TWO CURVES, ONE PRICE

Pricing & Hedging Interest Rate Derivatives Using Different Yield Curves For Discounting and Forwarding

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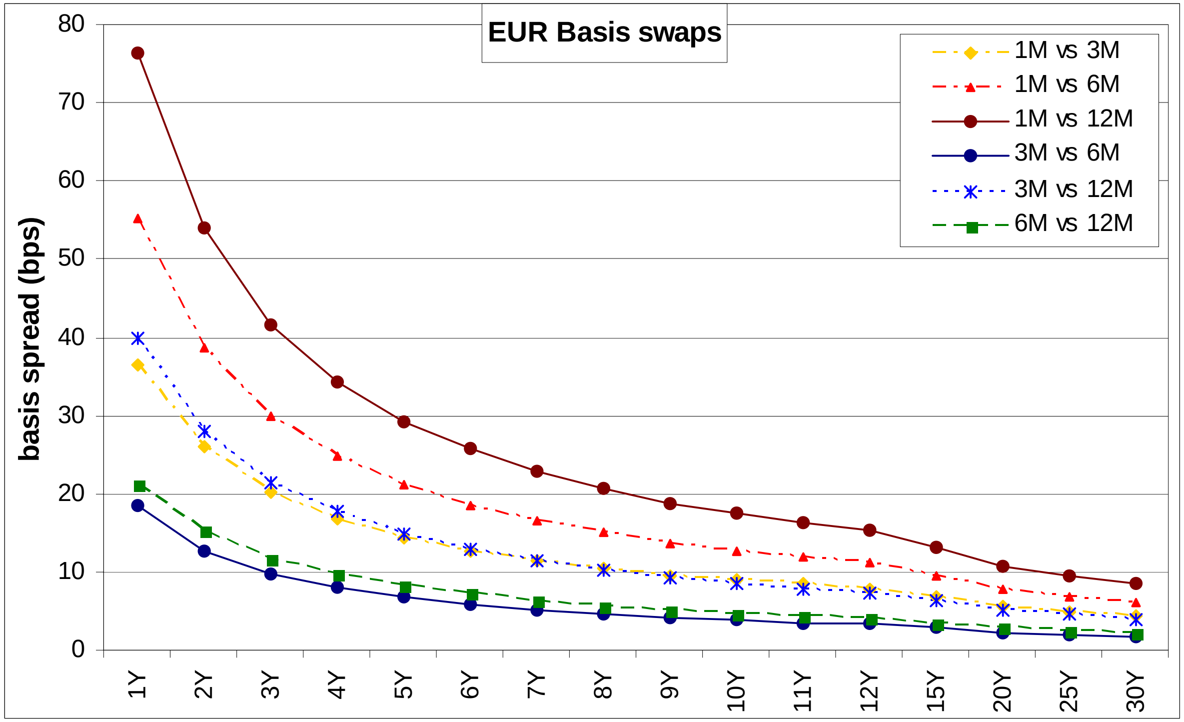
Summary

1: Context & Market Practices: *Single-Curve Pricing & Hedging IR Derivatives*

Pre credit-crunch single curve market practice:

- select ***a single set*** of the most convenient (e.g. liquid) vanilla interest rate instruments traded on the market with increasing maturities; for instance, a very common choice in the EUR market is a combination of short-term EUR ***deposit***, medium-term ***FRA/Futures*** on Euribor3M and medium-long-term ***swaps*** on Euribor6M;
- build ***a single yield curve*** $\mathcal{C}$ using the selected instruments plus a set of bootstrapping rules (e.g. pillars, priorities, interpolation, etc.);
- Compute, ***on the same curve*** $\mathcal{C}$, ***forward rates, cashflows, discount factors*** and work out the ***prices*** by summing up the discounted cashflows;
- compute the delta sensitivity and ***hedge*** the resulting delta risk using the suggested amounts (hedge ratios) of the ***same*** set of vanillas.

1: Context & Market Practices: *Market Evolution*



Quotations as of 16 Feb. 2009 (source: Reuters ICAPEUROBASIS)

Basis swaps (single currency) as 2 swaps:

1. $Euribor3M_T$ vs R_T^{3M}
2. $Euribor6M_T$ vs R_T^{6M}
3. $Basis_T^{3M6M} = R_T^{3M} - R_T^{6M}$

Other market evidences:

- the divergence between deposit (Euribor based) and OIS (Overnight Indexed Swaps, Eonia based) rates;
- The divergence between FRA rates and the corresponding forward rates implied by consecutive deposits *(see e.g. refs. [2], [6], [7].*

1: Context & Market Practices:
Multiple-Curve Pricing & Hedging IR Derivatives

Post credit-crunch multiple curve market practice:

No market standard ...Eonia?

