet collateralized in domestic-D USD , the definition of US Libor becomes a clear cut

$$L(t,T)=\mathbf{E}_{T_1}\left\{\ell(T)|\,\mathfrak{F}_t\right\}=\overline{\mathbf{E}}_{T_1}\left\{\ell(T)|\,\mathcal{F}_t\right\}. \tag{5.1}$$

Remark. The thought of collateralizing a US swaplet in anything but USD is not current, so the possibility is ignored!

But in a smaller economy where spreads are non-zero and $\overline{\mathbb{P}}_T\neq\mathbb{P}_T$, defining Libor consistently requires a model constraint. From the principle of invariance of Libor with respect to collateral, it is reasonable (and necessary to the stripping procedures in Sections-7 and 8 below) to ask that its Libor be the same whether it be collateralized in its own local D-currency or foreign-F USD currency. From Theorem-1, Corollary-2 and the change of measure (3.6) that requires

$$L(t,T)=\mathbf{E}_{T_1}\left\{\ell(T)|\,\mathfrak{F}_t\right\}=\overline{\mathbf{E}}_{T_1}\left\{\ell(T)|\,\mathcal{F}_t\right\},\qquad\text{that is} \tag{5.2}$$

$$\mathbf{E}_{T_1}\left\{\ell(T)|\,\mathfrak{F}_t\right\}=\mathbf{E}_{T_1}\left\{\mathcal{E}\left[\int_t^T\phi(s,T_1)\,dW_{T_1}(s)\right]\ell(T)\middle|\,\mathcal{F}_t\right\}.$$

The requirement (5.2) imposes a modeling constraint. For example, if Libor were modeled in standard lognormal Libor market model fashion with volatility $\xi(t,T)$ then

$$\ell(T)=L(T,T)=L(t,T)\,\mathcal{E}\left[\int_t^T\xi(s,T)\,dW_{T_1}(s)\right],$$

and we would need $\xi(t,T)$ to be orthogonal to $\phi(t,T_1)$

$$\xi(t,T)\,\phi(t,T_1)=0\quad\forall\,0\leq t\leq T. \tag{5.3}$$

6. Stripping single currency swaps

In this section we look at everything locally, and obtain .initial values for all discount functions, collateralized zeros and Libors in a single isolated economy

Two basic types of swap are available for local curve stripping, where ***local*** in regard to collateralization means domestic-D locally stripped swaps are collateralized in domestic-D currency, while foreign-F locally stripped swaps are collateralized in the foreign-F currency.

Overnight index swaps (OIS) accumulate the overnight rate (assumed to be the collateral rate $c(t)$) over a quarter and swap the difference against a market quoted OIS *par rate* κ_n set to make the present value of the whole swap zero. Applying the domestic version (2.2) of the FST Theorem, gives

$$\sum_{j=0}^{n-1}\mathbf{E}_0\left\{\exp\int_0^{T_{j+1}}[-c(s)]\,ds\left[\left(\exp\int_{T_j}^{T_{j+1}}c(s)\,ds-1\right)-\delta_j\kappa_n\right]\right\}=0.$$

These equations simplify to the following set of expressions for $D(0,T_{j+1})$

$$\kappa_n\sum_{j=0}^{n-1}\delta_jD(0,T_{j+1})=D(0,T_0)-D(0,T_n), \tag{6.1}$$

which can be bootstrapped to yield the collateralized zeros $D(0,T_{j+1})$.

Libor swaps exchange, at times $T_{j+1}\quad j=0,..,n-1$, the quarterly Libor rate settling to $\ell(T_j)$ at T_j against a market quoted *swaprate* ω_n set to make the present value of the whole swap zero. Hence, with forward Libor $L(t,T)$ defined as in Section-5,

$$\omega_n\sum_{j=0}^{n-1}\delta_j D(0,T_{j+1})=\sum_{j=0}^{n-1}\delta_j D(0,T_{j+1})\,\overline{\mathbf{E}}_{T_1}\{\ell(T_j)\}=\sum_{j=0}^{n-1}\delta_j D(0,T_{j+1})\,L(0,T_j)\,, \tag{6.2}$$

which can be bootstrapped to yield $L(0,T_j)$ because the $D(0,T_{j+1})$ have been determined by the OIS.

