BOOTSTRAPPING THE ILLIQUIDITY

Multiple Yield Curve Construction for market Coherent Discount and FRA Rates Estimation, Including Funding and Collateral

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Summary

1: Introduction
Classical vs modern pricing framework

The financial crisis begun in the second half of 2007 has triggered, among many consequences, a deep evolution phase of the classical framework adopted for trading derivatives. Credit and liquidity issues, in particular, was found to have macroscopical impacts on financial instruments, both plain vanillas and exotics. Today the market has not forgotten the lesson, and persistently shows the consequences of such effects. In particular, since August 2007, the primary interest rates of the interbank market, i.e. Libor, Euribor, Eonia, Eurepo, display large basis spreads. Similar divergences are also found between swap rates with different floating leg tenors. Recently, the market has included the effect of collateral agreements widely diffused among counterparties in the interbank market.

As a consequence, the standard no-arbitrage framework adopted to price derivatives has become obsolete. Classical relations described on standard textbooks and holding since decades had to be abandoned in one day. Also the idea of the construction of a single risk free yield curve reflecting at the same time the present cost of funding of future cash flows and the level of forward rates has been ruled out.

Thus the financial community has started the development of a modern theoretical framework, including a larger set of relevant risk factors and to review “from scratch” the no-arbitrage models used on the market for derivatives’ pricing and risk analysis.

PS: notice the similarity with the transition from classical to modern (relativistic quantum) physics.

1: Introduction
"Where is the garbage ?"

In April 2011, during the opening plenary panel at an international conference, a famous outstanding (equity) quant compared the transition from classical to modern (multi-curve) market to a boring problem, such as "*carrying the garbage out of the door when it's raining*".

Later In a conference session another (not so famous) quant began his talk, dedicated to CSA discounting, commenting that "*boring or not, we have to care about this problem, otherwise it's the garbage that enters the door*".

2: The Market Across The Credit Crunch
Libor interest rate [1]

Libor definition and mechanics *(source: www.bbalibor.com, September 2010)*

- Libor = London Interbank Offered rate,
  - first published in 1986,
  - sponsored by British Banker's Association (BBA, see *http://www.bbalibor.com*),
  - reference rate mentioned in ISDA standards for OTC transactions.
- Fixing mechanics:
  - each TARGET business day no later than 11:00 GMT each panel Bank submits to the calculation agent "*at what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am (GMT)?*" for 15 maturities (1d-12M) in a given currency; between 11:00-11:45 each bank can adjust its contribution.
  - At 11:45 GMT the calculation agent computes the rate fixings, for each maturity, as the average of rates submissions after discarding highest and lowest quartiles (25%) and publishes the results (Reuters page "Libor=").
  - Rate conventions: annualised rate, act/360, three decimal places, modified following, end of month.
  - Calculation agent: Reuters.
- Currencies: GBP, USD, JPY, CHF, CAD, AUD, EUR, DKK, SEK, NZD.

2: The Market Across The Credit Crunch

Libor interest rate [2]

Libor definition amplified

- the rate at which each bank submits must be formed from that bank's perception of its cost of funds in the interbank market;
- contributions must represent rates at which a bank would be offered funds in the London Money Market;
- contributions must be for the currency concerned, not the cost of producing one currency by borrowing in another currency and accessing the required currency via the foreign exchange markets;
- the rates must be submitted by members of staff at a bank with primary responsibility for management of a bank's cash, rather than a bank's derivative book;
- the definition of "funds" is: unsecured interbank cash or cash raised through primary issuance of interbank Certificates of Deposit.
- The rates are not necessarily based on actual transaction, because not all banks require funds each day, in size, in each currencies and maturities they quote. However, a bank is expected to know what its credit and liquidity risk profile is from rates at which it has dealt, and can construct a funding curve to predict accurately the correct rate for currencies or maturities in which it has not been active.

2: The Market Across The Credit Crunch

Libor interest rate [3]

Libor panels per currency											
Banks	**AUD**	**CAD**	**CHF**	**EUR**	**GBP**	**JPY**	**USD**	**DKK**	**NZD**	**SEK**	**Panels**
Abbey National plc				X	X						**2**
Bank of America							X				**1**
Bank of Montreal		X									**1**
Bank of Nova Scotia		X					X				**2**
Bank of Tokyo-Mitsubishi UFJ Ltd			X	X	X	X	X				**5**
Barclays Banks plc	X	X	X	X	X	X	X	X	X	X	**10**
BNP Paribas					X		X				**2**
Canadian Imperial Bank of Commerce		X									**1**
Citibank NA			X	X	X	X	X				**5**
Commonwealth Bank of Australia	X								X		**2**
Credit Agricole CIB					X	X	X				**3**
Credit Suisse			X	X			X				**3**
Deutsche Bank AG	X	X	X	X	X	X	X	X	X	X	**10**
HSBC		X	X	X	X	X	X	X	X	X	**9**
JP Morgan Chase	X		X	X	X	X	X	X	X	X	**9**
Lloyds Banking Group	X	X	X	X	X	X	X	X	X	X	**10**
Mizuho Corporate Bank				X	X	X					**3**
National Australia Bank Ltd	X								X		**2**
Rabobank		X	X	X	X	X	X	X		X	**8**
Royal Bank of Canada		X		X	X		X				**4**
Société Générale		X	X	X	X	X	X				**6**
Sumitomo Mitsui Banking Corporation						X	X				**2**
The Norinchukin Bank						X	X				**2**
The Royal Bank of Scotland Group	X	X	X	X	X	X	X	X	X	X	**10**
UBS AG	X	X	X	X	X	X	X	X		X	**9**
WestLB AG				X		X					**2**
Totals	**8**	**12**	**12**	**16**	**16**	**16**	**19**	**8**	**8**	**8**	
Last review	May 2011										

Source: www.bbalibor.org, September 2011

2: The Market Across The Credit Crunch
Libor interest rate [4]

Libor questioned during the crisis [1]

- Mar. 2008: the Bank for International Settlements reports that "*available data do not support the hypothesis that contributor banks manipulated their quotes to profit from positions based on fixings*“ (see J. Gyntelberg, P. Wooldridge, “*Interbank rate fixings during the recent turmoil*”, BIS Quarterly Review, Mar. 2008).
- Apr. 2008: Peng et al. from Citigroup (one of the largest Libor contributors) argue that “*...any Bank posting an high Libor level runs the risk of being perceived as needing funding*” (see Peng et al. “*Is Libor Broken*?”, Citi Fixed Income Strategies, Citigroup, Apr. 2008).
- Apr. 2008: the British Banker’s Association comments that Libor continues to be reliable, and that other proxies are not necessarily more sound than Libor at times of financial crisis.
- May 2008: the Wall Street Journal reports that some banks “*have been reporting significantly lower borrowing costs for the Libor, than what another market measure suggests they should be*” (see C. Mollenkamp, M. Whitehouse, The Wall Street Journal, 29 May 2008).
- Jun. 2008: Risk Magazine reports rumors that “*Libor rates are still not reflective of the true levels at which banks can borrow*” (see P. Madigan, “*Libor under attack*”, Risk, Jun. 2008).
- Oct. 2008: the International Monetary Fund reports that "*it appears that U.S. dollar Libor remains an accurate measure of a typical creditworthy bank’s marginal cost of unsecured U.S. dollar term funding*“ (see Global Financial Stability Report, Oct. 2008, ch. 2).
- Apr. 2010: an academic research paper reports evidences that Libor does not reflect the true bank’s borrowing costs (see C. Snider, T. Youle , “*Does the Libor reflect banks’ borrowing costs ?*”, SSRN working paper, 2 Apr. 2010, *http://ssrn.com/abstract=1569603*).

2: The Market Across The Credit Crunch
Libor interest rate [5]

Libor questioned during the crisis [2]

- Jun. 2012: Risk Magazine comments Barclays fined of 450 million USD by Commodity Futures Trading Commission (CFTC), Department of Justice (DOJ) and UK Financial Services Authority (FSA) for "*false, misleading or knowingly inaccurate submissions*" concerning Libor and Euribor in the period 2005-2009 (see P. Madigan, D. Wood, "*Libor manipulation lawsuits could cost banks tens of billions*", Risk, 28 Jun. 2012).
- Jun. 2012: FSA's publish "*The Wheatley review of LIBOR: final report*" (http://cdn.hm-treasury.gov.uk/wheatley_review_libor_finalreport_280912.pdf "*retaining Libor unchanged in its current state is not a viable option, given the scale of identified weaknesses and the loss of credibility that it has suffered*".
  The Wheatley reforms for strengthening the current Libor benchmark are evolving around four main themes:
  - the reform of the Libor fixing mechanism
  - the introduction of new rules and guidance for Libor contributions
  - the strengthening of the governance
  - changes to the regulatory framework.

2: The Market Across The Credit Crunch
Euribor interest rate [1]

Euribor definition and mechanics *(source: www.euribor.org, September 2010)*

- Euribor = Euro Interbank Offered Rate
  - first published on 30 Dec. 1998;
  - sponsored by the European Banking Federation (EBF) and by the Financial Markets Association (ACI).
- Fixing mechanics:
  - each TARGET business day no later than 10:45 CET each panel Bank submits to the calculation agent "*what rate do you believe one prime bank is quoting to another prime bank for interbank term deposits within the euro zone?*" for 16 maturities (T/N, 1w, 2w, 3w, 1M-12M); between 10:45-11:00 each bank can adjust its contribution.
  - At 11:00 CET the calculation agent computes the rate fixings, for each maturity, as the average of rates submissions after discarding highest and lowest 15% and publishes the results (Reuters page "Euribor=").
  - Rate conventions: spot (T+2) value, annualised rate, act/360, three decimal places, modified following, end of month.
  - Calculation agent: Reuters.
- Currencies: EUR

2: The Market Across The Credit Crunch

Euribor interest rate [2]

Euribor panel			
Bank	**Country**	**Bank**	**Country**
Erste Group Bank AG RZB Raiffeisen Zentralbank Österreich AG	Austria	Dexia Bank KBC	Belgium
Nordea Pohjola	Finland	AIB Group Bank of Ireland	Ireland
Banque Postale BNP - Paribas HSBC France Société Générale Natixis Crédit Agricole s.a. Crédit Industriel et Commercial CIC	France	Intesa Sanpaolo Unicredit Monte dei Paschi di Siena	Italy
		Banque et Caisse d'Épargne de l'État	Luxembourg
		RBS N.V. Rabobank ING Bank	Netherlands
Landesbank Berlin Bayerische Landesbank Girozentrale Deutsche Bank WestLB AG Commerzbank DZ Bank Deutsche Genossenschaftsbank Norddeutsche Landesbank Girozentrale Landesbank Baden-Württemberg Girozentrale Landesbank Hessen Thüringen Girozentrale	Germany	Caixa Geral De Depósitos (CGD)	Portugal
		Banco Bilbao Vizcaya Argentaria Confederacion Española de Cajas de Ahorros Banco Santander Central Hispano La Caixa Barcelona	Spain
		Barclays Capital Den Danske Bank Svenska Handelsbanken	Other EU Banks
National Bank of Greece	Greece	Bank of Tokyo - Mitsubishi J.P. Morgan Chase & Co. Citibank UBS (Luxembourg) S.A.	International Banks

Source: www.euribor.org, September 2011

2: The Market Across The Credit Crunch
Eonia interest rate

Eonia definition and mechanics *(source: www.euribor.org, September 2010)*

- Eonia = Euro Over Night Index Average
  - first published and sponsored as Euribor;
  - reference rate for overnight unsecured transactions in the Euro Market.
- Panel banks: same as Euribor.
- Fixing mechanics:
  - each TARGET business day no later than 18:30 CET each panel bank submits the total volume of overnight unsecured lending transactions of that day before 18:00 and the weighted average lending rate for these transactions for a single maturity (ON).
  - Between 18:30-18:45 (CET) the calculation agent computes the rate fixing as the average of all rates submissions (with no cuts) weighted with the corresponding transaction volumes and transmits the result to Reuters for publication within 18:45-19:00 (Reuters page "Eonia=").
  - Rate conventions: today value (T+0), annualised rate, act/360, three decimal places.
  - Calculation agent: European Central Bank.
- Overnight rates in other currencies:
  - USD: Federal Funds Effective Rate
  - GBP: SONIA = Sterling Over Night Index Average
  - CHF: SARON: Swiss Average Rate Over Night
  - JPY: Mutan rate

2: The Market Across The Credit Crunch
Repo interest rate

Eurepo definition and mechanics *(source: www.eurepo.org, March 2011)*

- Eurepo = Euro Repo (Repurchase Agreement Rate)
  - first published on 4 Mar. 2002;
  - sponsored by the European Banking Federation (EBF);
  - reference rate for Repo transaction in the Euro market
- Panel banks: 34 banks.
- Fixing mechanics:
  - each TARGET business day no later than 10:45 CET each panel Bank submits to the calculation agent "*the rate at which one prime bank offers funds in euro to another prime bank if in exchange the former receives from the latter the best collateral in terms of rating and liquidity within the Eurepo GC basket*" for 10 maturities (T/N, 1w, 2w, 3w, 1M, 2M, 3M, 6M, 9M, 12M); between 10:45-11:00 each bank can adjust its contribution.
  - At 11:00 CET the calculation agent computes the rate fixings, for each maturity, as the average of rates submissions after discarding highest and lowest 15% and publishes the results (Reuters page "EUREPO=").
  - Rate conventions: spot (T+2) value, annualised rate, act/360, three decimal places, modified following, end of month.
  - Calculation agent: Reuters.

2: The Market Across The Credit Crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [1]

Libor discussion

- Libor is based on:
  - o offered rates on unsecured funding;
  - o expectations, views and beliefs of the panel banks about borrowing rates in the currency money market (see e.g. P. Madigan, "*Libor under attack*", Risk, Jun. 2008).

- As any interest rate expectation, Libor includes informations on:
  - o the counterparty credit risk/premium,
  - o the liquidity risk/premium

  and thus its not a risk free rate, as already well known before the crisis (see e.g. B. Tuckman, P. Porfirio, "*Interest Rate Parity, Money Market Basis Swaps, and Cross-Currency Basis Swaps*", Lehman Brothers, Jun. 2003).

- Lending/borrowing Libor rates is tenor dependent: *"The age of innocence – when banks lent to each other unsecured for three months or longer at only a small premium to expected policy rates – will not quickly, if ever, return"* (M. King, Bank of England Governor, 21 Oct. 2008).

- The Libor panels may change over time, panel banks may be replaced by other banks with higher credit standing. Borrowers and lenders will not be Libor forever.

2: The Market Across The Credit Crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [2]

Eonia discussion

- Eonia is based on:
  - lending (offer side) rates on unsecured funding;
  - actual transaction executed by the panel banks in the Euro money market
- Eonia is used by ECB as a method of effecting and observing the transmission of the monetary policy actions in the unsecured Euro money market and thus it includes informations on:
  - the monetary policy effects,
  - the short term cost of liquidity expectations of the panel banks in the unsecured Euro money market;
- Eonia holds the shortest rate tenor available (one day) and carries low counterparty credit and liquidity risk, thus it is a good market proxy to a risk free rate.

See also Goldman Sachs, "*Overview of Eonia and Update on Eonia Swap Market*", Mar. 2010.

2: The Market Across The Credit Crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [3]

Eurepo discussion

- Same points as Eonia apply, but for the secured Euro money market
- Eurepo holds the shortest rate tenor available (one day) and carries the lowest counterparty credit and liquidity risk: thus it is the best market proxy to a risk free rate.
- Eonia and Eurepo are bracketed inside the interest rate corridor defined by the standing facilities provided by the european national banks to manage liquidity in the banking sector:
  - the marginal lending facility lets banks borrow liquidity from their national central bank against eligible assets: the marginal lending rate normallly defines a cap for the overnight market rates;
  - the deposit facility lets banks lend liquidity to their national central bank: the corresponding deposit rate normally definies a floor for the overnight market rates;
  - see graph later on.

