TWO CURVES, ONE PRICE

Pricing & Hedging Interest Rate Derivatives Using Different Yield Curves For Discounting and Forwarding

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Pre credit-crunch single curve market practice:

- select a single set of the most convenient (e.g. liquid) vanilla interest rate instruments traded on the market with increasing maturities; for instance, a very common choice in the EUR market is a combination of short-term EUR deposit, medium-term FRA/Futures on Euribor3M and medium-long-term swaps on Euribor6M;
- build a single yield curve $C$ using the selected instruments plus a set of bootstrapping rules (e.g. pillars, priorities, interpolation, etc.);
- Compute, on the same curve $C$, forward rates, cashflows, discount factors and work out the prices by summing up the discounted cashflows;
- compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the same set of vanillas.
1: Context & Market Practices: 

Market Evolution

Basis swaps (single currency) as 2 swaps:

1. Euribor$3M_T$ vs $R_T^{3M}$
2. Euribor$6M_T$ vs $R_T^{6M}$
3. $Basis_T^{3M6M} = R_T^{3M} - R_T^{6M}$

Other market evidences:
- the divergence between deposit (Euribor based) and OIS (Overnight Indexed Swaps, Eonia based) rates;
- The divergence between FRA rates and the corresponding forward rates implied by consecutive deposits (see e.g. refs. [2], [6], [7]).
Post credit-crunch multiple curve market practice:

- build a single discounting curve $C_d$ using the preferred procedure;
- build multiple distinct forwarding curves $C_{f1} \ldots C_{fn}$ using the preferred distinct selections of vanilla interest rate instruments each homogeneous in the underlying rate tenor (typically 1M, 3M, 6M, 12M);
- compute the forward rates with tenor $f$ on the corresponding forwarding curve $C_f$ and calculate the corresponding cashflows;
- compute the corresponding discount factors using the discounting curve $C_d$ and work out prices by summing the discounted cashflows;
- compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the corresponding set of vanillas.

No market standard ...Eonia?
1: Context & Market Practices: 

Rationale

- Apparently similar interest rate instruments with different underlying rate tenors are characterised, in practice, by *different liquidity and credit risk premia*, reflecting the *different views and interests of the market players*.

- Thinking in terms of more fundamental variables, e.g. a *short rate*, the credit crunch has acted as a sort of *symmetry breaking mechanism*: from a (unstable) situation in which an unique short rate process was able to model and explain the whole term structure of interest rates of all tenors, towards a sort of *market segmentation into sub-areas* corresponding to instruments with different underlying rate tenors, characterised, in principle, by *distinct dynamics*, e.g. different short rate processes.

- Notice that market segmentation was already present (and well understood) before the credit crunch (see e.g. ref. [3]), but not effective due to negligible basis spreads.
2: Multiple-Curve Framework: 

**General Assumptions**

1. There exist **multiple different interest rate sub-markets** \( M_X, x = \{d, f_1, \ldots, f_n\} \) characterized by the **same currency** and by distinct bank accounts \( B_X \) and **yield curves**

\[
C_x := \{ T \rightarrow P_x(t_0, T), T \geq t_0 \},
\]

2. The usual **no arbitrage relation**

\[
P_x(t, T_2) = P_x(t, T_1) \times P_x(t, T_1, T_2)
\]

holds in each interest rate market \( M_X \).

3. Simple compounded **forward rates** are defined as usual for \( t \leq T_1 < T_2 \)

\[
P_x(t, T_1, T_2) = \frac{P_x(t, T_2)}{P_x(t, T_1)} = \frac{1}{1 + F_x(t; T_1, T_2) \tau_x(T_1, T_2)}
\]

4. **FRA pricing** under \( Q_x^{T_2} \) forward measure associated to numeraire \( P_x(t, T_2) \)

\[
\text{FRA}_x(t; T_1, T_2, K) = P_x(t, T_2) \tau_x(T_1, T_2) \left\{ E_x^{Q_x^{T_2}} [L_x(t, T_1, T_2)] - K \right\}
\]

\[
= P_x(t, T_2) \tau_x(T_1, T_2) [F_x(t; T_1, T_2) - K]
\]
2: Multiple-Curve Framework: 

**Pricing Procedure**

1. assume $C_d$ as the **discounting curve** and $C_f$ as the **forwarding curve**;

2. calculate any relevant spot/forward rate **on the forwarding curve** $C_f$ as

$$F_f(t; T_{i-1}, T_i) = \frac{P_f(t, T_{i-1}) - P_f(t, T_i)}{\tau_f(T_{i-1}, T_i) P_f(t, T_i)}, \quad t \leq T_{i-1} < T_i,$$