No assumptions about the dynamics of the collateral rate $c(t)$ have been made in this strip, which produces the initial term structures $D(0,\cdot)$ and $L(0,\cdot)$ of the collateral and Libor rates.

To get the initial term structure $B(0,\cdot)$ of the risk-free rate one could assume, as for USD, that spreads are zero so $B(0,\cdot)=D(0,\cdot)$. Alternatively, if swaprate quotes ω_n^* for swaps collateralized in foreign-F USD currency exist, then from Corollary-2 and the Libor consistency equation (5.2)

$$\omega_n^*\sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})=\sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})\,\mathbf{E}_{T_1}\{\ell(T_j)\}=\sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})\,L(0,T_j)\,,$$

from which $B(0,T_{j+1})$ can be stripped using the already determined Libor $L(0,T_j)$.

Similar results hold for local stripping in the foreign-F economy.

7. USD XE swaps from the US side

In this section we look at everything from a US perspective, and obtain initial values for all discount functions, collateralized zeros and Libors. We base ourselves in the US domestic-D economy with the aim of present valuing swaps in USD. The choice of collateral for the XE-swap is either domestic-D (USD) currency or the foreign-F currency used on the other side of the XE swap. Ingredients for this kind of XE-swap include:

(1) From (1.3), domestic USD risk-free and collateral rates coincide $r(t)=c(t)$ and $\overline{\mathbb{P}}_T=\mathbb{P}_T$,
(2) The domestic US discount rate $B(0,T)$ and US Libor $L(0,T)$ are specified by local stripping of OIS and Libor swaps using (6.1) and (6.2) in the domestic-D (USD) economy.
(3) and $L^f(0,T)$ is specified from local stripping of OIS and Libor swaps using (6.1) and (6.2) in the foreign-F economy.

Specifically on the domestic-D USD side $B(0,T)=D(0,T)$ and $L(0,T)$ will be stripped from

$$\kappa_n\sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})=B(0,T_0)-B(0,T_n)\,,\quad \omega_n\sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})=\sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})\,L(0,T_j)\,,$$

and on the foreign-F side $D^f(0,T)$ and $L^f(0,T)$ will be stripped from

$$\kappa_n^f\sum_{j=0}^{n-1}\delta_j^f D^f(0,T_{j+1})=D^f(0,T_0)-D^f(0,T_n)\,,\quad \omega_n^f\sum_{j=0}^{n-1}\delta_j^f D^f(0,T_{j+1})=\sum_{j=0}^{n-1}\delta_j^f D^f(0,T_{j+1})\,L^f(0,T_j)\,. \tag{7.1}$$

It remains to tackle the XE swap with the foreign-F discount function $B^f(0,T)$ free to be specified.

7.1. **XE-swap is collateralized in the domestic-D (USD) currency.** Because of the dominant US financial position, collateralization in D(USD) is preferred. In this case Corollary-4 governs present valuing on the foreign side

$$h^f(t) = B^f(t,T)\,\mathbf{E}_T^f\left\{ h^f(T) \middle| \mathfrak{F}_t \right\},$$

with foreign-F Libors defined from a foreign settlements process $\ell^f(T)$ as in Section-5 by

$$L^f(t,T) = \mathbf{E}_{T_1}^f\left\{ \ell^f(T) \middle| \mathfrak{F}_t \right\}.$$

The XE stripping equation therefore becomes

$$P^f\left[B^f(0,T_0) - \sum_{j=0}^{n-1} \delta_j^f B^f(0,T_{j+1})\left[L^f(0,T_j)+b_n\right] - B^f(0,T_n)\right]S(0) \tag{7.2}$$
$$= P\left[B(0,T_0) - \sum_{j=0}^{n-1}\delta_j B(0,T_{j+1})\,L(0,T_j) - B(0,T_n)\right],$$

which can be directly stripped for the appropriate foreign-F discount curve $B^f(0,T)$.