2: The Market Across The Credit Crunch
Libor/Euribor/Eonia/Eurepo interest rates discussion [4]

	Libor	Euribor	Eonia	Eurepo
Definition	London InterBank Offered Rate	Euro InterBank Offered Rate	Euro OverNight Index Average	Euro Repurchase Ageement rate
Market	London Interbank	Euro Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Offer	Offer
Rate quotation specs	EURLibor = Euribor, Other currencies: minor differences (e.g. act/365, T+0, London calendar for GBPLibor).	TARGET calendar, settlement *T+2*, act/360, three decimal places, modified following, end of month, tenor variable.	TARGET calendar, settlement *T+1*, act/360, three decimal places, tenor 1d.	As Euribor
Maturities	1d-12m	1w, 2w, 3w,1m,…,12m	1d	T/N-12m
Publication time	12.30 CET	11:00 am CET	6:45-7:00 pm CET	As Euribor
Panel banks	8-20 banks (London based) per currency	42 banks from 15 EU countries + 4 international banks	Same as Euribor	34 EU banks plus some large international bank from non-EU countries
Calculation agent	Reuters	Reuters	European Central Bank	Reuters
Transactions based	No	No	Yes	No
Collateral	No (unsecured)	No (unsecured)	No (unsecured)	Yes (secured)
Counterparty risk	Yes	Yes	Low	Negligible
Liquidity risk	Yes	Yes	Low	Negligible
Tenor basis	Yes	Yes	No	No

2: The Market Across The Credit Crunch
How the market has changed: stylized facts

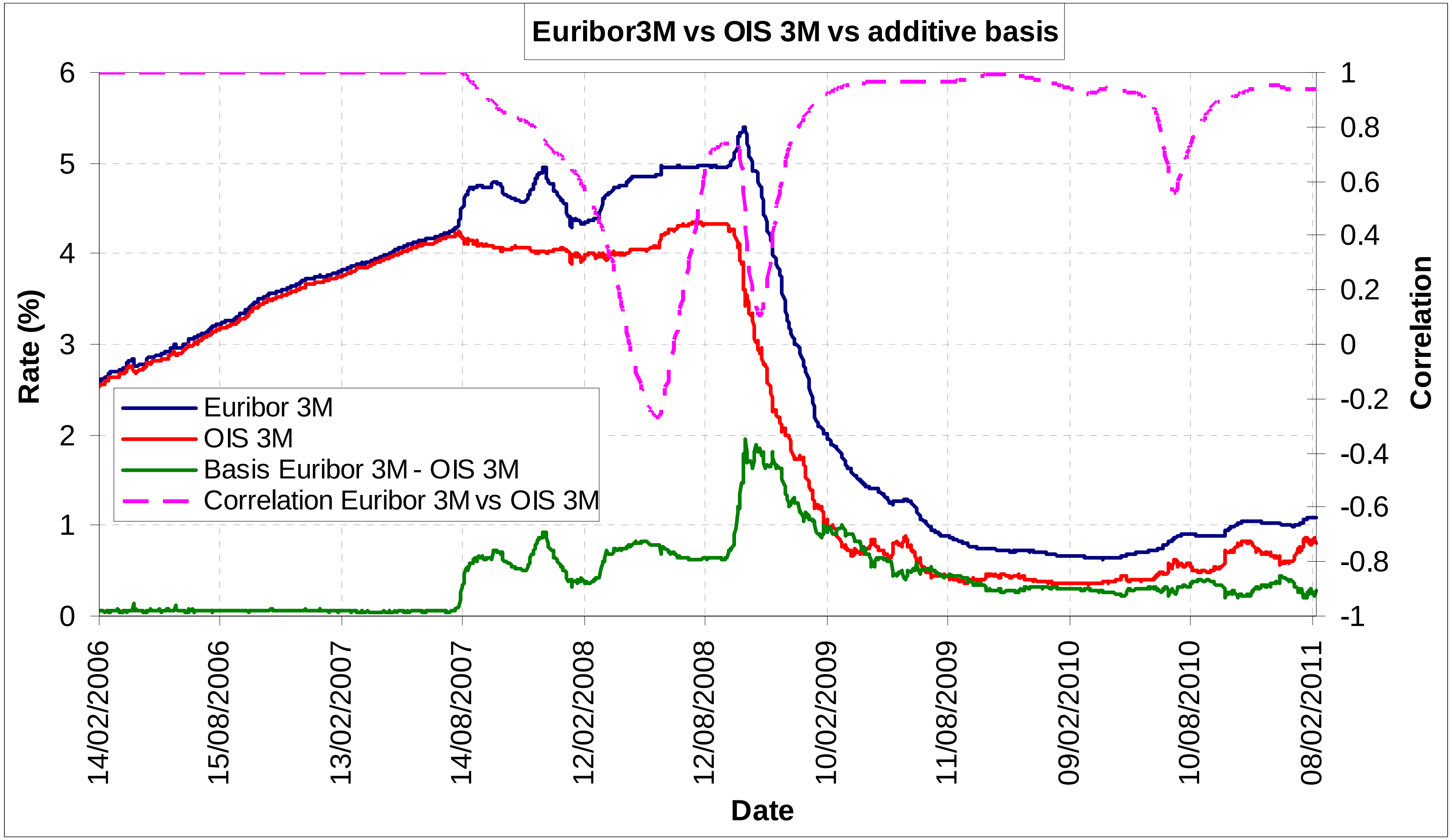
1. Divergence between deposit (Libor based) and OIS (Overnight based) rates.

2. Divergence between FRA rates and the corresponding forward rates implied by consecutive deposits.

3. Explosion of basis swap rates (based on Libor rates with different tenors).

4. Shift towards CSA discounting for collateralized cashflows: ICAP, Swapclear, CSA chaos, new standard ISDA CSA.

2: The Market Across The Credit Crunch
Spot rates



Spot EUR 3M OIS rates vs 3M Deposit rates

Quotations Feb. 2005 – Feb. 2011 (source: Bloomberg)