- build ***a single discounting curve*** $\mathcal{C}_d$ using the preferred procedure;
- build ***multiple distinct forwarding curves*** $\mathcal{C}_{f1...}$ $\mathcal{C}_{fn}$ using the preferred distinct selections of vanilla interest rate instruments each ***homogeneous*** in the underlying rate tenor (typically 1M, 3M, 6M, 12M);
- compute the forward rates with tenor *f* ***on the corresponding forwarding curve*** $\mathcal{C}_f$ and calculate the corresponding cashflows;
- compute the corresponding discount factors using the **discounting curve** $\mathcal{C}_d$ and work out prices by summing the discounted cashflows;
- compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the ***corresponding*** set of vanillas.

1: Context & Market Practices: *Rationale*

- Apparently similar interest rate instruments with different underlying rate tenors are characterised, in practice, by ***different liquidity and credit risk premia***, reflecting the ***different views and interests of the market players***.

- Thinking in terms of more fundamental variables, e.g. a ***short rate***, the credit crunch has acted as a sort of ***symmetry breaking mechanism***: from a (unstable) situation in which an unique short rate process was able to model and explain the whole term structure of interest rates of all tenors, towards a sort of ***market segmentation into sub-areas*** corresponding to instruments with different underlying rate tenors, characterised, in principle, by ***distinct dynamics***, e.g. different short rate processes.

- Notice that market segmentation was already present (and well understood) before the credit crunch (see e.g. ref. [3]), but not effective due to negligible basis spreads.

2: Multiple-Curve Framework: *General Assumptions*

1. There exist **multiple different interest rate sub-markets** M_x, $x = \{d, f_1, \ldots, f_n\}$ characterized by the **same currency** and by distinct **bank accounts** B_x and **yield curves**

$$C_x := \{T \to P_x(t_0, T), T \geq t_0\},$$

2. The usual **no arbitrage relation**

$$P_x(t, T_2) = P_x(t, T_1) \times P_x(t, T_1, T_2)$$

holds in each interest rate market M_x.

3. Simple compounded **forward rates** are defined as usual for $t \leq T_1 < T_2$

$$P_x(t, T_1, T_2) = \frac{P_x(t, T_2)}{P_x(t, T_1)} = \frac{1}{1 + F_x(t; T_1, T_2)\tau_x(T_1, T_2)},$$

4. **FRA pricing** under $Q_x^{T_2}$ forward measure associated to numeraire $P_x(t, T_2)$

$$\begin{aligned}\mathbf{FRA}_x(t; T_1, T_2, K) &= P_x(t, T_2)\tau_x(T_1, T_2)\left\{\mathrm{E}_t^{Q_x^{T_2}}[L_x(T_1, T_2)] - K\right\} \\ &= P_x(t, T_2)\tau_x(T_1, T_2)[F_x(t; T_1, T_2) - K],\end{aligned}$$

2: Multiple-Curve Framework: *Pricing Procedure*

1. assume $\mathcal{C}_d$ as the **discounting curve** and $\mathcal{C}_f$ as the **forwarding curve**;
2. calculate any relevant spot/forward rate **on the forwarding curve** $\mathcal{C}_f$ as

$$F_f(t;T_{i-1},T_i) = \frac{P_f(t,T_{i-1}) - P_f(t,T_i)}{\tau_f(T_{i-1},T_i)\,P_f(t,T_i)}, \quad t \leq T_{i-1} < T_i,$$

3. calculate cashflows c_i, $i = 1,\ldots,n$, as expectations of the *i-th* coupon payoff π_i with respect to the **discounting T_i - forward measure** $Q_d^{T_i}$

$$c_i := c(t,T_i,\pi_i) = E_t^{Q_d^{T_i}}[\pi_i];$$

4. calculate the price π at time t by discounting each cashflow c_i using the corresponding discount factor $P_d(t,T_i)$ obtained from the **discounting curve** $\mathcal{C}_d$ and summing,

$$\pi(t,\boldsymbol{T}) = \sum_{i=1}^{n} P(t,T_i)\,E_t^{Q_d^{T_i}}[\pi_i];$$

5. Price **FRAs** as

$$\mathbf{FRA}(t;T_1,T_2,K) = P_d(t,T_2)\,\tau_f(T_1,T_2)\left\{\mathrm{E}_t^{Q_d^{T_2}}\left[F_f(T_1;T_1,T_2)\right] - K\right\}$$