3. calculate cashflows $c_i, i = 1,\ldots,n$, as expectations of the $i$-th coupon payoff $\pi_i$ with respect to the **discounting $T_i$ - forward measure** $Q_{T_i}^{T_i}$

$$c_i := c(t, T_i, \pi_i) = E_{t}^{Q_{T_i}^{T_i}}[\pi_i];$$

4. calculate the price $\pi$ at time $t$ by discounting each cashflow $c_i$ using the corresponding discount factor $P_d(t, T_i)$ obtained from the **discounting curve** $C_d$ and summing,

$$\pi(t, T) = \sum_{i=1}^{n} P(t, T_i) E_{t}^{Q_{T_i}^{T_i}}[\pi_i];$$

5. **Price FRAs** as

$$\text{FRA}(t; T_1, T_2, K) = P_d(t, T_2) \tau_f(T_1, T_2) \left\{ E_{t}^{Q_{T_2}^{T_2}}[F_f(T_1; T_1, T_2)] - K \right\}$$
Multiple-Curve Framework: 

No Arbitrage and Forward Basis

Classic single-curve no arbitrage relations are broken: for instance, by specifying the subscripts $d$ and $f$ as prescribed above we obtain the two eqs.

$$P_d(t, T_2) = P_d(t, T_1) P_f(t, T_1, T_2),$$

$$P_f(t, T_1, T_2) = \frac{1}{1 + F_f(t; T_1, T_2) \tau_f(T_1, T_2)} = \frac{P_f(t, T_2)}{P_f(t, T_1)},$$

that clearly cannot hold at the same time. No arbitrage is recovered by taking into account the forward basis as follows

$$P_f(t, T_1, T_2) = \frac{1}{1 + F_f(t; T_1, T_2) \tau_f(T_1, T_2)} := \frac{1}{1 + [F_d(t; T_1, T_2) + BA_{fd}(t; T_1, T_2)] \tau_d(T_1, T_2)},$$

for which we obtain the following static expression in terms of discount factors

$$BA_{fd}(t; T_1, T_2) = \frac{1}{\tau_d(T_1, T_2)} \left[ \frac{P_f(t, T_1)}{P_f(t, T_2)} - \frac{P_d(t, T_1)}{P_d(t, T_2)} \right].$$
2: Multiple-Curve Framework: 
**Forward Basis Curves**

Forward basis (bps) as of end of day 16 Feb. 2009, daily sampled 3M tenor forward rates calculated on $C_{1M}, C_{3M}, C_{6M}, C_{12M}$ curves against $C_d$ taken as reference curve. Bootstrapping as described in ref. [2].

The richer term structure of the forward basis curves provides a sensitive indicator of the tiny, but observable, statiscal differences between different interest rate market sub-areas in the post credit crunch interest rate world, and a tool to assess the degree of liquidity and credit issues in interest rate derivatives' prices. Provided that…
2: Multiple-Curve Framework: 
Bad Curves

Left: 3M zero rates (red dashed line) and forward rates (blue continuous line). Right: forward basis. Linear interpolation on zero rates has been used. Numerical results from QuantLib (www.quantlib.org).

…smooth yield curves are used…Non-smooth bootstrapping techniques, e.g. linear interpolation on zero rates (still a diffused market practice), produce zero curves with no apparent problems, but ugly forward curves with a sagsaw shape inducing, in turn, strong and unnatural oscillations in the forward basis (see [2]).

“Two Curves, One Price” - Marco Bianchetti – Quant Congress Europe – London, 3-5 Nov. 2009
A second issue regarding no arbitrage arises in the double-curve framework:

\[
\text{FRA} (t; T_1, T_2, K) = P_d (t, T_2) \tau_f (T_1, T_2) \left\{ E_t^{Q_d} \left[ F_f (T_1 ; T_1, T_2) \right] - K \right\} \\
\neq P_d (t, T_2) \tau_f (T_1, T_2) \left[ F_f (T_1 ; T_1, T_2) - K \right]
\]

1. Double-curve-double-currency: 
   \( d = \text{domestic}, \ f = \text{foreign} \)
   
   \[
c_d (t) = x_{fd} (t) c_f (t), \\
x_{fd} (t_0) = x_{fd,0}.
\]

2. Double-curve-single-currency: 
   \( d = \text{discounting}, \ f = \text{forwarding} \)
   
   \[
c_d (t) = x_{fd} (t) c_f (t), \\
x_{fd} (t_0) = 1.
\]

Picture of no arbitrage definition of the forward exchange rate. Circuitation (round trip) \( \Rightarrow \) no money is created or destructed.
3: Foreign Currency Analogy: Quanto Adjustment