2: The Market Across The Credit Crunch

FRA rates [1]

```
KLIEMM
IRP       Tel. +49
CapMkt    Tel. +49
Fwds      Tel. +49
             Carl K
See <KLIEMMM2> f
16:52  30/12/10
          EUR
         CLOSED
 ON   0.28/0.38
 TN   0.50/1.00
 SN   0.40/0.50
 SW   0.53/0.63
 2W   0.56/0.66
 3W   0.62/0.72

 1M   0.70/0.80
 2M   0.82/0.92
 3M   0.92/1.02
 4M   0.97/1.07
 5M   1.05/1.15
 6M   1.14/1.24

 7M   1.19/1.29
 8M   1.23/1.33
 9M   1.28/1.38
10M   1.33/1.43
11M   1.37/1.47
 1Y   1.42/1.52

15M   1.60/1.85
18M   1.64/1.89
21M   1.70/1.95
 2Y   1.86/2.11
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16:16 30DEC10                ICAP LONDON                 UK69580          ICAPSHORT2
Contact Reuters EXEU           EURO Short Swaps / FRAs            +44 (0)20 7532 3530
           1M Swaps                     IMM Dated                         3m FRAs
  2x1     0.827-0.777      1y  MAR/MAR 1.213-1.163                1x4   1.037-0.987
  3x1     0.839-0.789      1y  JUN/JUN 1.329-1.279                2x5   1.055-1.005
  4x1     0.850-0.800      1y  SEP/SEP 1.477-1.427                3x6   1.080-1.030
  5x1     0.863-0.813      1y  DEC/DEC 1.646-1.596                4x7   1.108-1.058
  6x1     0.875-0.825      2y  MAR/MAR 1.518-1.468                5x8   1.137-1.087
  7x1     0.888-0.838      2y  JUN/JUN 1.671-1.621                6x9   1.166-1.116
  8x1     0.907-0.857      3y  MAR/MAR 1.855-1.805
  9x1     0.924-0.874                                                   6m FRAs
 10x1     0.941-0.891                IMM Fras                    1x7   1.264-1.214
 11x1     0.954-0.904                                            2x8   1.284-1.234
 12x1     0.969-0.919      1x7          1.273-1.223              3x9   1.307-1.257
                           2x8          1.292-1.242              4x10  1.332-1.282
 1y /3    1.158-1.108      3x9          1.320-1.270              5x11  1.357-1.307
 15m/3    1.209-1.159      4x10         1.344-1.294              6x12  1.391-1.341
 18m/3    1.270-1.220                                            12x18 1.649-1.599
 21m/3    1.345-1.295      0x3 Today    1.035-0.985              18x24 2.025-1.975
                           0x6 Today    1.259-1.209
 1y /6    1.347-1.297      0x3 Tom      1.033-0.983
 15m/6    1.352-1.302      0x6 Tom      1.260-1.210                     12m FRA
 18m/6    1.454-1.404                                            12x24 2.001-1.951
 21m/6    1.497-1.447                      ICAP OIS Fix Menu <ICAPOISFIX01>
ICAP Global Index <ICAP>                   Forthcoming changes <ICAPCHANG
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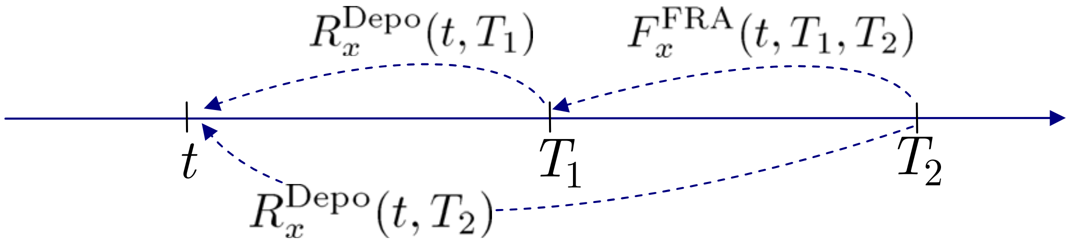
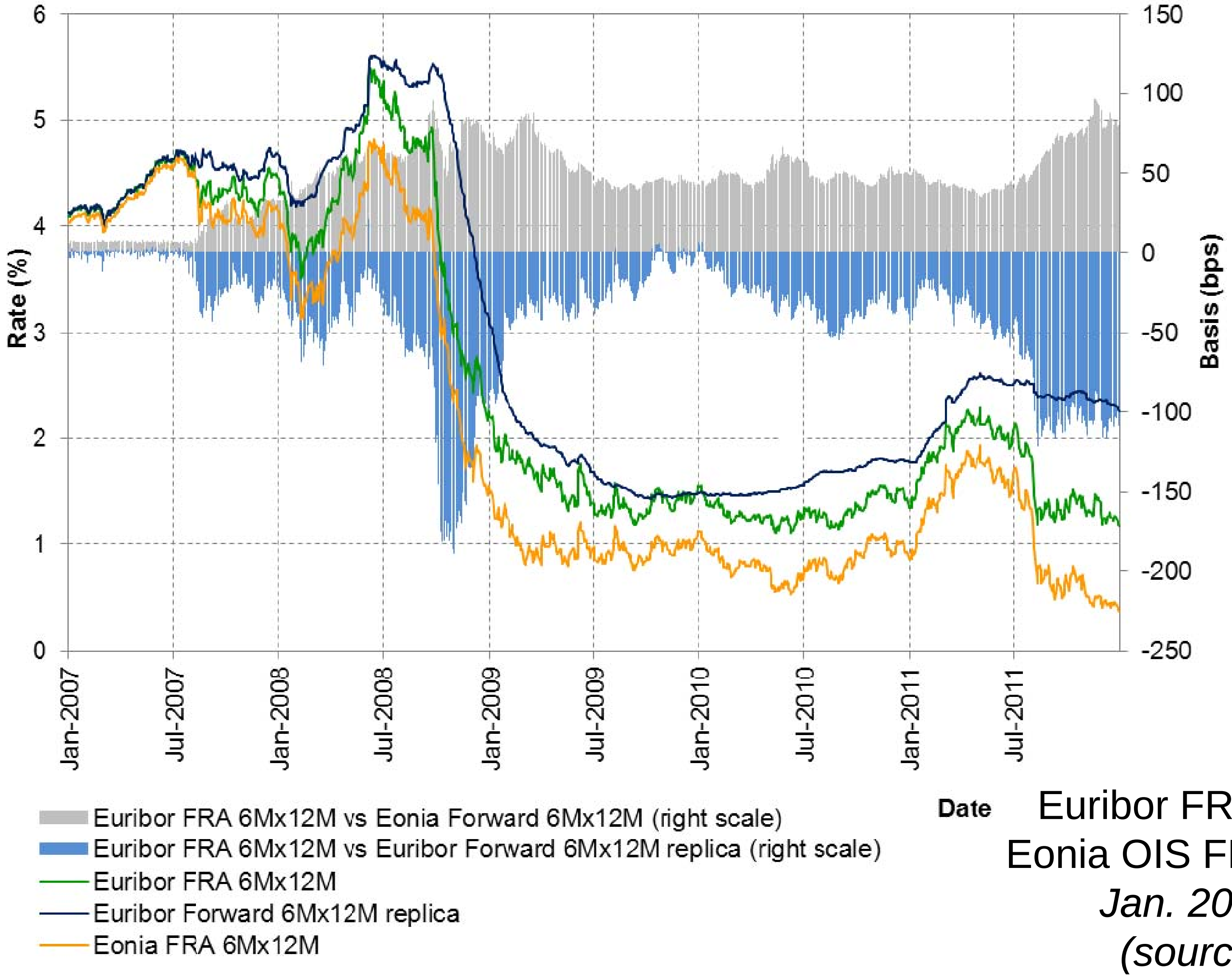
Check Euribor FRA replication (31 Dec 2010)			
Tenor	**FRA replica (%)**	**FRA market (%)**	**Difference (bp)**
1x4	1.101	1.012	8.9
2x5	1.273	1.03	24.3
3x6	1.420	1.055	36.5
4x7	1.556	1.083	47.3
5x8	1.603	1.112	49.1
6x9	1.631	1.141	49.0
1x7	1.323	1.239	8.4
2x8	1.433	1.259	17.4
3x9	1.520	1.282	23.8
4x10	1.636	1.307	32.9
5x11	1.695	1.332	36.3
6x12	1.764	1.366	39.8
12x18	2.335	1.624	71.1
18x24	2.613	2	61.3
12x24	2.469	1.976	49.3



$$\frac{1}{1 + R_x^{\mathrm{Depo}}(t, T_1)(T_1 - t)} \times \frac{1}{1 + F_x^{FRA}(t; T_1, T_2)(T_2 - T_1)} \neq \frac{1}{1 + R_y^{\mathrm{Depo}}(t, T_2)(T_2 - t)}$$

Market Euribor FRA vs Depo implicit forward rates.
Quotations 30 Dec. 2010 (source: Reuters)

2: The Market Across The Credit Crunch
FRA rates [2]



Euribor FRA/forward 6x12 vs Eonia OIS FRA 6x12*. Quotations Jan. 2007 – Dec. 2011 (source: Bloomberg)*

2: The Market Across The Credit Crunch
FRA rates [3]

16:16 30DEC10 ICAP LONDON
Contact Reuters EXEU EURO

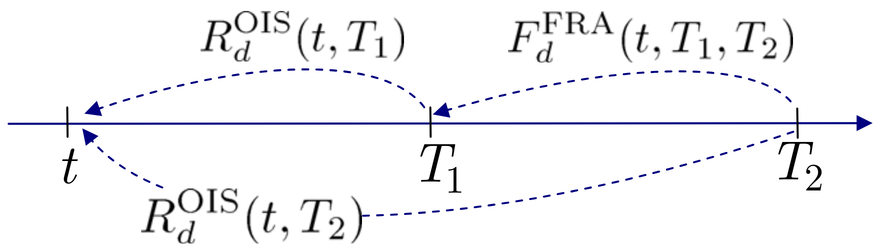
Eonia		Fwd EONIA	
1w	0.462-0.362	1X2	0.644-0.594
2w	0.456-0.356	2X3	0.686-0.636
3w	0.511-0.411	1x4	0.685-0.635
1m	0.527-0.477	2x5	0.724-0.674
2m	0.582-0.532	3x6	0.758-0.708
3m	0.619-0.569	6x12	0.892-0.842
4m	0.644-0.594		
5m	0.669-0.619	IMM Fra/Eonia	
6m	0.689-0.639		
7m	0.707-0.657	MAR	35.100-30.100
8m	0.726-0.676	JUN	34.200-29.200
9m	0.743-0.693	SEP	34.600-29.600
10m	0.761-0.711	DEC	35.300-30.300
11m	0.776-0.726		
12m	0.792-0.742		
Two Payments			
15m	0.849-0.799		
18m	0.914-0.864	All ICAP Euro pag	
21m	0.989-0.939		
2y	1.071-1.021		
3y	1.419-1.369		

ICAP Global Index <ICAP>

Check Eonia FRA replication (30 Dec. 2010)

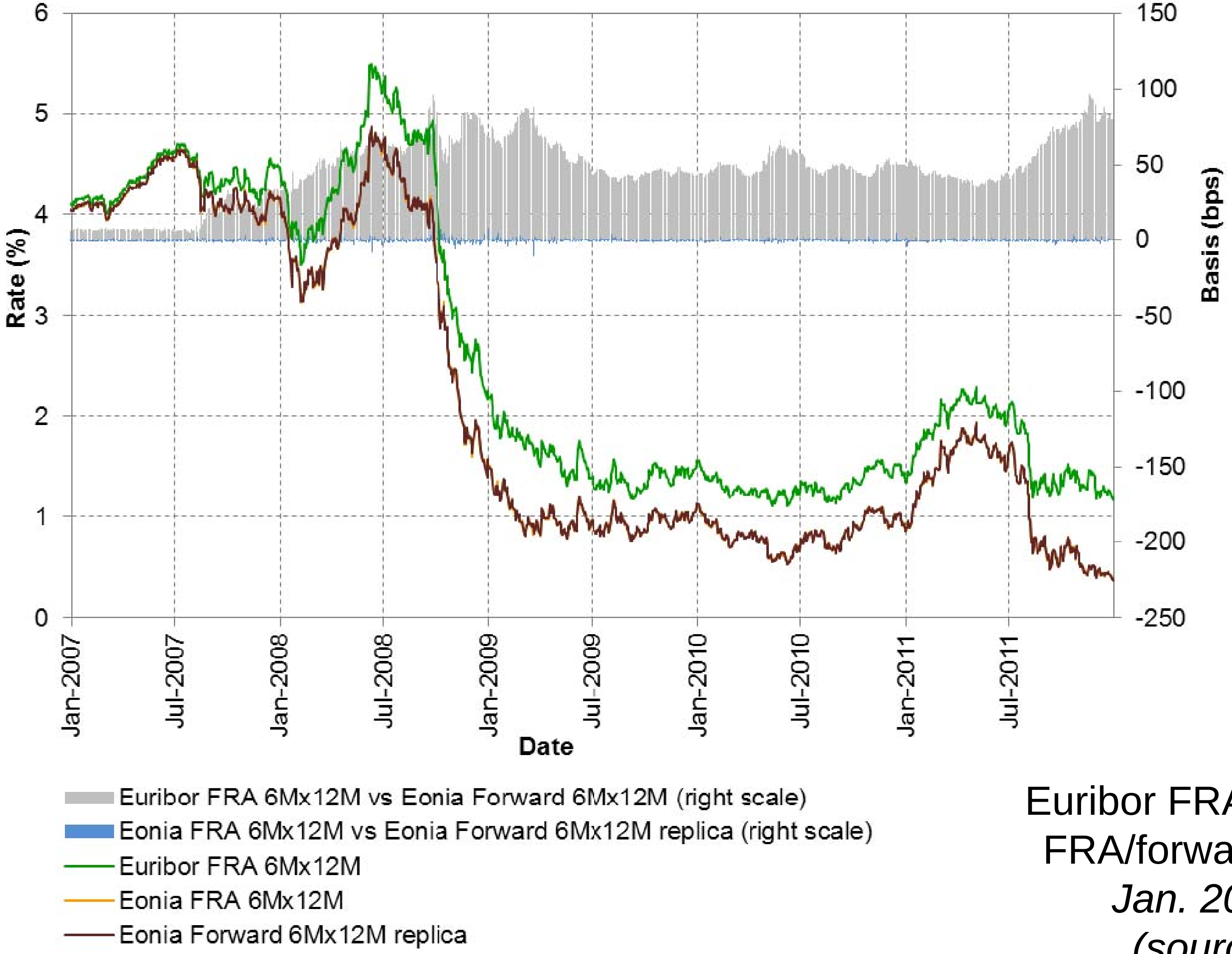
Tenor	FRA replica (%)	FRA market (%)	Difference (bp)
1x2	0.618	0.619	-0.1
2x3	0.664	0.661	0.3
1x4	0.659	0.66	-0.1
2x5	0.698	0.699	-0.1
3x6	0.732	0.733	-0.1
6x12	0.865	0.867	-0.2

Market Eonia FRA vs OIS implicit forward rates.
Quotations 30 Dec. 2010 (source: Reuters)



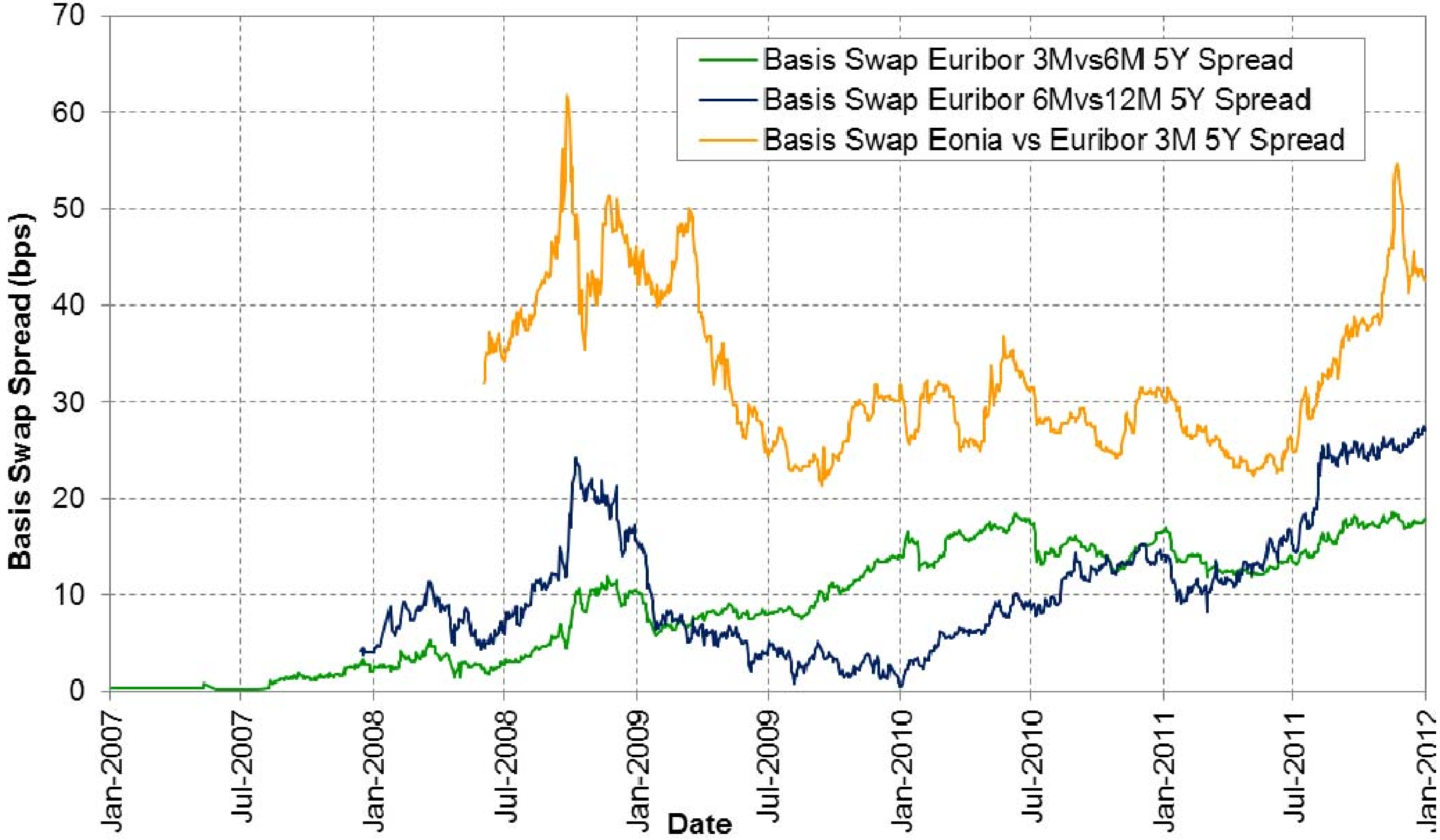
$$\frac{1}{1+R_d^{\text{OIS}}(t,T_1)(T_1-t)} \times \frac{1}{1+F_d^{FRA}(t;T_1,T_2)(T_2-T_1)} = \frac{1}{1+R_d^{\text{OIS}}(t,T_2)(T_2-t)}$$

2: The Market Across The Credit Crunch
FRA rates [4]



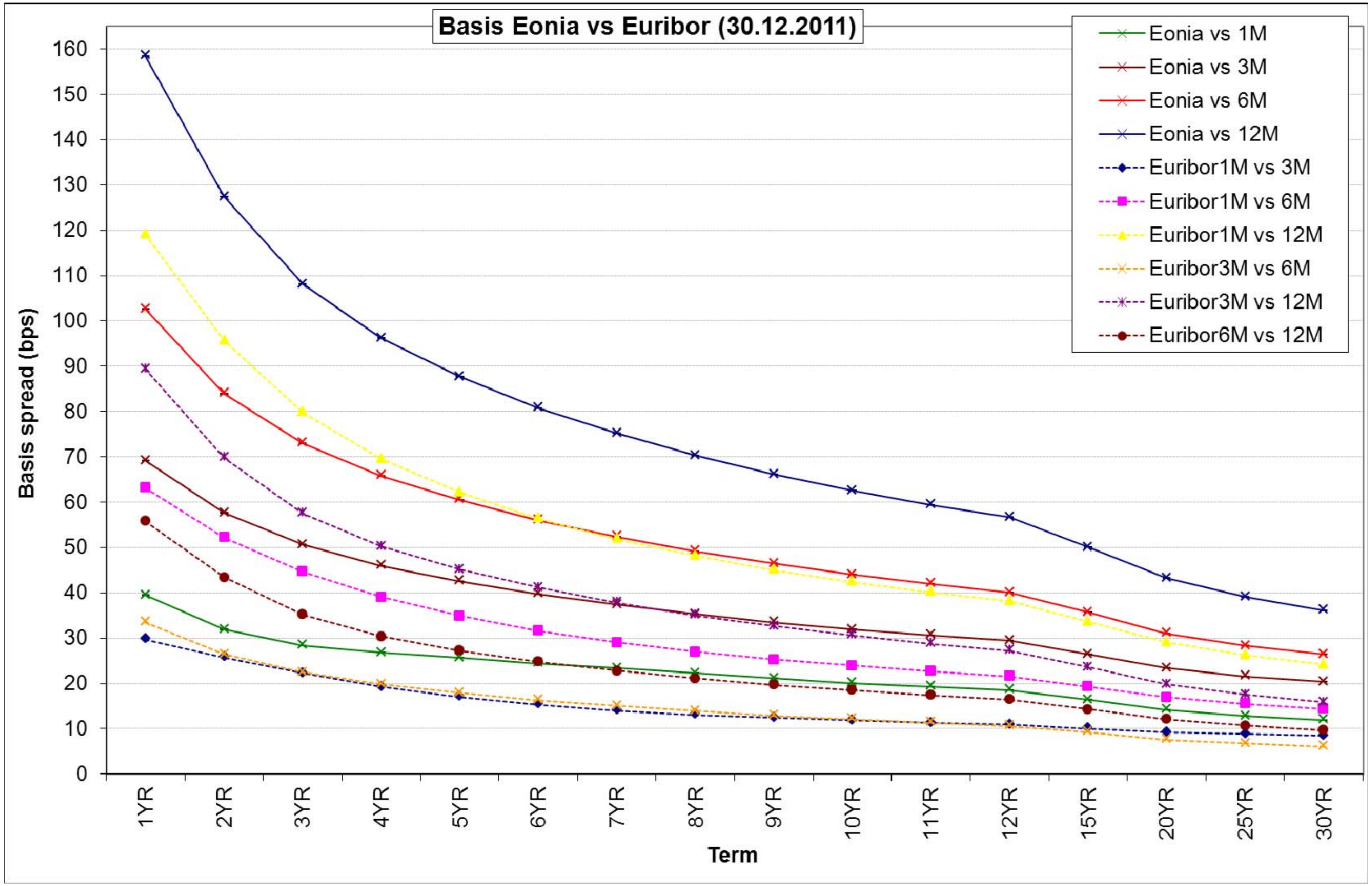
Euribor FRA 6x12 vs Eonia OIS FRA/forward 6x12. *Quotations Jan. 2007 – Dec. 2011 (source: Bloomberg)*

2: The Market Across The Credit Crunch
Basis Swap rates [1]



EUR Basis Swap 5Y, Euribor 3M vs 6M vs 12M vs Eonia,
Quotations Jan. 2007 – Dec. 2011 (source: Bloomberg)

2: The Market Across The Credit Crunch
Basis Swap rates [2]

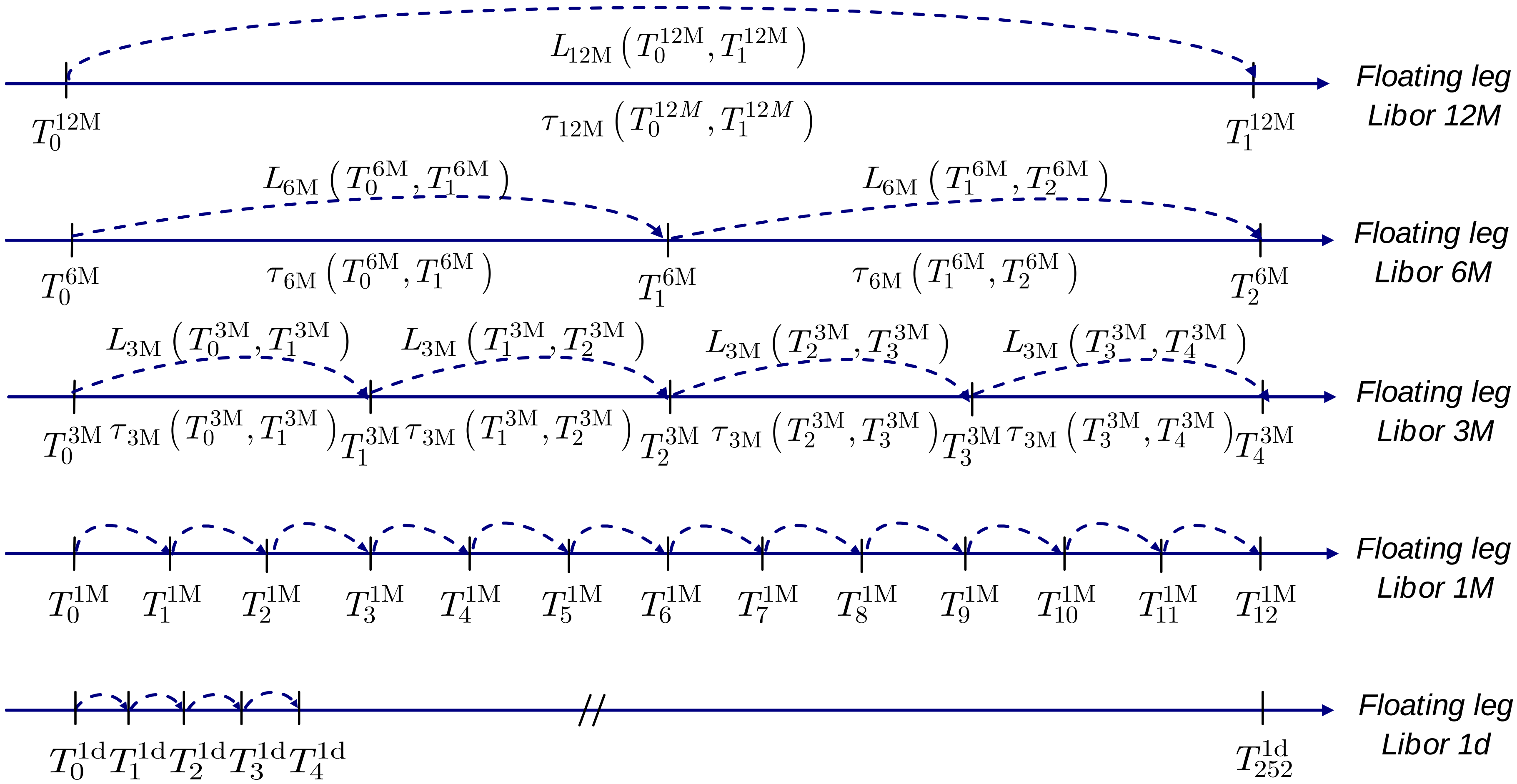


EUR Basis Swaps term structure

Quotations as of 30 Dec 2011 (source: Reuters, ICAP)

2: The Market Across The Credit Crunch

Basis Swap rates [3]



Picture of floating Swap legs with equal maturities ($T_1^{12M} = T_2^{6M} = T_4^{3M} = T_{12}^{1M} = T_{252}^{1d} = 1Y$ *) and different Libor tenors (12M, 6M, 3M, 1M, 1d from top to bottom).*

2: The Market Across The Credit Crunch
Counterparty risk and collateral [1]

Collateral mechanics: regulated vs OTC markets		
	Regulated markets	Over the counter markets
Collateralisation	All trades are collateralised	Not all trades are collateralised, it depends on the agreements between the counterparties
Financial instruments	highly standardised	highly customised
Clearing House	There is a Clearing House that acts as counterparty for any trade and establish settlement and margination rules	There is no Clearing House, direct interaction between the counterparties, ad hoc contracts are used
Settlement and margination execution	Daily settlement and margination, collateral in cash of main currencies or highly rated bonds (govies)	Most used contracts are: ✓ISDA Master Agreement ✓Credit Support Annex (CSA)
Collateral interest	Overnight rate	Depend on the agreements

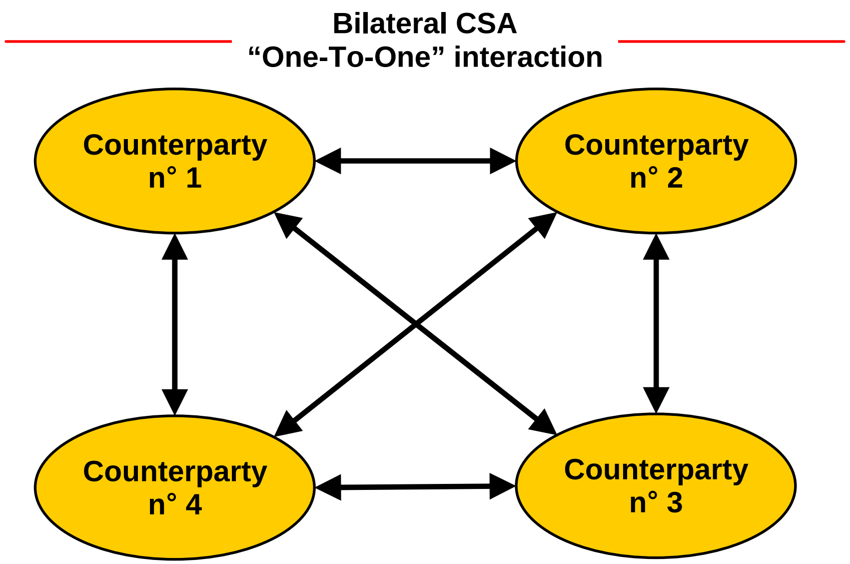
2: The Market Across The Credit Crunch
Counterparty risk and collateral [2]

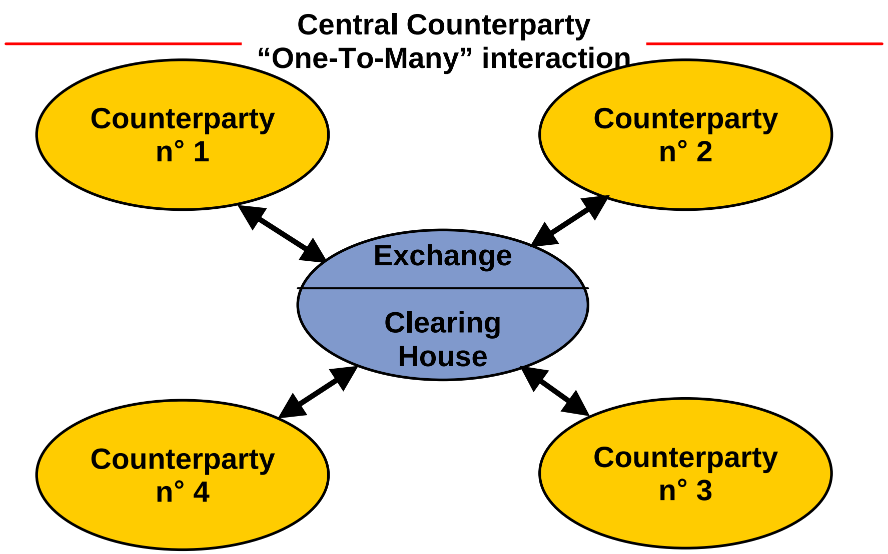
Bilateral CSA

OTC transactions under ISDA Master Agreement with CSA are bilateral agreements with direct obligations and collateral management.

Central Counterparty

The CCP reduces the settlement risks by:
- o netting transactions between multiple counterparties,
- o requiring collateral deposits
- o providing independent valuation of trades and collateral,
- o monitoring the credit worthiness of the clearing firms,
- o providing a guarantee fund





2: The Market Across The Credit Crunch
Counterparty risk and collateral [3]

CSA diffusion
(ISDA Margin Survey, 2010)

Table 3.2 Percent of trades subject to collateral agreements, by size of program

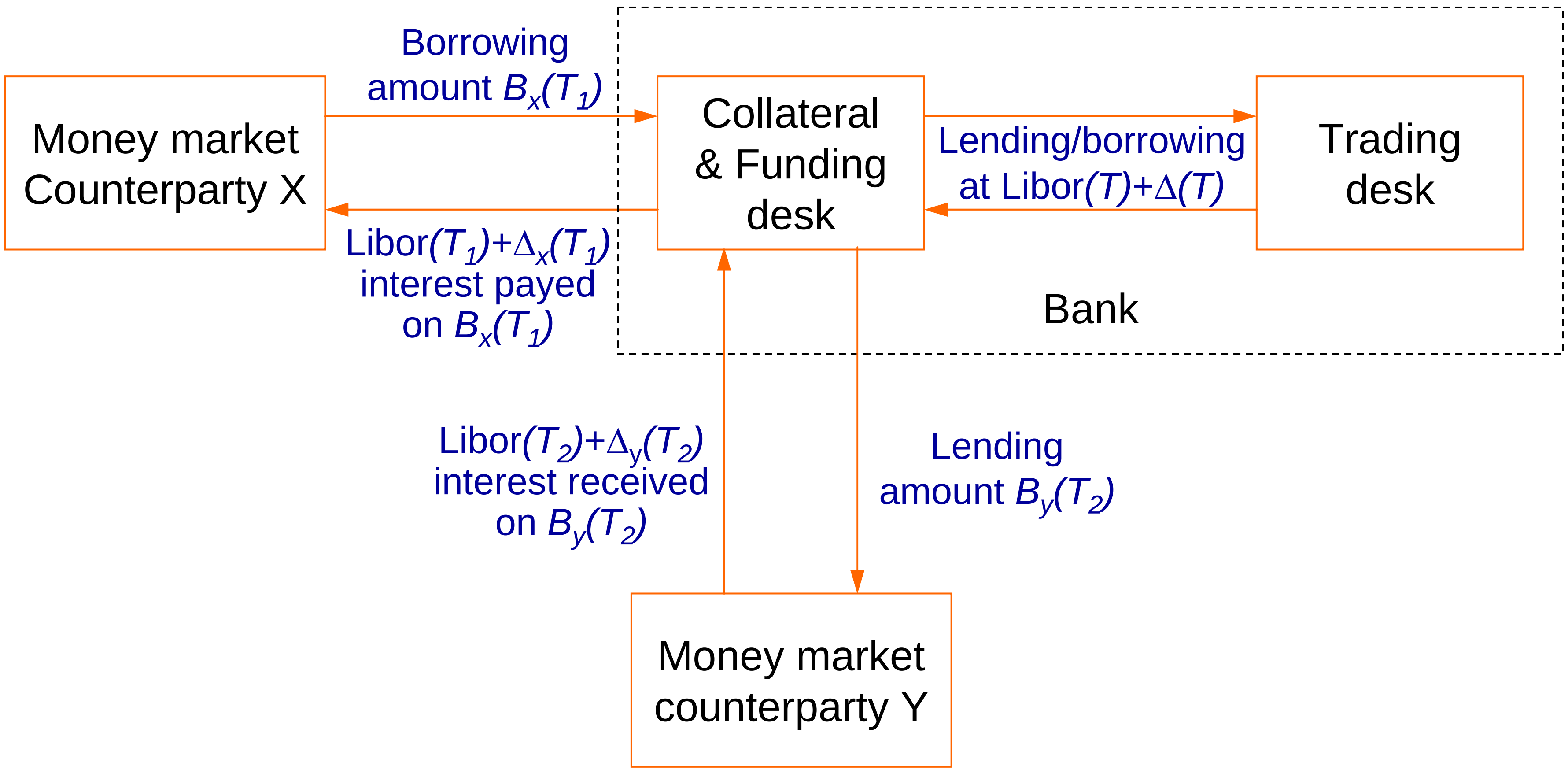
	Percent of trades						
	All OTC derivatives	Fixed Income derivatives	Credit derivatives	FX derivatives	Equity derivatives	Precious & base metals derivatives	Energy and other commodity derivatives
All Respondents	70	79	93	57	71	60	64
Large dealers	78	84	97	63	68	69	62
Medium and Small	68	77	91	54	72	52	65

Table 4.1 Frequency of portfolio reconciliation:percentange of trades reconciled at stated intervals

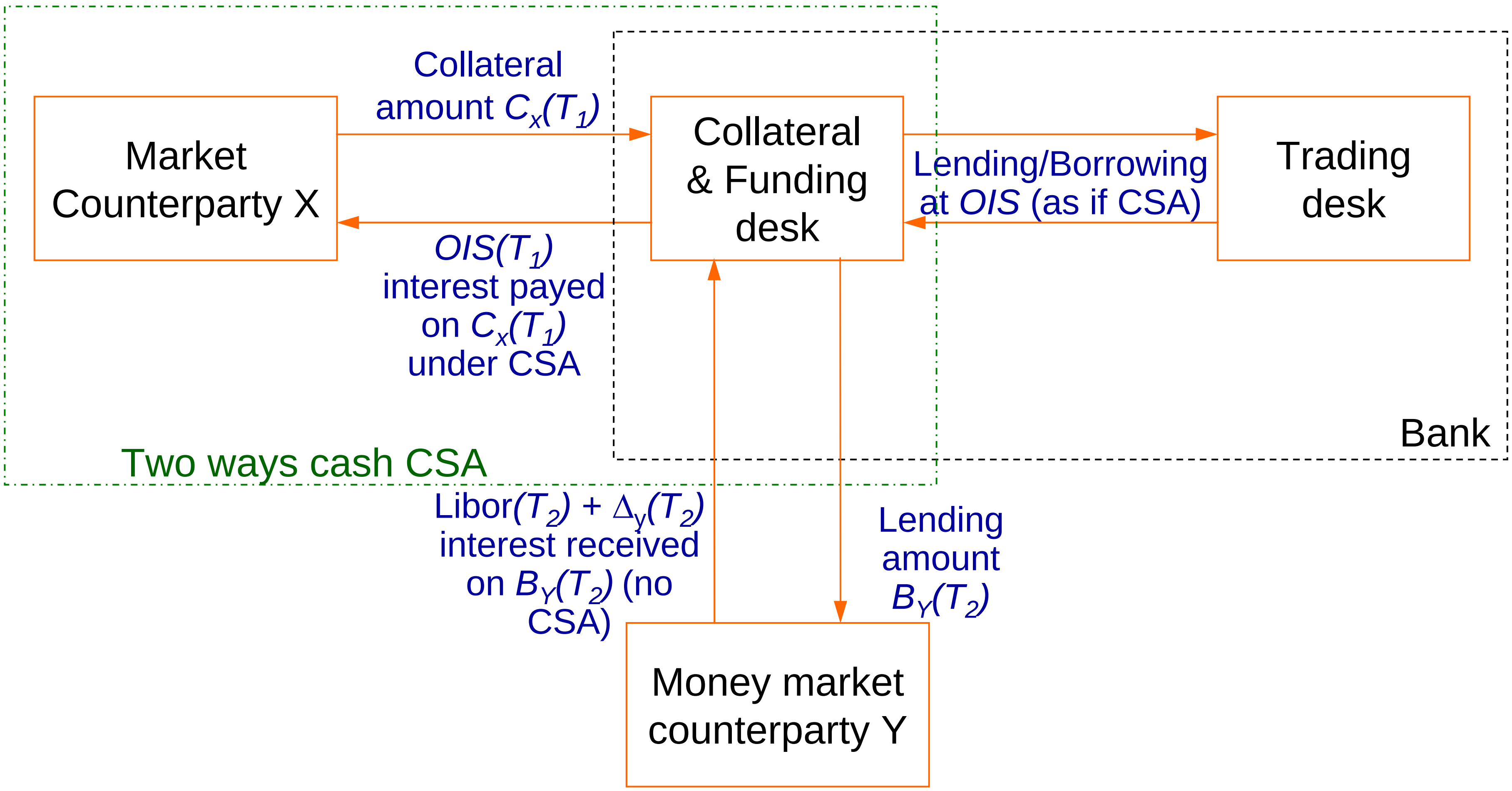
Percent of trades	**Daily**	**Weekly**	**Monthly**	**Ad hoc/ Dispute driven**
Total Sample	29	10	15	47
Fed 14	56	5	3	37

2: The Market Across The Credit Crunch
Counterparty risk and collateral [4]

Typical unsecured funding mechanics in a Bank (no CSA)



Typical secured funding mechanics in a Bank (Two Ways cash CSA)



2: The Market Across The Credit Crunch
From Libor to CSA discounting

At 2010 year end some Banks have given disclosure the switch to CSA discounting [1]:

- BNP: -108 MM EUR (IRS)
- Credit Agricole: -120 MM EUR (Fixed Income)
- Morgan Stanley: +176 MM USD (IRD)
- RBS: +127 MM GBP (???)
- UBS: +76 MM CHF (???)
- HSBC: not significant

"In the fourth quarter of 2010, the Company began using the overnight indexed swap ("OIS") curve as an input to value substantially all of its collateralized interest rate derivative contracts. The Company believes using the OIS curve, which reflects the interest rate typically paid on cash collateral, more accurately reflects the fair value of collateralized interest rate derivative contracts. Previously, the Company discounted these collateralized interest rate derivative contracts based on London Interbank Offered Rates ("LIBOR")." [2]

[1] M. Cameron, ""BNP Paribas takes €108 million on swaps after switch to OIS discounting", Risk, 6 May 2011.
[2] Morgan Stanley & Co. Inc., Consolidated Statement of Financial Condition as of Dec. 31, 2010 and Independent Auditors' report.

2: The Market Across The Credit Crunch
Conclusions Part 1

What we have understood up to now:

- The market has changed.
- Credit and liquidity risk are important.
- Collateral is important.
- Funding is important.
- We must be able to include these new elements in the pricing framework.

3: The Modern No Arbitrage Multiple-Curve Framework