2: Multiple-Curve Framework: *No Arbitrage and Forward Basis*

Classic single-curve no arbitrage relations are broken: for instance, by specifying the subscripts *d* and *f* as prescribed above we obtain the two eqs.

$$P_d(t, T_2) = P_d(t, T_1)\, P_f(t, T_1, T_2),$$

$$P_f(t, T_1, T_2) = \frac{1}{1 + F_f(t; T_1, T_2)\, \tau_f(T_1, T_2)} = \frac{P_f(t, T_2)}{P_f(t, T_1)},$$

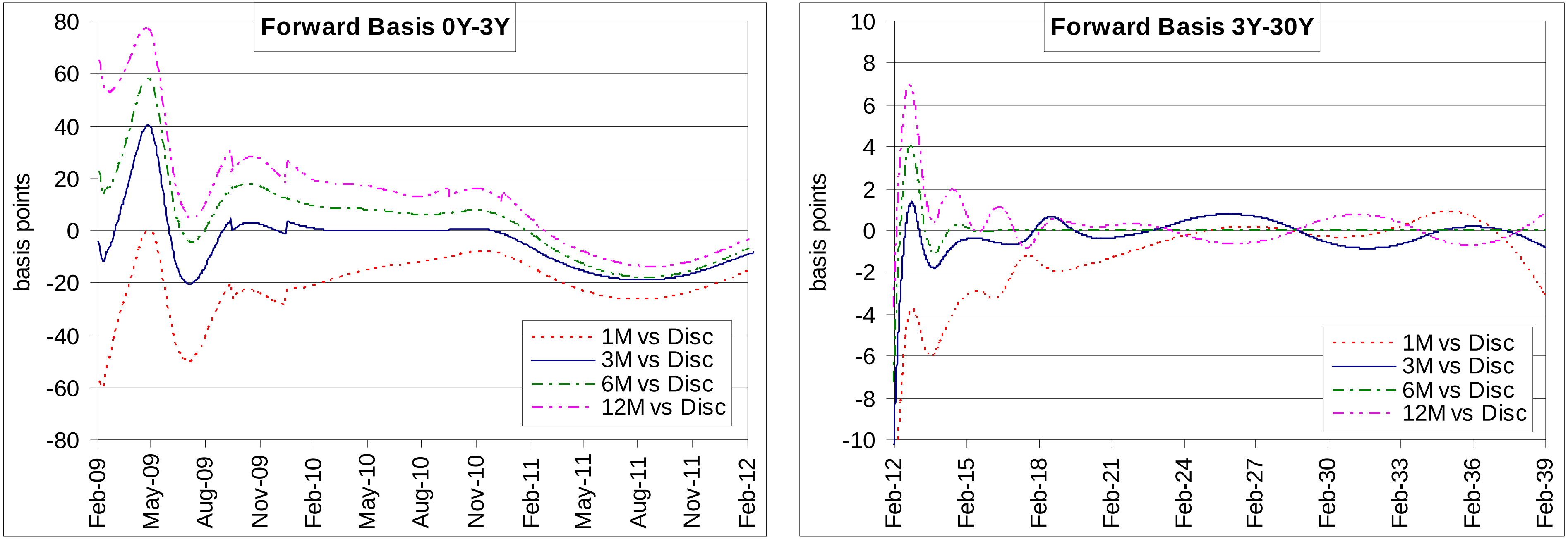
that clearly cannot hold at the same time. No arbitrage is recovered by taking into account the **forward basis** as follows

$$P_f(t, T_1, T_2) = \frac{1}{1 + F_f(t; T_1, T_2)\, \tau_f(T_1, T_2)} := \frac{1}{1 + \left[F_d(t; T_1, T_2) + BA_{fd}(t; T_1, T_2) \right] \tau_d(T_1, T_2)},$$

for which we obtain the following static expression in terms of discount factors

$$BA_{fd}(t; T_1, T_2) = \frac{1}{\tau_d(T_1, T_2)} \left[\frac{P_f(t, T_1)}{P_f(t, T_2)} - \frac{P_d(t, T_1)}{P_d(t, T_2)} \right].$$