7.2. **XE-swap is collateralized in foreign-F currency.** In this case applying the FST Theorem-1, we find (after a little manipulation) that present valuing on domestic and foreign sides is determined by

$$h_t = B(t,T)\,\mathbf{E}_T\left\{ h_T \exp\int_t^T y^f(s)\,ds \middle| \mathfrak{F}_t\right\}, \qquad h_t^f = B^f(t,T)\,\mathbf{E}_T^f\left\{ h_T^f \exp\int_t^T y^f(s)\,ds \middle| \mathfrak{F}_t\right\}.$$

Clearly, the XE stripping equation won't simplify much without assuming the foreign-F spread $y^f(s)$ is deterministic, in which case the term $\exp\int_t^T y^f(s)\,ds$ will cancel on domestic and foreign sides giving exactly the same XE stripping equation (7.2) for $B^f(0,T)$ as when collateral was posted in domestic-D (USD) currency.

8. USD XE swaps from the other side

In this section we look at a USD XE swap from the perspective of the smaller domestic-D economy with the aim of present valuing swaps in its domestic-D currency. The choice of collateral for the XE-swap is either domestic-D currency or foreign-F USD currency.

In this situation XE swaps are generally collateralized in the stronger foreign-F USD currency, foreign-F USD spreads are set zero, yet present values are required in domestic-D currency. So ingredients in this kind of XE swap include:

(1) $r^f(t) = c^f(t)$ and $\overline{\mathbb{P}}_T^f = \mathbb{P}_T^f$ on the foreign-F USD side
(2) From (6.1) and (6.2), the foreign-F USD discount $B^f(t,T)$ and Libor $L^f(0,T_j)$ is specified by American OIS and Libor swaps,
(3) Again from (6.1) and (6.2), domestic-D Libor $L(0,T_j)$ is specified by domestic-D OIS and Libor swaps.

Specifically, on the foreign-F USD side $B^f(t,T) = D^f(t,T)$and $L^f(0,T_j)$ will be stripped from

$$\kappa_n^f \sum_{j=0}^{n-1}\delta_j^f B^f(0,T_{j+1}) = B^f(0,T_0) - B^f(0,T_n), \quad \omega_n^f \sum_{j=0}^{n-1}\delta_j^f B^f(0,T_{j+1}) = \sum_{j=0}^{n-1}\delta_j^f B^f(0,T_{j+1})\,L^f(0,T_j),$$

and on the domestic-D side $D(0,T_j)$ and $L(0,T_j)$ will be stripped from

$$\kappa_n \sum_{j=0}^{n-1} \delta_j D(0,T_{j+1}) = D(0,T_0) - D(0,T_n), \quad \omega_n \sum_{j=0}^{n-1} \delta_j D(0,T_{j+1}) = \sum_{j=0}^{n-1} \delta_j D(0,T_{j+1}) L(0,T_j).$$

It remains to tackle the XE swap with the domestic-D discount function $B(0,T_j)$ free to be specified.

8.1. **Collateralized in foreign-F (USD) currency.** In this case Corollary-2 applies on the domestic side

$$h(t) = \mathbf{E}_0 \left\{ e^{-\int_t^T r(s)ds} h(T) \middle| \mathfrak{F}_t \right\} = B(t,T)\, \mathbf{E}_T \left\{ h(T) | \mathfrak{F}_t \right\}.$$

With the spread b_n now appearing on the domestic-D side (because its the smaller economy), the XE stripping equation therefore becomes

$$P^f \left[B^f(0,T_0) - \sum_{j=0}^{n-1} \delta_j^f B^f(0,T_{j+1}) L^f(0,T_j) - B^f(0,T_n) \right] S(0) \tag{8.1}$$
$$= P \left[B(0,T_0) - \sum_{j=0}^{n-1} \delta_j B(0,T_{j+1}) \left[L(0,T_j) + b_n \right] - B(0,T_n) \right],$$

for the domestic-D discount factors $B(0,T_j)$.