Restating the problem

We may identify two distinct modelling approaches:

1. Modeling the joint evolution of a default-free rate, plus counterparty's default times:
   - Interest rate risk: model one single risk free stochastic rate,
   - Credit risk: model the default of the interbank sector, not of a precise counterparty, taking into account that the Libor panel itself is not static but its composition changes over time, depending on the relative default probability of candidate Libor banks (it's itself stochastic !).
   - Correlations: we need a complex correlation matrix with rate/credit and credit/credit correlations.

2. Modeling the joint evolution of multiple distinct rates:
   this implies taking the approach of multiple-curves constructions to its logical consequences, and to introduce a generalised interest rate model where such distinct curves are modeled jointly.

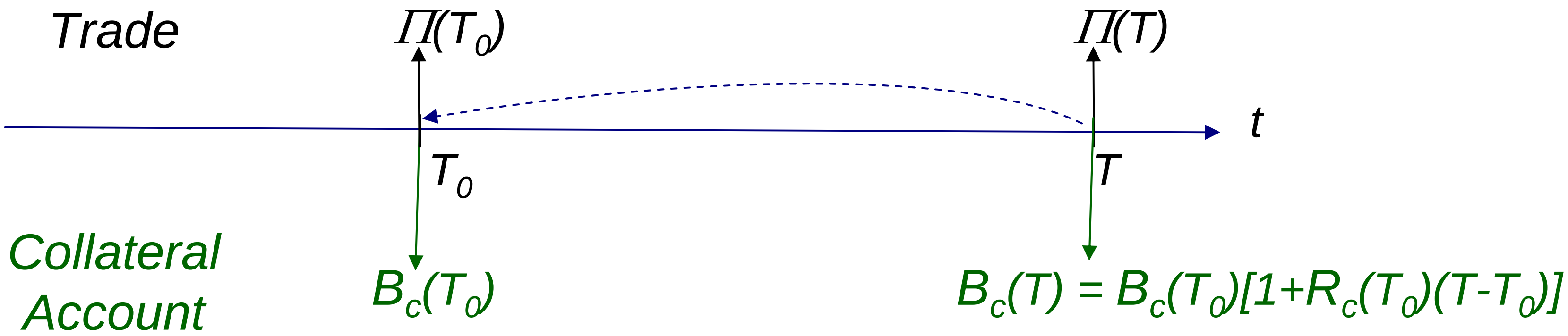
We will follow the second route, as described in the recent financial literature (see bibliography). In particular we will borrow mainly from recent papers, Kijima et al.(2008), F. Mercurio (2009-2010), Fujii et al. (2009-2010), V. Piterbarg (2010,2012).

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [1]

1. **Introducing cost of funding: a simple argument**
   Let's suppose a trade with a single cash flow, such that we receive/pay an amount $\Pi(T)$ at maturity T, corresponding to a present value $\Pi(T_0)$ at time T_0. Let's also suppose that the trade is under perfect collateral, with two margination dates, at T_0 and T. At time T_0 we post the amount $B_c(T_0)$ into the collateral account, where it grows at the collateral rate $R_c(T_0)$ up to maturity T. By no arbitrage and self-financing, we must have

$$B_c(T) = B_c(T_0)\left[1 + R_c(T_0)(T - T_0)\right] = \Pi(T),$$
$$B_c(T_0) = \Pi(T_0) = P_d(T_0, T)\Pi(T),$$
$$\Rightarrow P_d(T_0, T) = \frac{1}{\left[1 + R_c(T_0)(T - T_0)\right]}.$$

   Thus no arbitrage requires discounting at the collateral rate.



3: The Modern No Arbitrage Multiple-Curve Framework
Funding [2]

2. A formal proof: perfect collateral case

(adapted from V. Piterbarg, Risk, Feb 2010, and D. Brigo et al, Jul. 2012)

We consider a generic derivative Π depending on a single generic underlying *S(t)*, with payoff $\Pi[T,S]$ at time T. The price at time $t<T$ is given by $\Pi(t,S)$.

Thus our economy admits, in general, four financial instruments:

- the derivative $\Pi(t,S)$
- the underlying *S(t)*, with no dividends
- the collateral account $B_c(t)$ for the collateral associated with Π
- the funding account $B_f(t)$ for financing purposes

We also assume that the derivative Π is under perfect collateral, such that

$$\Pi(t,S) = B_c(t), \quad \forall\, t \leq T.$$

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [3]

In general, we may assume the following dynamics under the real measure Q_R

$$
\begin{aligned}
dS(t) &= \mu_S^R(t)S(t)dt + \sigma_S(t)S(t)dW^R(t),\\
dB_c(t) &= r_c(t)B_c(t)dt,\\
dB_f(t) &= r_f(t)B_f(t)dt,
\end{aligned}
$$

where $W^R(t)$ is a standard brownian motion under Q_R, $r_c(t)$ is the (instantaneous) collateral rate and $r_f(t)$ is the (instantaneous) funding rate.

Dropping obvious indexes, using Ito's Lemma and the first SDE above, we obtain, for the derivative Π,

$$
\begin{aligned}
d\Pi(t,S) &= \frac{\partial\Pi}{\partial t}dt + \frac{\partial\Pi}{\partial S}dS(t) + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}dS^2(t)\\
&= \left[\frac{\partial\Pi}{\partial t} + \mu(t)S(t)\frac{\partial\Pi}{\partial S} + \frac{1}{2}\sigma^2(t)S^2(t)\frac{\partial^2\Pi}{\partial S^2}\right]dt + \sigma(t)S(t)\frac{\partial\Pi}{\partial S}dW^R(t).
\end{aligned}
$$

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [4]

We now construct a replication strategy of the derivative Π, by setting up a replication portfolio V such that

$$V(t,S) = \Pi(t,S), \quad \forall\, t \leq T,$$

by combining appropriate amounts [θ_1, θ_2, θ_3] of the available assets [S, B_c, B_f],

$$\boldsymbol{V}(t,S) := \begin{bmatrix} S(t) \\ B_c(t) \\ B_f(t) \end{bmatrix}, \quad \boldsymbol{\theta}(t) := \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{bmatrix},$$

$$V(t,S) = \boldsymbol{\theta}(t)' \cdot \boldsymbol{V}(t,S) = \theta_1(t)S(t) + \theta_2(t)B_c(t) + \theta_3(t)B_f(t),$$

where:

- θ is the vector of the portfolio positions, or number of units, in each asset,
- $\boldsymbol{V}$ is the vector of the price processes of the assets,
- V is the (scalar) value of the replication portfolio,
- $\theta(t)'$ denotes vector transposition.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [5]

The assets and the replication portfolio are described, in general, by their (vector or scalar) price processes ***V****(t,S)*, dividend processes ***D****(t,S)*, and gain processes ***G****(t,S)*, such that

$$\boldsymbol{G}(t,S) := \boldsymbol{V}(t,S) + \boldsymbol{D}(t,S),$$
$$G(t,S) := V(t,S) - V(0,S) + D(t,S).$$

The gain processes of the assets, in SDE form, are given directly by the dynamics chosen before, as

$$d\boldsymbol{G}(t,S) := \begin{bmatrix} dS(t) \\ dB_c(t) \\ dB_f(t) \end{bmatrix} = \begin{bmatrix} \mu(t)S(t)dt + \sigma(t)S(t)dW^R(t) \\ r_c(t)B_c(t)dt \\ r_f(t)B_f(t)dt \end{bmatrix}.$$

The gain process of the replication portfolio is given, in SDE form, by

$$\begin{aligned} dG(t,S) :&= \boldsymbol{\theta}(t)' \cdot d\boldsymbol{G}(t,S) = \theta_1(t)dS(t) + \theta_2(t)dB_c(t) + \theta_3(t)dB_f(t) \\ &= \left[\mu(t)\theta_1(t)S(t) + r_c(t)\theta_2(t)B_c(t) + r_f(t)\theta_3(t)B_f(t)\right]dt \\ &+ \sigma(t)\theta_1(t)S(t)dW^R(t). \end{aligned}$$

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [6]

The dividend processes of the assets may now be obtained by difference, in SDE form, as

$$d\boldsymbol{D}(t,S) = d\boldsymbol{G}(t,S) - d\boldsymbol{V}(t,S) = \begin{bmatrix} dS(t) \\ dB_c(t) \\ dB_f(t) \end{bmatrix} - d\begin{bmatrix} S(t) \\ B_c(t) \\ B_f(t) \end{bmatrix} = \boldsymbol{0},$$

$$\boldsymbol{D}(0,S) = \boldsymbol{0}.$$

The dividend process for the replication portfolio is thus given by

$$D(t,S) := \boldsymbol{\theta}(t)' \cdot \boldsymbol{D}(t,S) = 0,$$
$$D(0,S) = 0.$$

This is consistent with the absence of dividends assumed at the beginning.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [7]

In particular, with zero dividends we also obtain

$$
\begin{aligned}
& dV(t,S) = dG(t,S), \\
& \Rightarrow\ d\left[\boldsymbol{\theta}(t)' \cdot \boldsymbol{V}(t,S)\right] = \boldsymbol{\theta}(t)' \cdot d\boldsymbol{G}(t,S) = \boldsymbol{\theta}(t)' \cdot d\boldsymbol{V}(t,S), \\
& \Rightarrow\ d\left[\theta_1(t)S(t)\right] = \theta_1(t)dS(t), \\
& \Rightarrow\ d\theta_1(t) = 0, \\
& \Rightarrow\ \theta_1(t) = \ \ constant.
\end{aligned}
$$

This is the well known feature of the classical Black-Scholes derivation:

- the position $\theta_1(t)S(t)$ in the risky asset *S* is self-financing in its own, because its variation $d[\theta_1(t)S(t)]$ is funded by the risky asset variation alone, $\theta_1(t)dS(t)$.
- The position is static, $\theta_1(t)= constant$.

We stress that this is a consequence of the absence of dividends. In general this equality does not hold (see later),

$$d\left[\boldsymbol{\theta}(t)' \cdot \boldsymbol{V}(t,S)\right] \neq \boldsymbol{\theta}(t)' \cdot d\boldsymbol{V}(t,S).$$

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [8]

We now impose the perfect collateral and replication conditions, and we obtain

$$
\begin{aligned}
V(t,S) &= \theta_1(t)S(t) + \boxed{\theta_2(t)B_c(t)} + \theta_3(t)B_f(t) \\
&= \theta_1(t)S(t) + \boxed{\Pi(t,S)} + \theta_3(t)B_f(t), \\
V(t,S) &= \Pi(t,S), \quad \forall\, t \leq T \\
&\Rightarrow \theta_2(t) = 1, \quad \theta_3(t) = 1, \quad B_f(t) = -\theta_1(t)S(t),
\end{aligned}
$$

consistently with the fact that the funding account B_f is used to finance the borrowing of $\theta_1(t)$ units of the underlying $S(t)$ at the funding rate $r_f(t)$.

The gain process of the replication portfolio becomes

$$
dG(t,S) = [\mu(t)\theta_1(t)S(t) + \underbrace{r_c(t)\Pi(t,S) - r_f(t)\theta_1(t)S(t)}_{d\Gamma(t,S)}]\,dt + \sigma(t)\theta_1(t)S(t)dW^R(t).
$$

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [9]

We observe at this stage that the cash amount $\Gamma(t,S)$ contained in the replication portfolio is split between:

- the collateral account $B_c(t)$, growing at the collateral rate $r_c(t)$,
- the amount $\theta(t)S(t)$, borrowed at the funding rate $r_f(t)$ to finance the purchase of $\theta(t)$ units of the underlying $S(t)$,

such that

$$\begin{aligned} d\Gamma(t,S) &= r_c(t)B_c(t)dt - r_f(t)\theta_1(t)S(t)dt \\ &= \left[r_c(t)\Pi(t,S) - r_f(t)\theta_1(t)S(t)\right]dt. \end{aligned}$$

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [10]

We now impose the self-financing condition. The replication strategy is said self-financing if its dividend process (in/out cash flows generated by the strategy) is null,

$$D(t, S) = G(t, S) - V(t, S) = 0.$$

We have just seen that this latter condition is already satisfied.
Combining the conditions above, we have

$$dG(t, S) = dV(t, S) = d\Pi(t, S).$$

3: The Modern No Arbitrage Multiple-Curve Framework

Funding [11]

Introducing in this latter equation the expressions of *dG(t,S)* and *dΠ(t,S)* obtained before, and rearranging terms we obtain the SDE

$$\left[\frac{\partial\Pi}{\partial t} + \mu(t)S(t)\left(\frac{\partial\Pi}{\partial S} - \theta_1(t)\right) + r_f(t)\theta_1(t)S(t) + \frac{1}{2}\sigma^2(t)S^2(t)\frac{\partial^2\Pi}{\partial S^2}\right]dt + \sigma(t)S(t)\left(\frac{\partial\Pi}{\partial S} - \theta_1(t)\right)dW^R(t) = r_c(t)\Pi(t)dt.$$