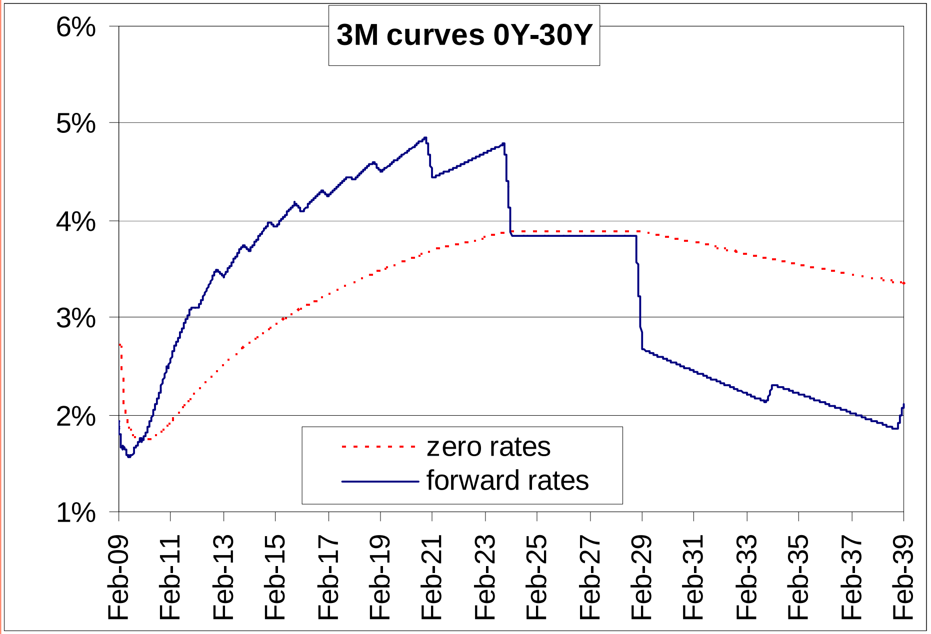
2: Multiple-Curve Framework: *Forward Basis Curves*

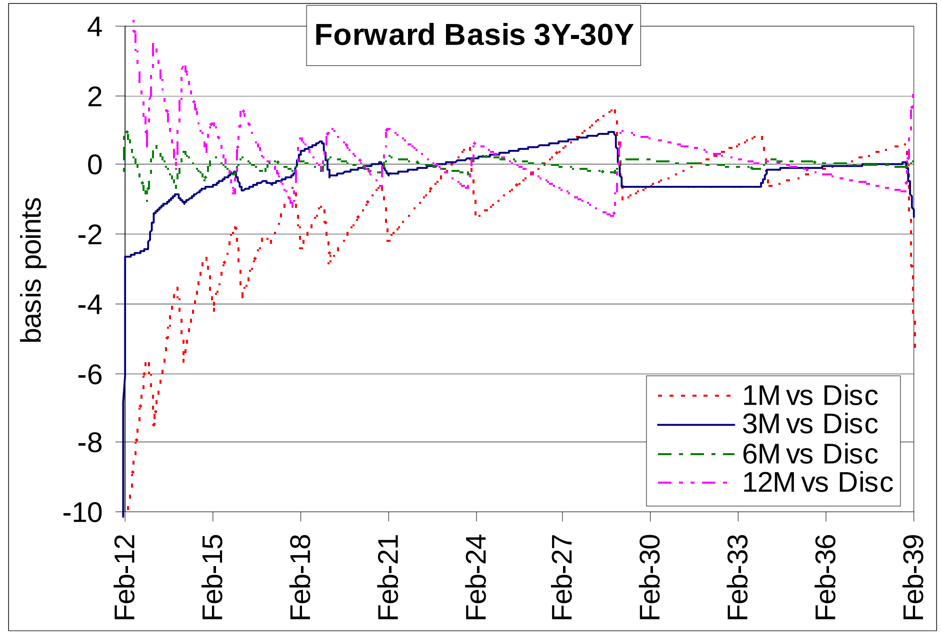


Forward basis (bps) as of end of day 16 Feb. 2009, daily sampled 3M tenor forward rates calculated on $\mathcal{C}_{1M}$, $\mathcal{C}_{3M}$, $\mathcal{C}_{6M}$, $\mathcal{C}_{12M}$ curves against $\mathcal{C}_{d}$ taken as reference curve. Bootstrapping as described in ref. [2].