1. Assume a lognormal martingale dynamic for the $C_f$ (foreign) forward rate

$$\frac{dF_f(t; T_1, T_2)}{F_f(t; T_1, T_2)} = \sigma_f(t) dW_f^{T_2}(t), \quad Q_f^{T_2} \leftrightarrow P_f(t, T_2) \leftrightarrow C_f;$$

2. since $x_{fd}(t) P_f(t, T)$ is the price at time $t$ of a $C_d$ (domestic) tradable asset, the forward exchange rate must be a martingale process

$$\frac{dX_{fd}(t, T_2)}{X_{fd}(t, T_2)} = \sigma_X(t) dW_X^{T_2}(t), \quad Q_d^{T_2} \leftrightarrow P_d(t, T_2) \leftrightarrow C_d,$$

with

$$dW_f^{T_2}(t) dW_X^{T_2}(t) = \rho_{fx}(t) dt;$$

3. by changing numeraire from $C_f$ to $C_d$ we obtain the modified dynamic

$$\frac{dF_f(t; T_1, T_2)}{F_f(t; T_1, T_2)} = \mu_f(t) dt + \sigma_f(t) dW_f^{T_2}(t), \quad Q_d^{T_2} \leftrightarrow P_d(t, T_2) \leftrightarrow C_d,$$

where

$$\mu_f(t) = -\sigma_f(t) \sigma_X(t) \rho_{fx}(t);$$

4. and the modified expectation including the (additive) quanto-adjustment

$$E_t^{Q_d^{T_2}} \left[ L_f(T_1, T_2) \right] = F_f(t; T_1, T_2) + QA_{fd}(t; T_1, \sigma_f, \sigma_X, \rho_{fx}),$$

where

$$QA_{fd}(t; T_1, \sigma_f, \sigma_X, \rho_{fx}) = F_f(t; T_1, T_2) \left[ \exp \int_t^{T_1} \mu_f(s) ds - 1 \right].$$
4: Pricing & Hedging IR Derivatives:

**Pricing Plain Vanillas [1]**

1. FRA:
   \[ \text{FRA}(t; T_1, T_2, K) = P_d(t, T_2) \tau_f(T_1, T_2) \]
   \[ \times \left[ F_f(t; T_1, T_2) + QA_{fd}(t, T_1, \sigma_f, \sigma_X, \rho_{fX}) - K \right] \]

2. Swaps:
   \[ \text{Swap}(t; T, S, K) = -\sum_{j=1}^{m} P_d(t, S_j) \tau_d(S_{j-1}, S_j)K_j \]
   \[ + \sum_{i=1}^{n} P_d(t, ST) \tau_f(T_{j-1}, T_j) \left[ F_f(t; T_{i-1}, T_i) + QA_{fd}(t, T_{i-1}, \sigma_{f,i}, \sigma_{X,i}, \rho_{fX,i}) \right]. \]

3. Caps/Floors:
   \[ \text{CF}(t; T, K, \omega) = \sum_{i=1}^{n} P_d(t, T_i) \tau_d(T_{i-1}, T_i) \]
   \[ \times \text{Black} \left[ F_f(t; T_{i-1}, T_i) + QA_{fd}(t, T_{i-1}, \sigma_{f,i}, \sigma_{X,i}, \rho_{fX,i}), K_i, \mu_{f,i}, \nu_{f,i}, \omega_i \right], \]

4. Swaptions:
   \[ \text{Swaption}(t; T, S, K, \omega) = A_d(t, S) \]
   \[ \times \text{Black} \left[ S_f(t; T, S) + QA_{fd}(t, T, S, \nu_f, \nu_Y, \rho_{fY}), K, \lambda_f, \nu_f, \omega \right]. \]
4: Pricing & Hedging IR Derivatives: 
*Pricing Plain Vanillas [2]*

We notice that the adjustment may be not negligible. Positive correlation implies negative adjustment, thus lowering the forward rates. The standard market practice, with no quanto adjustment, is thus not arbitrage free. In practice the adjustment depends on market variables not directly quoted on the market, making virtually impossible to set up arbitrage positions and locking today positive gains in the future.
4: Pricing & Hedging IR Derivatives: 