We finally impose the risk free condition

$$\theta_1(t) = \frac{\partial\Pi}{\partial S},$$

such that the stochastic (risky) term with $dW^R(t)$ disappears, and we obtain…

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [12]

…a generalised Black-Scholes PDE equation for the derivative's price *Π(t,S)*

$$\frac{\partial \Pi}{\partial t}(t,S) + r_f(t) S(t) \frac{\partial \Pi}{\partial S}(t,S) + \frac{1}{2}\sigma^2(t) S^2(t) \frac{\partial^2 \Pi}{\partial S^2}(t,S) = r_c(t) \Pi(t,S).$$

Using the Feynman-Kac theorem we may switch from the PDE representation to the SDE representation given by

$$\Pi(t,S) = \mathbb{E}_t^{Q_f}\left[D_c(t;T)\Pi(T,S)\right],$$

$$D_c(t;T) := \exp\left[-\int_t^T r_c(u)du\right],$$

$$dS(t) = r_f(t) S(t) dt + \sigma(t) S(t) dW^{Q_f}(t),$$

under the probability measure Q_f associated to the funding account $B_f(t)$.

We conclude that we discount at the collateral rate.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [13]

Remarks:

- The borrowing of the underlying S is normally realised through repo contracts and funded at the repo rate. Furthermore, equity underlyings' pay, in general, dividends. Thus a more general proof is necessary to deal with these real cases.

- Perfect collateral is a rather idealised CSA with the following characteristics:
  - cash collateral only
  - collateral currency = deal currency
  - fully symmetric
  - zero threshold
  - daily margination
  - flat overnight margination rate
  - instantaneous settlement

  Since real CSAs are far from being perfect, a more general proof is required to take into account imperfect collateral, such that $B_c(t) \neq \Pi(t,S)$, also in terms of currency.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [14]

- The probability measure Q_f introduced via Feynman-Kac is associated with the generic account $B_f(t)$ used for funding the hedging strategy.
  - In the classical financial world Q_f was associated with a Libor Bank account, reflecting the average funding rate on the interbank money market, with the underlying's dynamics

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW^{Q}(t),$$

    where the short rate $r(t)$ was associated with Libor, considered a good proxy of a risk free rate.
  - Nowadays, in the modern financial world, Q_f must be interpreted simply as the measure associated with the funding account $B_f(t)$. We stress that this account is, in general, not risk free. In the special case of funding at overnight, assuming that overnight rate = collateral rate, we have $B_f(t) = B_c(t)$, $Q_f = Q_c$, and the dynamics for S becomes

$$dS(t) = r_c(t)S(t)dt + \sigma(t)S(t)dW^{Q_c}(t),$$

    where the overnight/collateral short rate $r_c(t)$ is considered a good proxy of a risk free rate.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [15]

- No collateral: set in the proof $B_c(t) \to 0 \; \forall t$ and obtain

$$\frac{\partial \Pi}{\partial t}(t,S) + r_f(t) S(t) \frac{\partial \Pi}{\partial S}(t,S) + \frac{1}{2}\sigma^2(t) S^2(t) \frac{\partial^2 \Pi}{\partial S^2}(t,S) = r_f(t)\Pi(t,S)$$

$$\Pi(t,S) = \mathbb{E}_t^{Q_f}\left[D_f(t;T)\Pi(T,S)\right],$$

$$D_f(t;T) := \exp\left[-\int_t^T r_f(u)\,du\right],$$

$$dS(t) = r_f(t) S(t) dt + \sigma(t) S(t) dW^{Q_f}(t),$$

under the probability measure Q_f associated to the funding account $B_f(t)$. Hence, we discount at the funding rate.

Clearly, without collateral (but even with collateral), there is a counterparty risk that we do not consider here.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding [16]

Collateralised trades (CSA)	Uncollateralised trades (no CSA)
Assuming perfect collateralisation: ■ Fully symmetric ■ Zero threshold ■ Cash collateral only ■ Collateral currency = deal currency ■ Daily margination ■ Flat overnight margination rate ■ Immediate settlement $\Rightarrow$ By no arbitrage **discounting rate** **= funding rate** **= collateral rate**	Assuming no collateralisation: ■ Funding on the money market (Deposits) or securities market (Bonds, etc.) $\Rightarrow$ By no arbitrage **discounting rate** **= funding rate**
Overnight or OIS discounting	**“Libor+” discounting**

See also J. Hull, A. White, “*LIBOR vs OIS: The Derivatives Discounting Dilemma*” Apr. 2012, *www.defaultrisk.com*

3: The Modern No Arbitrage Multiple-Curve Framework
Pricing vanilla FRAs [1]

1. Standard FRA:
   the payoff at time T_i of the standard FRA tied to risky Libor $L_x(T_{i-1},T_i)$ is

$$\mathbf{FRA}_{\text{Std}}(T_i;\mathbf{T},K,\omega) = N\omega\left[L_x(T_{i-1},T_i) - K\right]\tau_x(T_{i-1},T_i),$$

   The price at time $t<T_{i-1}$ is given by

$$\begin{aligned}\mathbf{FRA}_{\text{Std}}(t;\mathbf{T},K,\omega) &= P_d(t;T_i)\mathbb{E}_t^{Q_d^{T_i}}\left[\mathbf{FRA}_{\text{Std}}(T_i;\mathbf{T},K,\omega)\right]\\ &= N\omega P_d(t;T_i)\left\{\mathbb{E}_t^{Q_d^{T_i}}\left[L_x(T_{i-1},T_i)\right] - K\right\}\tau_x(T_{i-1},T_i),\end{aligned}$$

   and we can define the generalised FRA rate as

$$\begin{aligned}\tilde{R}_{x,\text{Std}}^{\text{FRA}}(t;\mathbf{T}) &:= \tilde{F}_{x,i}(t) := \mathbb{E}_t^{Q_d^{T_i}}\left[L_x(T_{i-1},T_i)\right]\\ &\neq R_{\text{Std}}^{\text{FRA}}(t;T_{i-1},T_i)\end{aligned}$$

   such that the standard FRA price can be written as

$$\mathbf{FRA}_{\text{Std}}(t;\mathbf{T},K,\omega) = N\omega P_d(t;T_i)\left[\tilde{F}_{x,i}(t) - K\right]\tau_x(T_{i-1},T_i).$$

3: The Modern No Arbitrage Multiple-Curve Framework

Pricing vanilla FRAs [2]

Properties of the generalised FRA rate:

1. at fixing date T_{i-1} it coincides with the Libor rate

$$\tilde{F}_{x,i}(T_{i-1}) = L_x(T_{i-1}, T_i).$$

2. It is a martingale under the T_i- forward discounting measure $Q_d^{T_i}$ associated to the numeraire $P_d(t;T_i)$:

$$\tilde{F}_{x,i}(t) := \mathbb{E}_t^{Q_d^{T_i}}\left[L_x(T_{i-1}, T_i)\right] = \mathbb{E}_t^{Q_d^{T_i}}\left[\tilde{F}_{x,i}(T_{i-1})\right],$$

3. In the single curve limit it recovers the classical single-curve value

$$\tilde{F}_{x,i}(t) := \mathbb{E}_t^{Q_d^{T_i}}\left[L_x(T_{i-1}, T_i)\right] \underset{d \simeq x}{\longrightarrow} \mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1}, T_i)\right]$$

$$= \mathbb{E}_t^{Q^{T_i}}\left[F(T_{i-1}; T_{i-1}, T_i)\right] = F(t; T_{i-1}, T_i) := F_i(t),$$

thanks to the (single-curve) martingality property of the forward rate $F(t;T_{i-1},T_i)$ under the forward measure Q^{T_i}.

4. FRA contracts are quoted on the market in terms of the FRA rates, thus it is "what you read on the screen". A FRA rate term structure can be stripped from FRA quotations.

5. We do not even need to talk about "forward rates" anymore: the FRA rate itself is the basic building block of the new theoretical interest rate framework.

3: The Modern No Arbitrage Multiple-Curve Framework

Pricing vanilla FRAs [3]

16:16 30DEC10 ICAP LONDON UK69580 ICAPSHORT2

Contact Reuters EXEU EURO Short Swaps / FRAs +44 (0)20 7532 3530

1M Swaps	
2x1	0.827-0.777
3x1	0.839-0.789
4x1	0.850-0.800
5x1	0.863-0.813
6x1	0.875-0.825
7x1	0.888-0.838
8x1	0.907-0.857
9x1	0.924-0.874
10x1	0.941-0.891
11x1	0.954-0.904
12x1	0.969-0.919
1y /3	1.158-1.108
15m/3	1.209-1.159
18m/3	1.270-1.220
21m/3	1.345-1.295
1y /6	1.347-1.297
15m/6	1.352-1.302
18m/6	1.454-1.404
21m/6	1.497-1.447

IMM Dated		
1y	MAR/MAR	1.213-1.163
1y	JUN/JUN	1.329-1.279
1y	SEP/SEP	1.477-1.427
1y	DEC/DEC	1.646-1.596
2y	MAR/MAR	1.518-1.468
2y	JUN/JUN	1.671-1.621
3y	MAR/MAR	1.855-1.805

IMM Fras	
1x7	1.273-1.223
2x8	1.292-1.242
3x9	1.320-1.270
4x10	1.344-1.294
0x3 Today	1.035-0.985
0x6 Today	1.259-1.209
0x3 Tom	1.033-0.983
0x6 Tom	1.260-1.210

3m FRAs	
1x4	1.037-0.987
2x5	1.055-1.005
3x6	1.080-1.030
4x7	1.108-1.058
5x8	1.137-1.087
6x9	1.166-1.116

6m FRAs	
1x7	1.264-1.214
2x8	1.284-1.234
3x9	1.307-1.257
4x10	1.332-1.282
5x11	1.357-1.307
6x12	1.391-1.341
12x18	1.649-1.599
18x24	2.025-1.975

12m FRA	
12x24	2.001-1.951

ICAP Global Index <ICAP> ICAP OIS Fix Menu <ICAPOISFIX01>

Forthcoming changes <ICAPCHANG

3: The Modern No Arbitrage Multiple-Curve Framework
Pricing vanilla FRAs [4]

	FRA pricing formulas
Classical (single-curve)	$\mathbf{FRA}_{\text{Std}}(t;T_{i-1},T_i,K,\omega) = N\omega P(t;T_i)\left[F_i(t)-K\right]\tau(T_{i-1},T_i),$ $R^{\text{FRA}}_{\text{Std}}(t;\mathbf{T}) = F_i(t) = \mathbb{E}^{Q^{T_i}}_t\left[L(T_{i-1},T_i)\right],$ $\mathbf{FRA}_{\text{Mkt}}(t;T_{i-1},T_i,K,\omega) = \mathbf{FRA}_{\text{Std}}(t;T_{i-1},T_i,K,\omega),$ $R^{\text{FRA}}_{\text{Mkt}}(t,\mathbf{T}) = R^{\text{FRA}}_{\text{Std}}(t,\mathbf{T}).$
Modern (multiple-curve)	$\mathbf{FRA}_{\text{Std}}(t;T_{i-1},T_i,K,\omega) = N\omega P_d(t;T_i)\left[\tilde{F}_{x,i}(t)-K\right]\tau_x(T_{i-1},T_i),$ $\tilde{R}^{\text{FRA}}_{x,\text{Std}}(t;\mathbf{T}) = \tilde{F}_{x,i}(t) := \mathbb{E}^{Q_d^{T_i}}_t\left[L_x(T_{i-1},T_i)\right],$ $\mathbf{FRA}_{\text{Mkt}}(t;\mathbf{T},K,\omega) = N\omega P_d(t;T_{i-1})\left[1-\frac{1+\tau_x(T_{i-1},T_i)K}{1+\tau_x(T_{i-1},T_i)\tilde{F}_{x,i}(t)}e^{C_x^{\text{FRA}}(t;T_{i-1})}\right],$ $\tilde{R}^{\text{FRA}}_{x,\text{Mkt}}(t;\mathbf{T}) = \frac{1}{\tau_x(T_{i-1},T_i)}\left\{\left[1+\tau_x(T_{i-1},T_i)\tilde{F}_{x,i}(t)\right]e^{C_x^{\text{FRA}}(t;T_{i-1})}-1\right\}.$

3: The Modern No Arbitrage Multiple-Curve Framework

Pricing vanilla Swaps

We stress that the legs of a spot-starting swap do not need to be worth par (when a fictitious exchange of notionals is introduced at maturity). However, this is not a problem, since the only requirement for quoted spot-starting swaps is that their initial total value must be equal to zero.

	Swap pricing formulas