The richer term structure of the forward basis curves provides a sensitive indicator of the tiny, but observable, statical differences between different interest rate market sub-areas in the post credit crunch interest rate world, and a tool to assess the degree of liquidity and credit issues in interest rate derivatives' prices. Provided that…

2: Multiple-Curve Framework:
Bad Curves





Left: 3M zero rates (red dashed line) and forward rates (blue continuous line). Right: forward basis. Linear interpolation on zero rates has been used. Numerical results from QuantLib (www.quantlib.org).

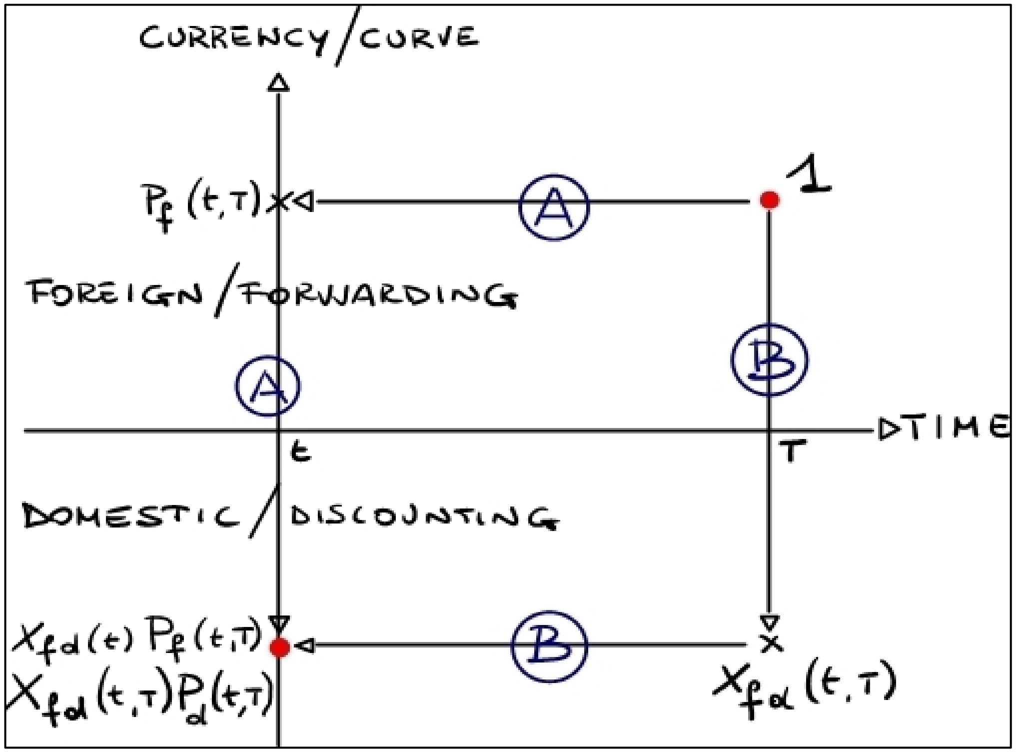
…smooth yield curves are used…Non-smooth bootstrapping techniques, e.g. linear interpolation on zero rates (still a diffused market practice), produce zero curves with no apparent problems, but ugly forward curves with a sagsaw shape inducing, in turn, strong and unnatural oscillations in the forward basis (see [2]).

3: Foreign Currency Analogy: *Spot and Forward Exchange Rates*

A second issue regarding no arbitrage arises in the double-curve framework:

$$\mathbf{FRA}(t; T_1, T_2, K) = P_d(t, T_2)\,\tau_f(T_1, T_2)\left\{\mathrm{E}_t^{Q_d^{T_2}}\left[F_f(T_1; T_1, T_2)\right] - K\right\}$$

$$\neq P_d(t, T_2)\,\tau_f(T_1, T_2)\left[F_f(T_1; T_1, T_2) - K\right]$$



Picture of no arbitrage definition of the forward exchange rate. Circuitation (round trip) ⇒ no money is created or destructed.