**Hedging**

1. Given any portfolio of interest rate derivatives with price \( \Pi(t, T, R_{mkt}^x) \), compute delta risk with respect to both curves \( \mathcal{C}_d \) and \( \mathcal{C}_f \):

\[
\Delta \pi(t, T, R_{mkt}^x) = \Delta \pi_d(t, T, R_{d}^{mkt}) + \Delta \pi_f(t, T, R_{f}^{mkt})
\]

\[
= \sum_{j=1}^{N_d} \frac{\partial \Pi(t, T, R_{mkt}^x)}{\partial R_d^{mkt}(T_j)} + \sum_{j=1}^{N_f} \frac{\partial \Pi(t, T, R_{mkt}^x)}{\partial R_f^{mkt}(T_j)},
\]

2. eventually aggregate it on the subset of most liquid market instruments (hedging instruments);

3. calculate hedge ratios:

\[
h_{x,j} = \frac{\partial \Pi(t, T, R_{mkt}^x)}{\partial R_x^{mkt}(T_j)} / \delta_{x,j}^{mkt},
\]

\[
\delta_{x,j}^{mkt} = \frac{\partial \pi_{x,j}^{mkt}(t)}{\partial R_x^{mkt}(T_j)}, \quad x = f, d.
\]
Both the forward basis and the quanto adjustment discussed above find a simple financial explanation in terms of counterparty risk.

If we identify:
- \( P_d(t, T) = \text{default free zero coupon bond} \),
- \( P_f(t, T) = \text{risky zero coupon bond} \) emitted by a risky counterparty for maturity \( T \) and with recovery rate \( R_f \),
- \( \tau(t)>t = (\text{random}) \text{ counterparty default time observed at time } t \),
- \( q_d(t, T) = E_t^{Q_d} \{1_{[\tau(t)>T]}\} = \text{default probability after time } T \text{ expected at time } t \),

we obtain the following expressions

\[
P_f(t, T) = P_d(t, T) R(t; t, T, R_f),
\]

\[
F_f(t; T_1, T_2) = \frac{1}{\tau_f(T_1, T_2)} \left[ \frac{P_d(t, T_1) R(t; t, T_1, R_f)}{P_d(t, T_2) R(t; t, T_2, R_f)} - 1 \right],
\]

where:

\[
R(t; T_1, T_2, R_f) = R_f + (1 - R_f) E_t^{Q_d} [q_d(T_1, T_2)].
\]
If \( L_d(T_1, T_2), L_f(T_1, T_2) \) are the risk free and the risky Xibor rates underlying the corresponding derivatives, respectively, we obtain:

\[
\text{FRA}_f (t; T_1, T_2, K) = \frac{P_d(t, T_1)}{R(t; T_1, T_2, R_f)} \left[ 1 + K \tau_f(T_1, T_2) \right] P_d(t, T_2),
\]

\[
\text{BA}_{fd} (t; T_1, T_2) = \frac{1}{\tau_d(T_1, T_2)} \frac{P_d(t, T_1)}{P_d(t, T_2)} \left[ \frac{R(t; t, T_1, R_f)}{R(t; t, T_2, R_f)} - 1 \right],
\]

\[
\text{QA}_{fd} (t; T_1, T_2) = \frac{1}{\tau_f(T_1, T_2)} \frac{P_d(t, T_1)}{P_d(t, T_2)} \left[ \frac{1}{R(t; T_1, T_2, R_f)} - \frac{R(t; t, T_1, R_f)}{R(t; t, T_2, R_f)} \right],
\]

That is, the forward basis and the quanto adjustment expressed in terms of risk free zero coupon bonds \( P_d(t, T) \) and of the expected recovery rate.
6: Pros & Cons, Other Approaches:

**PROs**
- Simple and familiar framework, no additional effort, just analogy.
- Straightforward interpretation in terms of counterparty risk.

**CONs**
- Unobservable exchange rate and parameters.
- Plain vanilla prices acquire volatility and correlation dependence.

- M. Henrard: “ab-initio parsimonious” model [5]
- M. Morini: full credit model [7]
7: Conclusions

1. We have reviewed the pre and post credit crunch market practices for pricing & hedging interest rate derivatives.
2. We have shown that in the present double-curve framework standard single-curve no arbitrage conditions are broken and can be recovered taking into account the forward basis; once a smooth bootstrapping technique is used, the richer term structure of the calculated forward basis curves provides a sensitive indicator of the tiny, but observable, statical differences between different interest rate market sub-areas.
3. Using the foreign-currency analogy we have computed the no arbitrage generalised double-curve-single-currency market-like pricing expressions for basic interest rate derivatives, including a quanto adjustment arising from the change of numeraires naturally associated to the two yield curves. Numerical scenarios show that the quanto adjustment can be non negligible.
4. Both the forward basis and the quanto adjustment have a simple interpretation in terms of counterparty risk, using a simple credit model with a risk-free and a risky zero coupon bonds.
7: Main references