Classical (single-curve)	$$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega) = N\omega\left[R^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) - K\right]A_d(t,\mathbf{S}),$$ $$R^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) = \frac{\sum_{j=1}^{m} P(t,T_j)F(t;T_{j-1},T_j)\tau_L(T_{j-1},T_j)}{A(t,\mathbf{S})} \simeq \frac{P(t,T_0)-P(t,T_m)}{A(t,\mathbf{S})}.$$
Modern (multiple-curve)	$$\mathbf{Swap}(t;\mathbf{T},\mathbf{S},K,\omega) = N\omega\left[\tilde{R}_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) - K\right]A_d(t,\mathbf{S}),$$ $$\tilde{R}_x^{\text{Swap}}(t;\mathbf{T},\mathbf{S}) = \frac{\sum_{j=1}^{m} P_d(t,T_j)\tilde{F}_{x,j}(t)\tau_x(T_{j-1},T_j)}{A_d(t,\mathbf{S})}.$$

3: The Modern No Arbitrage Multiple-Curve Framework:
Pricing Overnight Indexed Swaps

	OIS pricing formulas
Classical (single-curve)	$$\mathbf{OIS}(t;\mathbf{T},K,\omega) = N\omega\left[R^{\text{OIS}}(t;\mathbf{T}) - K\right]A(t;\mathbf{T}),$$ $$R^{\text{OIS}}(t;\mathbf{T}) = \frac{P(t;T_0) - P(t;T_n)}{A(t;\mathbf{T})}$$
Modern (multiple-curve)	$$\mathbf{OIS}(t;\mathbf{T},K,\omega) = N\omega\left[R_d^{\text{OIS}}(t;\mathbf{T}) - K\right]A_d(t;\mathbf{T}),$$ $$R_d^{\text{OIS}}(t;\mathbf{T}) = \frac{P_d(t;T_0) - P_d(t;T_n)}{A_d(t;\mathbf{T})}.$$

3: The Modern No Arbitrage Multiple-Curve Framework

Pricing vanilla Basis Swap

	Basis Swap pricing formulas
Classical (single-curve)	$\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S},\omega)=\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)=0$
Modern (multiple-curve)	$\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y,\boldsymbol{S},\omega)=\dfrac{\mathbf{Swap}_{\text{float}}(t;\boldsymbol{T}_x)-\mathbf{Swap}_{\text{float}}(t;\boldsymbol{T}_y)}{N\omega A_d(t;\boldsymbol{S})},$ $\Delta(t;\boldsymbol{T}_x,\boldsymbol{T}_y)=\dfrac{\mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_x,\omega)-\mathbf{Swap}_{\text{Float}}(t;\boldsymbol{T}_y,\omega)}{N\omega A_d(t;\boldsymbol{T}_y)}.$

3: The Modern No Arbitrage Multiple-Curve Framework

Pricing vanilla Caps/Floors/Swaptions

	Caplet/floorlet pricing formulas
Classical (single-curve)	$NP(t,T_i)\tau(T_{i-1},T_i)\text{Black}\left[F_i(t),K,v(t;T_{i-1}),\omega\right]$
Modern (multiple-curve)	$NP_d(t,T_i)\tau_x(T_{i-1},T_i)\text{Black}\left[\tilde{F}_{x,i}(t),K,\tilde{v}_x(t;T_{i-1}),\omega\right]$

	Swaption pricing formulas
Classical (single-curve)	$NA(t,\mathbf{S})\text{Black}\left[R^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,v(t,\mathbf{T},\mathbf{S}),\omega\right]$
Modern (multiple-curve)	$NA_d(t,\mathbf{S})\text{Black}\left[\tilde{R}_x^{\text{Swap}}(t,\mathbf{T},\mathbf{S}),K,\tilde{v}_x(t,\mathbf{T},\mathbf{S}),\omega\right]$

3: The Modern No Arbitrage Multiple-Curve Framework

Modern multiple curves market practice [1]

In case of vanilla linear derivatives the modern procedure is as follows:

1. build a single discounting curve $\mathcal{C}_d$ using the preferred bootstrapping procedure;
2. build multiple distinct forwarding curves $\mathcal{C}_{f1} \ldots \mathcal{C}_{fn}$ using distinct selections of vanilla interest rate instruments, each homogeneous in the underlying rate tenor (typically 1M, 3M, 6M, 12M);
3. compute the FRA/Swap rates with tenor *f* on the corresponding forwarding curve $\mathcal{C}_f$ and calculate the corresponding cash flows;
4. compute the corresponding discount factors using the discounting curve $\mathcal{C}_d$ and work out prices by summing the discounted cashflows;
5. compute the delta sensitivity and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the corresponding set of vanillas.

3: The Modern No Arbitrage Multiple-Curve Framework

Modern multiple curves market practice [2]

In case of volatility derivatives the procedure above must be extended as follows:

1. build multiple distinct volatility surfaces $\Sigma_{f1} \ldots \Sigma_{fn}$ using distinct selections of vanilla interest rate options, each homogeneous in the underlying rate tenor, typically 1M, 3M, 6M, 12M for Euribor rate and swap rate volatilities;

2. compute the FRA/Swap rates and volatilities with tenor *f* on the corresponding curves $\mathcal{C}_f$ and volatility surfaces Σ_{f1} , and calculate the corresponding cashflows;

3. compute the corresponding discount factors using the discounting curve $\mathcal{C}_d$ and work out prices by summing the discounted cashflows;

4. compute the delta and vega sensitivities and hedge the resulting delta and vega risk using the suggested amounts (hedge ratios) of the corresponding set of vanillas.

3: The Modern No Arbitrage Multiple-Curve Framework
Modern multiple curves market practice [3]

In case of non-vanilla derivatives the procedure above must be extended as follows:

- Choose the fundamental variables:
  - Multiple short rates ⇒ multi-curve short rate models.
  - Multiple instantaneous forward rates ⇒ multi-curve HJM models.
  - Multiple discrete FRA rates ⇒ multi-curve Black's model, SABR, Libor Market Model.
  - Multiple forward Swap rates ⇒ multi-curve Black's model, SABR, Swap Market Model.
- Assume a dynamics for the time evolution of the fundamental variables.
- Derive (arbitrage free) pricing formulas for plain vanilla instruments.
- Calibrate the model parameters to the market quotes of a chosen set of plain vanilla instruments (calibration instruments).
- Price other derivatives using the calibrated model.
- Derive sensitivities and hedge ratios with respect to a choosen subset of calibration instruments (hedging instruments).

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curve models

But research is just at the beginning:

- Libor Market Models: see e.g. F. Mercurio, "*A Libor Market Model with Stochastic Basis*", Risk Magazine, Dec. 2010 and refs. therein.
- HJM models: see e.g N. Moreni, A. Pallavicini, "*Parsimonious HJM Modelling for Multiple Yield-Curve Dynamics",* Oct. 2010, SSRN working paper, *http://ssrn.com/abstract=1699300*
- Short rate models: see e.g. C. Kenyon, "*Post Shock Short-Rate Pricing*", Risk Magazine, Oct. 2010.
- HJM Model with credit risk: see e.g. S. Crepey, Z. Grbac, H. Nguyen, "*A defaultable HJM multiple-curve term structure model*", 9 Oct. 2011.

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves construction

- **Main issues:**
  - Interpretation of the market quotes
  - Selection of bootstrapping instruments
  - Bootstrapping formulas
  - Bootstrapping results
  - Choice of interpolation
  - Turn of year effect
  - Multiple delta hedging

- **Main references** (see bibliography):
  - F. Ametrano and M. Bianchetti, "*Bootstrapping the Illiquidity: Multiple Yield Curves Construction For Market Coherent Forward Rates Estimation*", in "*Modeling Interest Rates: Latest Advances for Derivatives Pricing*", edited by F. Mercurio, Risk Books, 2009.
  - …

"When dealing with curves, nothing ever goes straight"
Andrea Bugin, (now) Head of Financial Engineering, Banca IMI

3: The Modern No Arbitrage Multiple-Curve Framework

Multiple curves construction: bootstrapping instruments

EUR yield curves market bootstrapping instruments				
Discount curve	1M tenor curve	3M tenor curve	6M tenor curve	12M tenor curve
■ OIS 1D-2Y ■ Basis Eonia-Euribor3M	■ Depo 1M ■ Swap 1M, 2M-12M ■ Basis swap 6M-1M	■ Depo 3M + 2 synthetics ■ FRA 3M tod/tom. ■ FRA 3M from 1x4 to 1x9 ■ IMM/serial Futures 3M ■ Basis swap 6M-3M	■ Depo 6M + 5 synthetics ■ FRA 6M tod/tom. ■ FRA 6M 1x7-18x24 ■ FRA 6M IMM 1x7-4x10 ■ Swaps 6M	■ Depo 12M + 11 synthetics ■ FRA 12x24 ■ Basis swap 6M-12M

- We select multiple distinct sets of bootstrapping instruments homogeneous in the underlying rate tenor, no “mixing apples and oranges”.

- The choice of the bootstrapping instruments is subject to many practical trading and risk management considerations: liquidity, bid-ask spreads, transaction costs, information to be included or not in the curve, etc.

3: The Modern No Arbitrage Multiple-Curve Framework

Multiple curves construction: market data [1]

OIS curve (Eonia):

16:16 30DEC10 ICAP LONDON UK69580 ICAPSHORT1
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	Eonia		Fwd EONIA		ECB Dates		Eonia v 3m E'bor A/360
1w	0.462-0.362	1X2	0.644-0.594	JAN	0.607-0.557	1Yr	+037.5/+032.5
2w	0.456-0.356	2X3	0.686-0.636	FEB	0.656-0.606	18M	+036.5/+031.5
3w	0.511-0.411	1x4	0.685-0.635	MAR	0.697-0.647	2Yr	+036.1/+031.1
1m	0.527-0.477	2x5	0.724-0.674	APR	0.735-0.685	3Yr	+035.0/+030.0
2m	0.582-0.532	3x6	0.758-0.708	MAY	0.771-0.721	4Yr	+034.1/+029.1
3m	0.619-0.569	6x12	0.892-0.842	JUN	0.795-0.745	5Yr	+033.2/+028.2
4m	0.644-0.594					6Yr	+032.2/+027.2
5m	0.669-0.619	IMM Fra/Eonia				7Yr	+031.2/+026.2
6m	0.689-0.639					8Yr	+030.2/+025.2
7m	0.707-0.657	MAR	35.100-30.100			9Yr	+029.3/+024.3
8m	0.726-0.676	JUN	34.200-29.200			10Y	+028.2/+023.2
9m	0.743-0.693	SEP	34.600-29.600			11Y	+027.3/+022.3
10m	0.761-0.711	DEC	35.300-30.300			12Y	+026.5/+021.5
11m	0.776-0.726					15Y	+024.4/+019.4
12m	0.792-0.742					20Y	+022.4/+017.4
Two Payments						25Y	+021.4/+016.4
15m	0.849-0.799					30Y	+020.7/+015.7
18m	0.914-0.864						
21m	0.989-0.939						
2y	1.071-1.021						
3y	1.419-1.369						

All ICAP Euro pages to close at 5.30pm on 30th Dec (2 Swaps)

ICAP OIS Fix Menu <ICAPOISFIX01>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANG

16:21 30DEC10 ICAP UK69580 ICAPEUR02

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	Eonia IRS ACT/360	Euro Swap vs 3M Euribor 30/360	Euro Swap vs 1M Euribor 30/360
1YR	0.804-0.734	1.159-1.109	0.994-0.924
2YR	1.081-1.011	1.428-1.378	1.265-1.195
3YR	1.429-1.359	1.769-1.719	1.609-1.539
4YR	1.745-1.675	2.081-2.031	1.922-1.852
5YR	2.026-1.956	2.356-2.306	2.201-2.131
6YR	2.260-2.190	2.584-2.534	2.432-2.362
7YR	2.459-2.389	2.776-2.726	2.627-2.557