1. Double-curve-double-currency:
   ⇒ *d = domestic, f = foreign*

$$c_d(t) = x_{fd}(t)\,c_f(t),$$
$$X_{fd}(t, T)\,P_d(t, T) = x_{fd}(t)\,P_f(t, T),$$
$$x_{fd}(t_0) = x_{fd,0}.$$

2. Double-curve-single-currency:
   ⇒ *d = discounting, f=forwarding*

$$c_d(t) = x_{fd}(t)\,c_f(t),$$
$$X_{fd}(t, T)\,P_d(t, T) = x_{fd}(t)\,P_f(t, T),$$
$$x_{fd}(t_0) = 1.$$

3: Foreign Currency Analogy: *Quanto Adjustment*

1. Assume a lognormal martingale dynamic for the $\mathcal{C}_f$ **(foreign) forward rate**

$$\frac{dF_f(t;T_1,T_2)}{F_f(t;T_1,T_2)} = \sigma_f(t)\,dW_f^{T_2}(t), \quad Q_f^{T_2} \leftrightarrow P_f(t,T_2) \leftrightarrow C_f;$$

2. since $x_{fd}(t)P_f(t,T)$ is the price at time *t* of a $\mathcal{C}_d$ **(domestic)** tradable asset, the forward exchange rate must be a **martingale process**

$$\frac{dX_{fd}(t,T_2)}{X_{fd}(t,T_2)} = \sigma_X(t)\,dW_X^{T_2}(t), \quad Q_d^{T_2} \leftrightarrow P_d(t,T_2) \leftrightarrow C_d,$$

with

$$dW_f^{T_2}(t)\,dW_X^{T_2}(t) = \rho_{fX}(t)\,dt;$$

3. by **changing numeraire** from $\mathcal{C}_f$ to $\mathcal{C}_d$ we obtain the **modified dynamic**

$$\frac{dF_f(t;T_1,T_2)}{F_f(t;T_1,T_2)} = \mu_f(t)\,dt + \sigma_f(t)\,dW_f^{T_2}(t), \quad Q_d^{T_2} \leftrightarrow P_d(t,T_2) \leftrightarrow C_d,$$

$$\mu_f(t) = -\sigma_f(t)\sigma_X(t)\rho_{fX}(t);$$

4. and the modified **expectation** including the (additive) **quanto-adjustment**

$$E_t^{Q_d^{T_2}}\left[L_f(T_1,T_2)\right] = F_f(t;T_1,T_2) + QA_{fd}\left(t;T_1,\sigma_f,\sigma_X,\rho_{fX}\right),$$

$$QA_{fd}\left(t;T_1,\sigma_f,\sigma_X,\rho_{fX}\right) = F_f(t;T_1,T_2)\left[\exp\int_t^{T_1}\mu_f(s)\,ds - 1\right].$$

4: Pricing & Hedging IR Derivatives: *Pricing Plain Vanillas [1]*

1. FRA:

$$\mathbf{FRA}(t;T_1,T_2,K) = P_d(t,T_2)\tau_f(T_1,T_2)$$
$$\times\left[F_f(t;T_1,T_2) + QA_{fd}\left(t,T_1,\sigma_f,\sigma_X,\rho_{fX}\right) - K\right]$$

2. Swaps:

$$\mathbf{Swap}(t;\boldsymbol{T},\boldsymbol{S},\boldsymbol{K}) = -\sum_{j=1}^{m} P_d(t,S_j)\tau_d(S_{j-1},S_j)K_j$$
$$+\sum_{i=1}^{n} P_d(t,ST)\tau_f(T_{j-1},T_j)\left[F_f(t;T_{i-1},T_i) + QA_{fd}\left(t,T_{i-1},\sigma_{f,i},\sigma_{X,i},\rho_{fX,i}\right)\right].$$