8.2. **Collateralized in domestic-D currency.** In this case applying the FST Theorem-1, we find (after a little manipulation) that present valuing on domestic and foreign sides is determined by

$$h_t = B(t,T)\, \mathbf{E}_T \left\{ h_T \exp \int_t^T y(s)\, ds \middle| \mathfrak{F}_t \right\}, \qquad h_t^f = B^f(t,T)\, \mathbf{E}_T^f \left\{ h_T^f \exp \int_t^T y(s)\, ds \middle| \mathfrak{F}_t \right\}.$$

Similarly to SubSection-7.2, assuming the domestic spread $y(s)$ is deterministic causes the term $\exp \int_t^T y(s)\, ds$ to cancel on domestic and foreign sides giving exactly the same XE stripping equation (8.1) for $B(0,T_j)$ as when collateral was posted in foreign-F (USD) currency.

9. FX forwards

9.1. **Uncollateralized case.** Recall how FX forwards were priced when uncollateralized and Libor was the discount curve. If $S_T(t)$ was the time-t value of the contract, entered into at zero cost, delivering $1\,\mathsf{F} = S(T)\,\mathsf{D}$ at time-T, then by pricing parity

$$S_T(t)\, \mathbf{E}_0 \left\{ e^{-\int_t^T r(s)ds} \middle| \mathfrak{F}_t \right\} = \mathbf{E}_0 \left\{ e^{-\int_t^T r(s)ds} S(T) \middle| \mathfrak{F}_t \right\} = S(t)\, \mathbf{E}_0^f \left\{ e^{-\int_t^T r(s)ds}\, 1 \middle| \mathfrak{F}_t \right\},$$

which simplified to the ***standard uncollateralized form*** for the FX forward

$$S_T(t) = \frac{B^f(t,T)}{B(t,T)} S(t). \tag{9.1}$$

9.2. **Collateralized case.** When collateralized in ***domestic***-D currency, apply the FST Theorem-1 to the domestic and foreign sides of the pricing parity equation to get

$$S_T(t)\,\mathbf{E}_0\left\{ e^{-\int_t^T c(s)ds} \middle| \mathfrak{F}_t \right\} = \mathbf{E}_0\left\{ e^{-\int_t^T c(s)ds} S(T) \middle| \mathfrak{F}_t \right\}$$
$$= \mathbf{E}_0^f\left\{ e^{-\int_t^T c^f(s)ds} \exp\left[-\int_t^T \left[y^f(s) - y(s) \right] ds \right] \middle| \mathfrak{F}_t \right\} S(t),$$

giving the following formula for the FX forward

$$S_T(t) = \frac{B^f(t,T)\,\mathbf{E}_T^f\left\{ \exp\int_t^T y(s)\,ds \middle| \mathfrak{F}_t \right\}}{D(t,T)} S(t), \tag{9.2}$$

In a similar fashion, when collateralized in ***foreign***-F currency

$$S_T(t) = \frac{D^f(t,T)}{B(t,T)\,\mathbf{E}_T\left\{ \exp\int_t^T y^f(s)\,ds \middle| \mathfrak{F}_t \right\}} S(t), \tag{9.3}$$

Our ***pricing principal*** that FX forwards should be invariant under choice of collateral requires that (9.2) and (9.3) be identical. The simplest way to ensure that is to make the spreads $y(t)$ and $y^f(t)$ deterministic, when the FX forward takes the form (9.1).

Alternatively, in the setting of US dollar XE swaps in Section-8 we have $y^f(s) = 0$ so if the FX forward is collateralized in foreign-F (USD) currency, equation (9.3) immediately gives the standard uncollateralized form (9.1).