8YR	2.631-2.561	2.939-2.889	2.795-2.725
9YR	2.770-2.700	3.072-3.022	2.930-2.860
10YR	2.894-2.824	3.187-3.137	3.048-2.978
11YR	3.006-2.936	3.290-3.240	3.156-3.086
12YR	3.106-3.036	3.383-3.333	3.253-3.183
15YR	3.310-3.240	3.569-3.519	3.449-3.379
20YR	3.405-3.335	3.645-3.595	3.541-3.471
25YR	3.344-3.274	3.572-3.522	3.481-3.411
30YR	3.239-3.169	3.459-3.409	3.376-3.306

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Source: Reuters

3: The Modern No Arbitrage Multiple-Curve Framework

Multiple curves construction: market data [2]

IRS curves (Euribor):

16:16 30DEC10 ICAP UK69580 ICAPEURO

Euribor vs 6 mth — 3/6 basis Spot Starting Date

1 Yr	1.347-1.297	16Yrs	3.704-3.654	1 Yr	+19.0
2 Yrs	1.610-1.560	17Yrs	3.724-3.674	2 Yrs	+18.2
3 Yrs	1.946-1.896	18Yrs	3.735-3.685	3 Yrs	+17.7
4 Yrs	2.256-2.206	19Yrs	3.737-3.687	4 Yrs	+17.1
5 Yrs	2.529-2.479	20Yrs	3.733-3.683	5 Yrs	+16.6
6 Yrs	2.749-2.699			6 Yrs	+15.9
7 Yrs	2.934-2.884	21Yrs	3.724-3.674	7 Yrs	+15.2
8 Yrs	3.089-3.039	22Yrs	3.710-3.660	8 Yrs	+14.5
9 Yrs	3.215-3.165	23Yrs	3.693-3.643	9 Yrs	+13.8
10Yrs	3.323-3.273	24Yrs	3.673-3.623	10Yrs	+13.1
		25Yrs	3.651-3.601		
11Yrs	3.419-3.369				
12Yrs	3.505-3.455	26Yrs	3.627-3.577	10X12	0.202/0.162
13Yrs	3.575-3.525	27Yrs	3.603-3.553	10X15	0.371/0.331
14Yrs	3.631-3.581	28Yrs	3.578-3.528	10X20	0.430/0.390
15Yrs	3.674-3.624	29Yrs	3.554-3.504	10X25	0.348/0.308
		30Yrs	3.531-3.481	10X30	0.228/0.188
		35Yrs	3.438-3.388	10X35	0.135/0.095
		40Yrs	3.378-3.328	10X40	0.075/0.035
		50Yrs	3.335-3.285	10X50	0.032/-0.008
		60Yrs	3.285-3.235	10X60	-0.018/-0.058

All ICAP Euro pages to close at 5.30pm on 30th Dec

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16:16 30DEC10 ICAP LONDON UK69580 ICAPSHORT2

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1M Swaps		IMM Dated			3m FRAs	
2x1	0.827-0.777	1y	MAR/MAR	1.213-1.163	1x4	1.037-0.987
3x1	0.839-0.789	1y	JUN/JUN	1.329-1.279	2x5	1.055-1.005
4x1	0.850-0.800	1y	SEP/SEP	1.477-1.427	3x6	1.080-1.030
5x1	0.863-0.813	1y	DEC/DEC	1.646-1.596	4x7	1.108-1.058
6x1	0.875-0.825	2y	MAR/MAR	1.518-1.468	5x8	1.137-1.087
7x1	0.888-0.838	2y	JUN/JUN	1.671-1.621	6x9	1.166-1.116
8x1	0.907-0.857	3y	MAR/MAR	1.855-1.805		
9x1	0.924-0.874				6m FRAs	
10x1	0.941-0.891	IMM Fras			1x7	1.264-1.214
11x1	0.954-0.904				2x8	1.284-1.234
12x1	0.969-0.919	1x7		1.273-1.223	3x9	1.307-1.257
		2x8		1.292-1.242	4x10	1.332-1.282
1y /3	1.158-1.108	3x9		1.320-1.270	5x11	1.357-1.307
15m/3	1.209-1.159	4x10		1.344-1.294	6x12	1.391-1.341
18m/3	1.270-1.220				12x18	1.649-1.599
21m/3	1.345-1.295	0x3 Today		1.035-0.985	18x24	2.025-1.975
		0x6 Today		1.259-1.209		
1y /6	1.347-1.297	0x3 Tom		1.033-0.983		
15m/6	1.352-1.302	0x6 Tom		1.260-1.210	12m FRA	
18m/6	1.454-1.404				12x24	2.001-1.951
21m/6	1.497-1.447					

ICAP OIS Fix Menu <ICAPOISFIX01>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANG

16:16 30DEC10 ICAP UK69580 ICAPEUROBASIS

EUR Basis Swaps (as 2 Swaps)

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These are indicative mids priced out of a spot starting date

All prices are Euribor vs Euribor quoted Bond Basis

	3M vs 6M	1M vs 3M	1M vs 6M	6M vs 12M	3M vs 12M
1YR	19.0	17.5	36.5	21.3	40.2
2YR	18.2	17.3	35.5	18.5	36.7
3YR	17.7	17.1	34.7	17.1	34.8
4YR	17.1	16.8	34.0	15.5	32.6
5YR	16.6	16.5	33.1	14.0	30.6
6YR	15.9	16.2	32.1	12.9	28.9
7YR	15.2	15.9	31.1	12.1	27.3
8YR	14.5	15.5	30.0	11.4	25.9
9YR	13.8	15.2	29.0	10.9	24.7
10YR	13.1	14.9	28.1	10.4	23.6
11YR	12.4	14.5	26.9	10.1	22.5
12YR	11.7	14.1	25.8	9.7	21.4
15YR	10.0	13.1	23.1	9.0	19.0
20YR	8.3	11.5	19.8	8.4	16.7
25YR	7.3	10.2	17.5	8.0	15.4
30YR	6.7	9.4	16.0	7.8	14.4
40YR	5.7				
50YR	5.2				

All ICAP Euro pages to close at 5.30pm on 30th Dec

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Source: Reuters

3: The Modern No Arbitrage Multiple-Curve Framework

Multiple curves construction: market data [1]

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16:52 30/12/10

	EUR CLOSED	USD CLOSED	FWDS CLOSED	GBP CLOSED	JPY CLOSED	CHF CLOSED	
ON	0.28/0.38	0.35/0.55	-0.040/-0.010	0.50/0.65		0.00/0.25	ON
TN	0.50/1.00	0.50/1.00	1.150/ 1.350	0.50/1.00	-0.01/0.14	0.10/0.35	TN
SN	0.40/0.50	0.30/0.50	0.030/ 0.060	0.50/0.75	-0.01/0.14	0.01/0.26	SN
SW	0.53/0.63	0.40/0.65	0.10/ 0.25	0.65/0.90	0.05/0.20	0.02/0.22	SW
2W	0.56/0.66	0.40/0.65	0.05/ 0.25	0.70/0.95	0.05/0.20	0.03/0.23	2W
3W	0.62/0.72	0.40/0.65	-0.35/ -0.10	0.75/1.00	0.05/0.20	0.04/0.24	3W
1M	0.70/0.80	0.55/0.75	-0.70/ -0.40	0.80/1.05	0.15/0.35	0.08/0.28	1M
2M	0.82/0.92	0.60/0.80	-2.30/ -1.90	0.85/1.10	0.20/0.40	0.18/0.38	2M
3M	0.92/1.02	0.65/0.85	-4.20/ -3.70	0.95/1.20	0.30/0.50	0.25/0.45	3M
4M	0.97/1.07	0.70/0.90	-6.30/ -5.55	1.00/1.25	0.30/0.55	0.31/0.51	4M
5M	1.05/1.15	0.74/0.94	-8.60/ -7.85	1.10/1.35	0.35/0.60	0.36/0.56	5M
6M	1.14/1.24	0.80/1.00	-10.80/ -9.80	1.20/1.45	0.40/0.65	0.40/0.60	6M
7M	1.19/1.29	0.84/1.04	-13.60/-12.10	1.25/1.50	0.45/0.70	0.45/0.65	7M
8M	1.23/1.33	0.88/1.08	-16.20/-14.70	1.30/1.55	0.45/0.70	0.49/0.69	8M
9M	1.28/1.38	0.94/1.14	-17.10/-15.10	1.35/1.60	0.45/0.70	0.53/0.73	9M
10M	1.33/1.43	0.98/1.18	-18.90/-16.40	1.40/1.65	0.45/0.70	0.57/0.77	10M
11M	1.37/1.47	1.01/1.21	-20.90/-18.40	1.45/1.70	0.45/0.70	0.60/0.80	11M
1Y	1.42/1.52	1.05/1.25	-22.50/-19.50	1.55/1.80	0.50/0.75	0.63/0.83	1Y
15M	1.60/1.85	1.14/1.44	-29.00/-23.00	1.64/2.14	0.49/0.74	0.66/1.16	15M
18M	1.64/1.89	1.23/1.53	-30.00/-23.00	1.74/2.24	0.50/1.00	0.70/1.20	18M
21M	1.70/1.95	1.34/1.64	-32.00/-24.00	1.87/2.37	0.49/0.99	0.76/1.26	21M
2Y	1.86/2.11	1.46/1.76	-34.00/-24.00	2.13/2.63	0.49/0.99	0.89/1.39	2Y
3Y	2.20/2.50	1.96/2.36	-10.00/ 10.00	2.62/3.12	0.53/1.03	1.21/1.71	3Y
4Y	2.52/2.82	2.46/2.86	55.00/ 85.00	3.03/3.53	0.58/1.08	1.51/2.01	4Y
5Y	2.80/3.10	2.91/3.31	150.00/190.00	3.38/3.88	0.67/1.17	1.78/2.28	5Y
7Y	3.30/3.70	3.67/4.17	CENTRALBANKNEWS	3.96/4.46	0.96/1.46	2.23/2.73	7Y
10Y	3.74/4.14	4.31/4.81	EZBnext13/01/11	4.47/4.97	1.34/1.84	2.61/3.11	10Y
12Y	3.95/4.35	4.58/5.08	1.00%v.07/05/09	4.70/5.20	1.56/2.06	2.77/3.27	12Y
15Y	4.14/4.54	4.85/5.35	FOMCnxt26/01/11	4.87/5.37	1.82/2.32	2.87/3.37	15Y
20Y	4.19/4.59	5.06/5.56	0-.25%v16/12/08	4.93/5.43	2.07/2.57	2.79/3.29	20Y
25Y	4.04/4.44	5.09/5.59	BoEnext13/01/11	4.89/5.39	2.14/2.64	2.72/3.22	25Y
30Y	3.82/4.22	5.14/5.64	0.50%v.05/03/09	4.81/5.31	2.15/2.65	2.62/3.12	30Y

Source: Reuters

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves construction: interpretation of market quotes

We assume that:

- the OTC inter-dialer market is fully collateralized under an ideal CSA with daily margination and overnight margination rate;

- the market uses coherently an overnight discounting curve.

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves construction: bootstrapping results

Notice that pricing formulas depend on (generalised) spot/FRA rates and thus they allow the bootstrapping of a FRA rate yield curve directly, without recurring to a zero coupon bond curve.
The familiar zero coupon bond and zero rate yield curves can be obtained from the FRA rate curve using the standard definitions

$$\begin{aligned}\tilde{F}_{x,i}(t) &:= \frac{1}{\tau_x(T_{i-1},T_i)}\left[\frac{P_x(t;T_{i-1})}{P_x(t;T_i)}-1\right],\\ P_x(t;T_i) &:= \exp\left[-z_x(t,T_i)\tau_z(t,T_i)\right].\end{aligned}$$

The bootstrapping terminates with the construction of the set of yield curves

$$\begin{aligned}&\mathcal{C}_x^F(t_0) := \left\{T \longrightarrow \tilde{F}_x(t_0;T,T+\Delta_x T),\ T \geq t_0\right\},\ \text{FRA rate curve},\\ &\mathcal{C}_x^z(t_0) := \left\{T \longrightarrow z_x(t_0,T),\ T \geq t_0\right\},\ \text{zero rate curve},\\ &\mathcal{C}_x^P(t_0) := \left\{T \longrightarrow P_x(t_0;T_i),\ T \geq t_0\right\},\ \text{zero coupon bond curve},\\ &x = \{on, 1M, 3M, 6M, 12M, ...\}.\end{aligned}$$

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves construction: bootstrapping formulas [1]

The yield curve bootstrapping formulas are just the (modern) pricing formulas discussed before, applied to the plain vanilla instruments quoted on the market and selected as bootstrapping instruments.

Instrument	**Quotation**	**Pricing formula**
Deposits	Spot rate	$L_x(t, T_i)$
FRAs	(Market) FRA rate	$\tilde{R}^{\text{FRA}}_{x,\text{Mkt}}(t; \mathbf{T}) = \dfrac{\left[1 + \tau_x(T_{i-1}, T_i)\tilde{F}_{x,i}(t)\right] e^{C^{\text{FRA}}_x(t; T_{i-1})} - 1}{\tau_x(T_{i-1}, T_i)}$
Futures	Futures price	$\textbf{Futures}(t; \mathbf{T}) = N\left\{1 - \left[\tilde{F}_{x,i}(t) + C^{\text{Fut}}_x(t, T_{i-1})\right]\right\}$
OIS	OIS rate	$R^{\text{OIS}}_d(t; \mathbf{T}) = \dfrac{P_d(t; T_0) - P_d(t; T_n)}{A_d(t; \mathbf{T})}$
Swaps	Swap rate	$\tilde{R}^{\text{Swap}}_x(t; \mathbf{T}, \mathbf{S}) = \dfrac{\sum_{j=1}^{m} P_d(t, T_j)\tilde{F}_{x,j}(t)\tau_x(T_{j-1}, T_j)}{A_d(t, \mathbf{S})}$
Basis Swaps	Basis swap rate	$\Delta(t; \boldsymbol{T}_x, \boldsymbol{T}_y, \boldsymbol{S}, \omega) = \tilde{R}^{\text{Swap}}_x(t; \boldsymbol{T}_x, \boldsymbol{S}) - \tilde{R}^{\text{Swap}}_y(t; \boldsymbol{T}_y, \boldsymbol{S})$

3: The Modern No Arbitrage Multiple-Curve Framework

Multiple curves construction: bootstrapping formulas [2]

In particular, suppose that:

- *T = [T_0, T_1,…, T_n]* be the time grid of the market data selected as bootstrapping instruments (pillars),
- $R^{\mathrm{mkt}}(T_0, T_i)$ is the market rate quoted for the bootstrapping instrument associated to pillar *I*,
- We have already bootstrapped the yield curve until pillar *i-1* and we want to compute the curve at pillar *i*.

Then, the bootstrapping algorithm proceeds as follows, for each typology of bootstrapping instruments:

- **Deposits**: $L_x(T_0, T_i)$,

$$P_x(T_0, T_i) = \frac{1}{1 + L_x(T_0, T_i)\tau_x(T_0, T_i)}$$