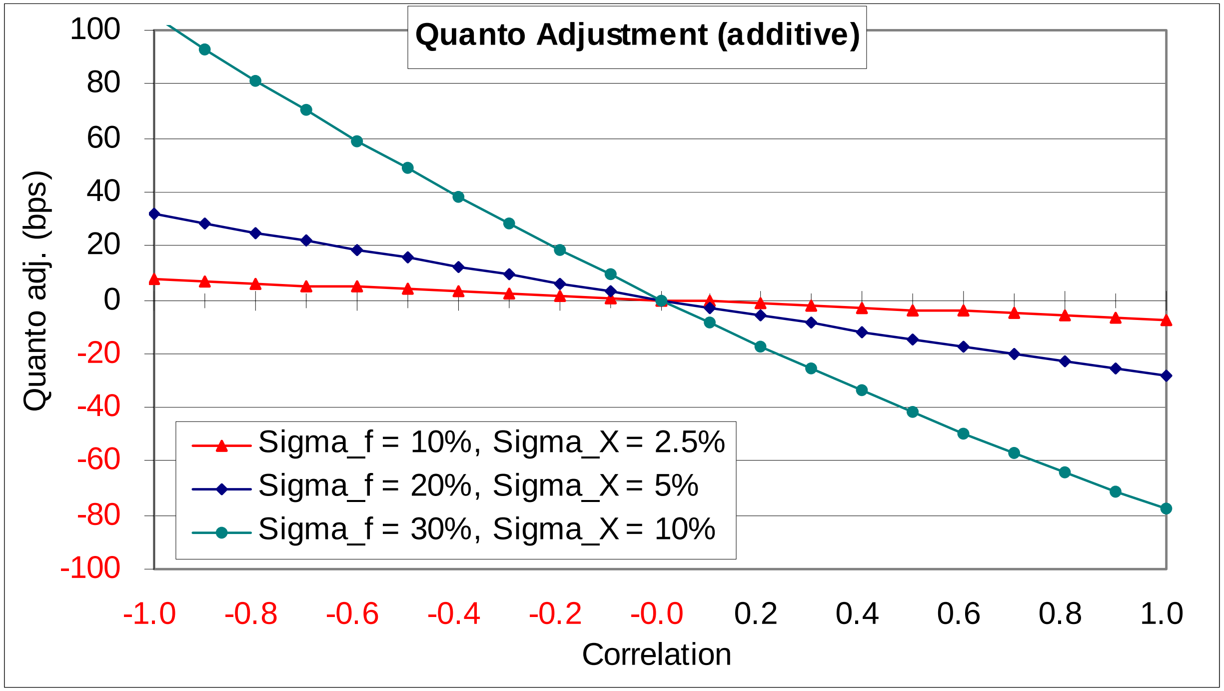
3. Caps/Floors:

$$\mathbf{CF}(t;\boldsymbol{T},\boldsymbol{K},\boldsymbol{\omega}) = \sum_{i=1}^{n} P_d(t,T_i)\tau_d(T_{i-1},T_i)$$
$$\times Black\left[F_f(t;T_{i-1},T_i) + QA_{fd}\left(t,T_{i-1},\sigma_{f,i},\sigma_{X,i},\rho_{fX,i}\right),K_i,\mu_{f,i},v_{f,i},\omega_i\right],$$

4. Swaptions:

$$\mathbf{Swaption}(t;\boldsymbol{T},\boldsymbol{S},K,\omega) = A_d(t,\boldsymbol{S})$$
$$\times Black\left[S_f(t;\boldsymbol{T},\boldsymbol{S}) + QA_{fd}\left(t,\boldsymbol{T},\boldsymbol{S},\nu_f,\nu_Y,\rho_{fY}\right),K,\lambda_f,v_f,\omega\right].$$

4: Pricing & Hedging IR Derivatives: *Pricing Plain Vanillas [2]*



Numerical scenarios for the (additive) quanto adjustment, corresponding to three different combinations of (flat) volatility values as a function of the correlation. The time interval is fixed to T_1-t=10 years and the forward rate to 3%.

We notice that **the adjustment may be not negligible**. **Positive correlation implies negative adjustment, thus lowering the forward rates**. The standard market practice, with no quanto adjustment, is thus **not arbitrage free**. In practice the adjustment depends on market variables not directly quoted on the market, making virtually impossible to set up arbitrage positions and locking today positive gains in the future.

4: Pricing & Hedging IR Derivatives: *Hedging*

1. Given any portfolio of interest rate derivatives with price $\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)$, compute delta risk with respect to **both curves** $\mathcal{C}_d$ **and** $\mathcal{C}_f$:

$$\begin{aligned}\Delta^{\pi}\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right) &= \Delta_d^{\pi}\left(t,\boldsymbol{T},\boldsymbol{R}_d^{mkt}\right)+\Delta_f^{\pi}\left(t,\boldsymbol{T},\boldsymbol{R}_f^{mkt}\right)\\ &= \sum_{j=1}^{N_d}\frac{\partial\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)}{\partial R_d^{mkt}\left(T_j\right)}+\sum_{j=1}^{N_f}\frac{\partial\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)}{\partial R_f^{mkt}\left(T_j\right)},\end{aligned}$$

2. eventually aggregate it on the subset of most liquid market instruments (**hedging instruments**);

3. calculate **hedge ratios**:

$$h_{x,j} = \frac{\partial\Pi\left(t,\boldsymbol{T},\boldsymbol{R}^{mkt}\right)}{\partial R_x^{mkt}\left(T_j\right)}\Bigg/\delta_{x,j}^{mkt},$$

$$\delta_{x,j}^{mkt} = \frac{\partial\pi_{x,j}^{mkt}\left(t\right)}{\partial R_x^{mkt}\left(T_j\right)},\quad x=f,d.$$

5: No Arbitrage and Counterparty Risk: *A Simple Credit Model* *(adapted from ref. [6])*

Both the forward basis and the quanto adjustment discussed above find a simple financial explanation in terms of counterparty risk.