To get (9.2) equal then requires the model constraint

$$\mathbf{E}_T^f\left\{ \exp\int_t^T y(s)\,ds \middle| \mathfrak{F}_t \right\} = \mathbf{E}_T\left\{ \exp\int_t^T y(s)\,ds \middle| \mathfrak{F}_t \right\}. \tag{9.4}$$

Recalling the spread model (3.5)

$$\exp\int_t^T y(s)\,ds = \frac{D(t,T)}{B(t,T)}\mathcal{E}\left[\int_t^T \phi(s,T)\,dW_T(s) \right],$$

for the spread $y(t)$, and changing measures from $\mathbb{P}_T^f$ to $\mathbb{P}_T$ (see [7]) with

$$\mathbb{P}_T^f = \mathcal{E}\left(\int_0^t \nu_T(s)\,dW_T(s) \right) \mathbb{P}_T,$$

where $\nu_T(t)$ is the volatility of $S_T(t)$, the constraint (9.4) is

$$\mathbf{E}_T\left\{ \mathcal{E}\left(\int_0^t \nu_T(s)\,dW_T(s) \right) \mathcal{E}\left[\int_t^T \phi(s,T)\,dW_T(s) \right] \middle| \mathfrak{F}_t \right\} = 1,$$

which, similarly to the modeling constraint (5.3) on Libor, requires the volatility vectors $\nu_T(s)$ and $\phi(s,T)$ to be orthogonal

$$\nu_T(t)\,\phi(t,T) = 0 \quad \forall 0 \le t \le T. \tag{9.5}$$

The same modeling constraint is obtained if the above FX forward analysis is carried out in the context of a US-centric swap as in Section-7.

Today's actual FX forward will therefore be

$$S_T(0) = \frac{B^f(0,T)}{B(0,T)} S(0)$$

with the discount functions $B(0,T)$ and $B^f(0,T)$ determined by one of the swap strips detailed in Sections-7 and -8.

10. Conclusion

We have shown how to jointly strip US swaps, other country swaps, and their mutual cross-economy swaps in the framework of a consistent arbitrage free interest rate model. The strips ***cleanly*** produced initial values for discount functions $B(0,\cdot)$, collateralized zeros $D(0,\cdot)$ and Libors $L(0,\cdot)$ in both economies, and also FX forwards $S_T(0)$, without having to resort to the ***discard and replace*** methods needed to fit uncollateralized swaps.

The underlying dynamic arbitrage-free interest rate model is an extension of standard HJM, in which volatilities can be stochastic. This model is therefore quite general and easily articulated into forms like the Libor market model.

The mechanism for fitting the XE basis spread $'b'$ was to zero US spreads, but let the non-US discount and collateral rates differ forming a strictly non-zero spread $'y'$. That the two spreads $'b'$ and $'y'$ must be intimately related, could help in calibrating the model, and be an interesting point of research.

The other country non-zero spread $'y'$ imposes conditions via the two model constraints (5.3) and (9.5) and creates a simplification problem in Section-7.2 and Section-8.2. This confluence of forces seems to demand a deterministic $'y'$ to rid us of these irritations. And mature reflection supports this simplification when we realize that it frees us to model Libor and FX forwards with few restrictions; e.g. Libor can still be at a positive stochastic spread to the OIS or risk-free rates even though $'y'$ is deterministic.

References

[1] Brace A (2008) Engineering BGM *ISBN 13:978-1-58488-968-7*

[2] Numerix LLC (2012) Primer: The FST theorem for pricing with foreign collateral, *Numerix Primers*

[3] Pallavicini A, Perini D, Brigo D (2011) Funding value adjustment; a consistent framework including CVA, DVA, collateral, netting rules and re-hypothecation `http://arxiv.org/abs/1112.1521v2`

[4] Fujii, M , Shimada, Y, Takahashi, A (2010) A Note on Construction of Multiple Swap Curves with and without Collateral. CARF Working Paper Series No. CARF-F-154. Available at SSRN: `http://ssrn.com/abstract=1440633`

[5] Fujii, M , Shimada, Y, Takahashi, A (2010) Modeling interest rate term structures under collateralization and its implications. Available at SSRN: `http://ssrn.com/abstract=1681910`

[6] Fujii M, Takahashi A (2011) Choice of collateral currency. *Risk*, January 2011.

[7] Numerix LLC (2011) Primer: The HJM Interest Rate Framework, *Numerix Primers*

[8] Piterbarg V (2010) Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, February 2010.