If we identify:

- $P_d(t,T)$ = **default free zero coupon bond**,
- $P_f(t,T)$ = **risky zero coupon bond** emitted by a risky counterparty for maturity T and with **recovery rate** R_f,
- $\tau(t)>t$ = (random) counterparty default time observed at time t,
- $q_d(t,T) = E_t^{Q_d}\left\{1_{[\tau(t)>T]}\right\}$ = default probability after time T expected at time t,

we obtain the following expressions

$$P_f(t,T) = P_d(t,T)\,R\big(t;t,T,R_f\big),$$

$$F_f(t;T_1,T_2) = \frac{1}{\tau_f(T_1,T_2)}\left[\frac{P_d(t,T_1)}{P_d(t,T_2)}\frac{R\big(t;t,T_1,R_f\big)}{R\big(t;t,T_2,R_f\big)} - 1\right],$$

where:

$$R\big(t;T_1,T_2,R_f\big) = R_f + \big(1-R_f\big)E_t^{Q_d}\left[q_d(T_1,T_2)\right].$$

5: No Arbitrage and Counterparty Risk: *A Simple Credit Model [2]*

If $L_d(T_1,T_2)$, $L_f(T_1,T_2)$ are the risk free and the risky Xibor rates underlying the corresponding derivatives, respectively, we obtain:

$$\mathbf{FRA}_f\left(t;T_1,T_2,K\right)=\frac{P_d\left(t,T_1\right)}{R\left(t;T_1,T_2,R_f\right)}-\left[1+K\tau_f\left(T_1,T_2\right)\right]P_d\left(t,T_2\right),$$

$$BA_{fd}\left(t;T_1,T_2\right)=\frac{1}{\tau_d\left(T_1,T_2\right)}\frac{P_d\left(t,T_1\right)}{P_d\left(t,T_2\right)}\left[\frac{R\left(t;t,T_1,R_f\right)}{R\left(t;t,T_2,R_f\right)}-1\right],$$

$$QA_{fd}\left(t;T_1,T_2\right)=\frac{1}{\tau_f\left(T_1,T_2\right)}\frac{P_d\left(t,T_1\right)}{P_d\left(t,T_2\right)}\left[\frac{1}{R\left(t;T_1,T_2,R_f\right)}-\frac{R\left(t;t,T_1,R_f\right)}{R\left(t;t,T_2,R_f\right)}\right],$$

That is, the forward basis and the quanto adjustment expressed in terms of risk free zero coupon bonds $P_d(t,T)$ and of the expected recovery rate.

6: Pros & Cons, Other Approaches:

PROs

- Simple and familiar framework, no additional effort, just analogy.
- Straightforward interpretation in terms of counterparty risk.

CONs

- Unobservable exchange rate and parameters.
- Plain vanilla prices acquire volatility and correlation dependence.

- M. Henrard: “ab-initio parsimonious” model [5]
- F. Mercurio: generalised Libor Market Model [6]
- M. Morini: full credit model [7]
- F. Kijima et al: DLG model [8]

7: Conclusions

1. We have reviewed the **pre and post credit crunch market practices for pricing & hedging interest rate derivatives**.
2. We have shown that in the present **double-curve framework standard single-curve no arbitrage conditions are broken** and can be recovered taking into account the **forward basis**; once a smooth bootstrapping technique is used, the richer term structure of the calculated forward basis curves provides a sensitive indicator of the tiny, but observable, statical differences between different interest rate market sub-areas.
3. Using the foreign-currency analogy we have computed the **no arbitrage generalised double-curve-single-currency market-like pricing expressions** for basic interest rate derivatives, including a **quanto adjustment** arising from the change of numeraires naturally associated to the two yield curves. Numerical scenarios show that the **quanto adjustment can be non negligible.**
4. Both the forward basis and the quanto adjustment have a simple interpretation in terms of **counterparty risk**, using a simple credit model with a risk-free and a risky zero coupond bonds.

7: Main references

[1] M. Bianchetti, “*Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Using Different Yield Curves for Discounting and Forwarding*”, (2009), available at SSRN: *http://ssrn.com/abstract=1334356*.

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