- **FRA** $\tilde{F}_{x,i}(t) = \frac{\left[1 + \tilde{R}_x^{\mathrm{FRA}}(T_0; T_{i-1}, T_i)\tau_x(T_{i-1}, T_i)\right] e^{-C_x^{\mathrm{FRA}}(T_0;T_{i-1})} - 1}{\tau_x(T_{i-1}, T_i)} \simeq \tilde{R}_x^{\mathrm{FRA}}(T_0; T_{i-1}, T_i),$

$$P_x(T_0, T_i) = \frac{P_x(T_0, T_{i-1})e^{-C_x^{\mathrm{FRA}}(T_0;T_{i-1})}}{1 + \tilde{R}_x^{\mathrm{FRA}}(T_0; T_{i-1}, T_i)\tau_x(T_{i-1}, T_i)} \simeq \frac{P_x(T_0, T_{i-1})}{1 + \tilde{R}_x^{\mathrm{FRA}}(T_0; T_{i-1}, T_i)\tau_x(T_{i-1}, T_i)}$$

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves construction: bootstrapping formulas [3]

o **Futures**:

$$\tilde{F}_{x,i}(t) = 1 - \frac{\mathbf{Futures}(t;\mathbf{T})}{100} - C_x^{\mathrm{Fut}}(t, T_{i-1}),$$

$$P_x(T_0, T_i) = \frac{P_x(T_0, T_{i-1})}{1 + \left[1 - \frac{\mathbf{Futures}(t;\mathbf{T})}{100} - C_x^{\mathrm{Fut}}(t, T_{i-1})\right] \tau_x(T_{i-1}, T_i)}.$$

o **OIS**:

$$F_{d,i}(T_0) = \frac{1}{\tau_R(T_{i-1}, T_i)} \left\{ \frac{P_d(T_0; T_{i-1}) \left[1 + R_i^{\mathrm{OIS}}(T_0)\tau_R(T_{i-1}, T_i)\right]}{\left[R_{i-1}^{\mathrm{OIS}}(T_0) - R_i^{\mathrm{OIS}}(T_0)\right] A_{d,i-1}(T_0) + P_d(T_0; T_{i-1})} - 1 \right\},$$

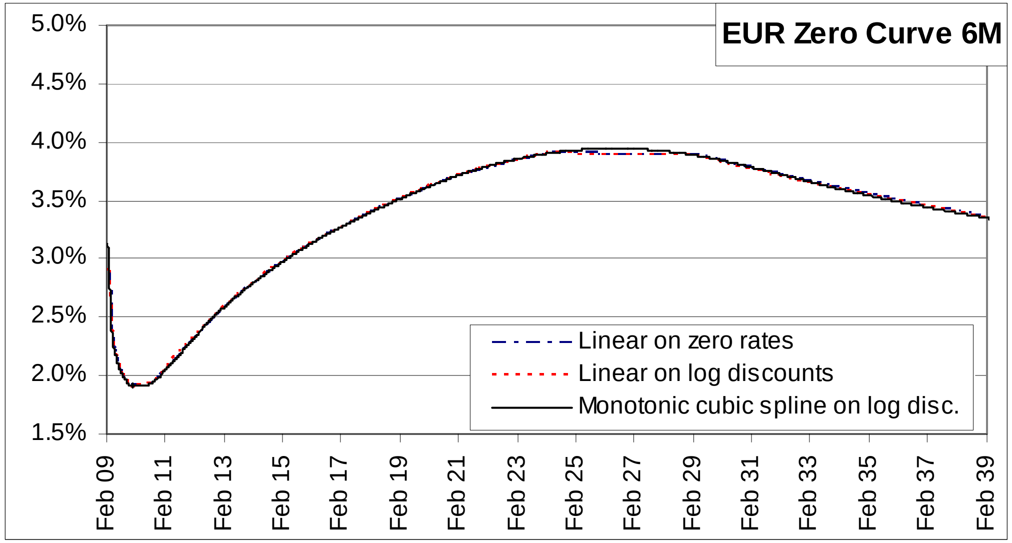
$$P_d(T_0; T_i) = \frac{\left[R_{i-1}^{\mathrm{OIS}}(T_0) - R_i^{\mathrm{OIS}}(T_0)\right] A_{d,i-1}(T_0) + P_d(T_0; T_{i-1})}{1 + R_i^{\mathrm{OIS}}(T_0)\tau_R(T_{i-1}, T_i)}.$$

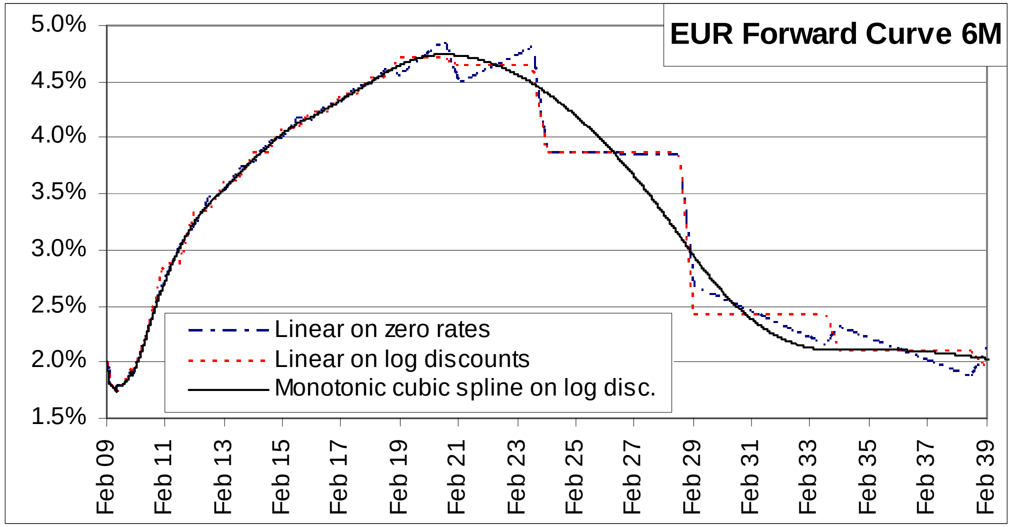
o **Swap:**

$$\tilde{F}_{x,i}(T_0) = \frac{\tilde{R}_{x,i}^{\mathrm{Swap}}(T_0)A_{d,i}(T_0) - \tilde{R}_{x,i-1}^{\mathrm{Swap}}(T_0)A_{d,i-1}(T_0)}{P_d(T_0, T_i)\tau_x(T_{i-1}, T_i)},$$

$$P_x(T_0, T_i) = \frac{P_d(T_0, T_i)P_x(T_0, T_{i-1})}{\tilde{R}_{x,i}^{\mathrm{Swap}}(T_0)A_{d,i}(T_0) - \tilde{R}_{x,i-1}^{\mathrm{Swap}}(T_0)A_{d,i-1}(T_0) + P_d(T_0, T_i)}$$

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves construction: interpolation





Examples of “bad” (but very popular…) interpolation schemes.

Upper panel: zero rate curves (Euribor6M) bootstrapped using different interpolation schemes display similar smooth behaviours. Closer inspection reveals non differentiable points (discontinuous first derivative) at the interpolation sites (the larger the gap between two consecutive quotes, the larger may be the discontinuity)

Lower panel: forward rate curves (daily forwards with 6M tenor) reveal different non-smooth behaviours, with ugly oscillations larger than 100bp. The monotonic cubic spline interpolation on log discounts (continuous black line) is clearly the smoothest choice.

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves, multiple deltas, multiple hedging [1]

1. Given any portfolio of interest rate derivatives with price $\Pi\left(t, \boldsymbol{T}, \boldsymbol{R}^{mkt}\right)$, compute the delta risk with respect to all curves $\boldsymbol{C} = \left\{C_d, C_{f_1}, ..., C_{f_n}\right\}$ as

$$\Delta^{\pi}\left(t, \boldsymbol{T}, \boldsymbol{R}^{mkt}\right) = \sum_{k=1}^{N_C} \Delta^{\pi}\left(t, \boldsymbol{T}, \boldsymbol{R}_k^{mkt}\right) = \sum_{k=1}^{N_C} \sum_{j=1}^{N_k^R} \frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}_k^{mkt}\right)}{\partial R_{k,j}^{mkt}},$$

where N_C is the number of yield curves involved and $\boldsymbol{R}_k^{mkt}$ is the vector of N_k^R bootstrapping instruments quotes (yield curve pillars) associated with yield curve $\mathcal{C}_k$.

2. Take properly into account all the delta components due to multiple curve bootstrapping: in particular, the curves $\left\{C_{f_1}, ..., C_{f_n}\right\}$ depend directly on their corresponding bootstrapping instruments with tenor *f*, but also indirectly on the discounting curve,

$$\Delta^{\pi}\left(t, \boldsymbol{T}, \boldsymbol{R}_d^{mkt}\right) = \sum_{j=1}^{N_d^R} \sum_{\alpha=1}^{N_d^Z} \frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}_d^{mkt}\right)}{\partial z_\alpha^d} \frac{\partial z_\alpha^d}{\partial R_{d,j}^{mkt}},$$

$$\Delta^{\pi}\left(t, \boldsymbol{T}, \boldsymbol{R}_f^{mkt}\right) = \sum_{j=1}^{N_f^R} \sum_{\alpha=1}^{N_f^Z} \frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}_f^{mkt}\right)}{\partial z_\alpha^f} \frac{\partial z_\alpha^f}{\partial R_{f,j}^{mkt}} + \sum_{j=1}^{N_d^R} \sum_{\alpha=1}^{N_f^Z} \frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}_f^{mkt}\right)}{\partial z_\alpha^f} \frac{\partial z_\alpha^f}{\partial R_{d,j}^{mkt}},$$

"One price, two curves, three deltas" (see M. Henrard, 2009)

where $\boldsymbol{z}^f$ is the vector of N_f^Z zero rates pillars in the zero rate curve $\mathcal{C}_f$.

3: The Modern No Arbitrage Multiple-Curve Framework
Multiple curves, multiple deltas, multiple hedging [2]

3. Possibly, aggregate the delta sensitivity on the selected subset *H* of the most liquid market instruments used for hedging $\boldsymbol{R}^H = \left\{ R_1^H, ..., R_{N_H}^H \right\}$ (hedging instruments);

$$\Delta^\pi\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right) \simeq \sum_{j=1}^{N_H} \frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right)}{\partial R_j^H},$$

4. Calculate hedge ratios: $h_j\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right) = \frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right)}{\partial R_j^H} \Big/ \delta_j^H(t),$

$$\delta_j^H\left(t, T_j, R_j^H\right) = \frac{\partial \pi_j^H\left(t, T_j, R_j^H\right)}{\partial R_j^H}.$$

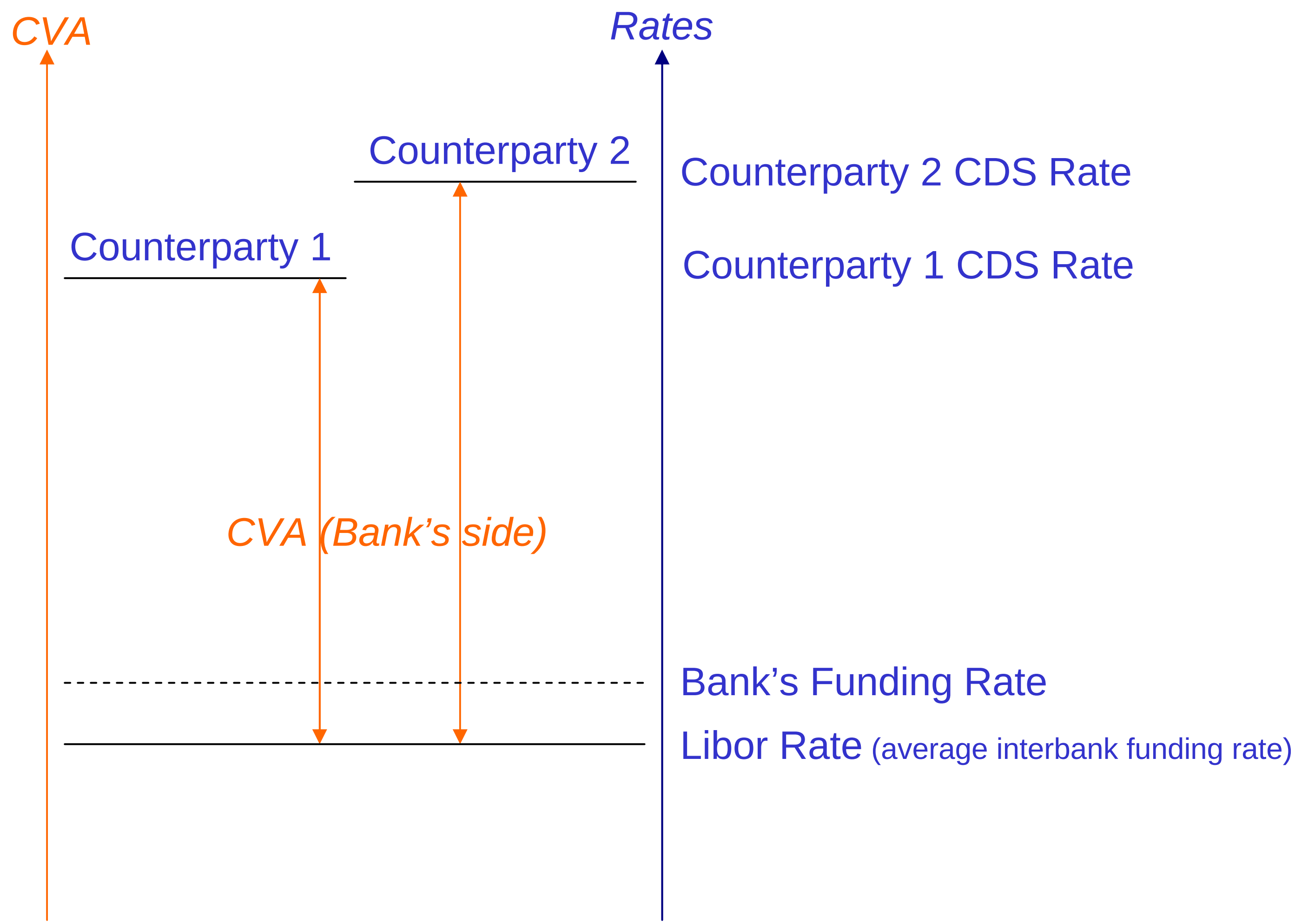
where $\pi_j^H(t)$ is the market price (unit nominal) of the corresponding hedging instrument, such that the hedged portfolio has zero delta

$$\Pi^{tot}\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right) = \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right) - \sum_{j=1}^{N_H} h_j(t) \pi_j^H\left(t, T_j, R_j^H\right),$$

$$\Delta^{tot}\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right) \simeq \sum_{k=1}^{N_H} \left[\frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right)}{\partial R_k^H} - \sum_{j=1}^{N_H} h_j(t) \frac{\partial \pi_j^H\left(t, T_j, R_j^H\right)}{\partial R_k^H} \right]$$

$$= \sum_{k=1}^{N_H} \left[\frac{\partial \Pi\left(t, \boldsymbol{T}, \boldsymbol{R}^H\right)}{\partial R_k^H} - h_k(t) \delta_k^H(t) \right] = 0,$$

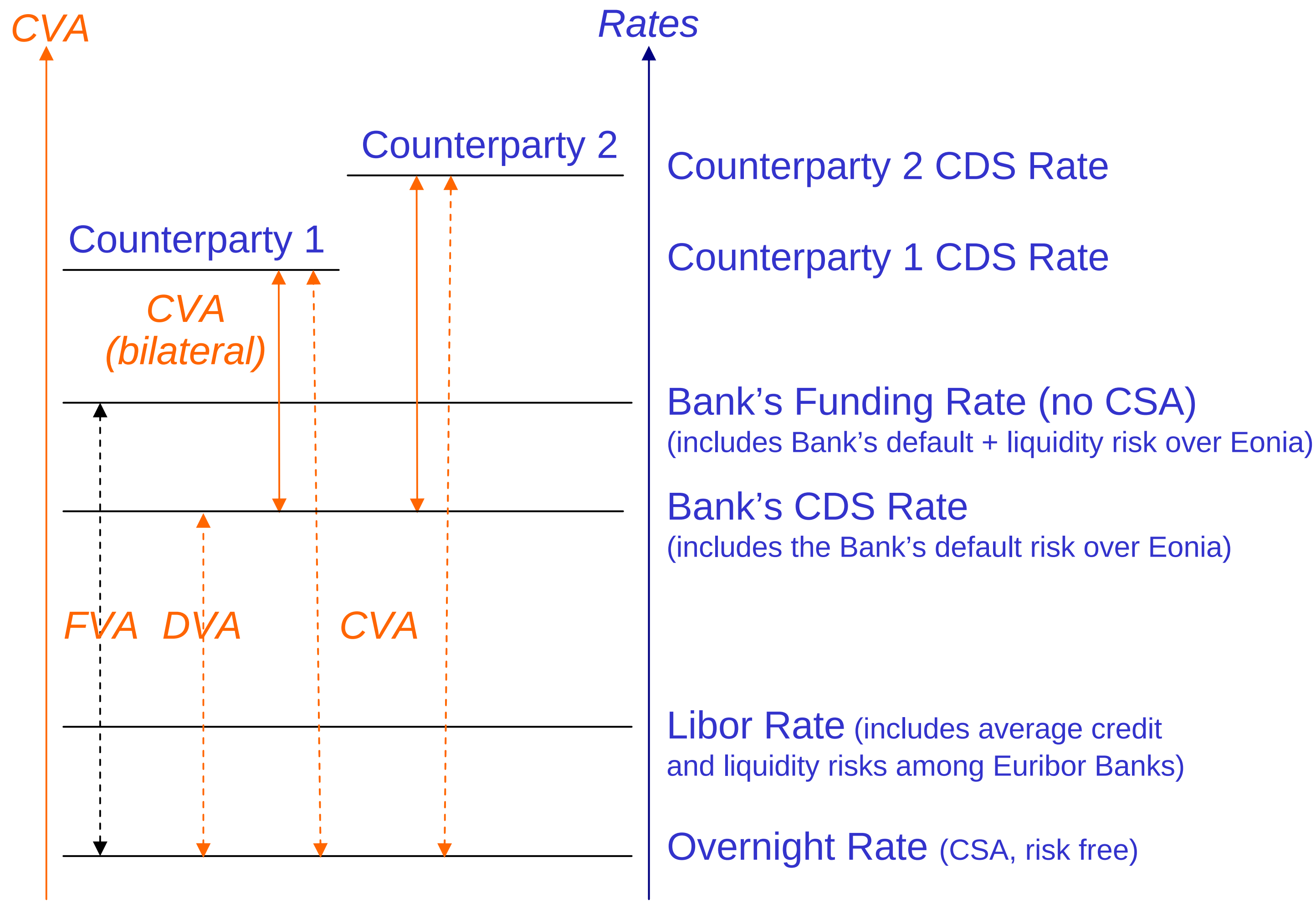
3: The Modern No Arbitrage Multiple-Curve Framework
Funding issues: CVA/DVA/FVA puzzle [1]



Classical pre-credit crunch Libor discounting for interbank counterparties
+ CVA for non-interbank counterparties.

3: The Modern No Arbitrage Multiple-Curve Framework

Funding issues: CVA/DVA/FVA puzzle [2]



Modern post-credit crunch CSA Discounting + CVA + DVA.

Problem of overlapping DVA vs FVA.

3: The Modern No Arbitrage Multiple-Curve Framework
Funding issues: CVA/DVA/FVA puzzle [3]

Credit Value Adjustment (CVA), Debt Value Adjustment (DVA) and Funding Value Adjustment (FVA) are presently the main issue in the modern interest rate market
A consistent pricing framework is still under development.

See e.g.

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3: The Modern No Arbitrage Multiple-Curve Framework

The role of quants

CSA-discounting is a typical complex problem in which a simple no-arbitrage pricing issue generates many consequences that propagate all around. In such a situation quant people play a critical role, being called to introduce financial modelling into other areas of the bank, traditionally free of pricing issues. They also have the chance both to learn on the job how the banks actually work and to show that they are not just technicians addicted for math and computers.

Key actions

- In order not to be arbitraged out, each institution must adopt CSA discounting for collateralized deals, at least for asset classes where the market indicates that most of the banks have switched their pricing to the new method.
- In times of rising funding costs, no bank can afford to ignore its funding costs. The economic valuation of uncollateralized deals should be linked to the actual funding spread of each bank. A business model which does not allow to fully price in its actual cost of funding is not feasible in the long-term.
- CSA or funding related valuation is not a pure playground for quants, but rather a topic that evokes questions about transfer pricing, steering of risk and, most importantly, the business model of each bank.

4: Conclusions

1. We have reviewed the changes in the interest rate market across the credit crunch

2. We have shown how to build a modern, self-consistent interest rate market framework, and in particular we have revisited:
   - the general pricing formula including funding and collateral
   - The modern pricing formulas of vanilla linear instruments
   - the modern multiple-curve bootstrapping

3. We have addressed some important issues connected to the switch towards CSA discounting, in particular the CVA/DVA/FVA puzzle.

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- L. B. G. Andersen, V. V. Piterbarg, "*Interest Rate Modeling",* Atlantic Financial Press, 2010.

Websites:

- Euribor, Eonia, Eurepo official website: *http://www.euribor.org*
- Libor official website: *http://www.bbalibor.com*

Regulators:

- International Accounting Standards Board (IASB), International Financial Reporting Standards (IFRS) 13 - Fair Value Measurement, www.ifrs.org

5: Main References:
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I. Euribor ACI – The Financial Markets Association, "€onia Swap Index", Nov. 2009, http://www.euribor.org
II. Bank for International Settlements, "*International banking and financial market developments*", Mar. 2008 Quarterly Review, *http://www.bis.org/publ/qtrpdf/r_qt0803.htm.*
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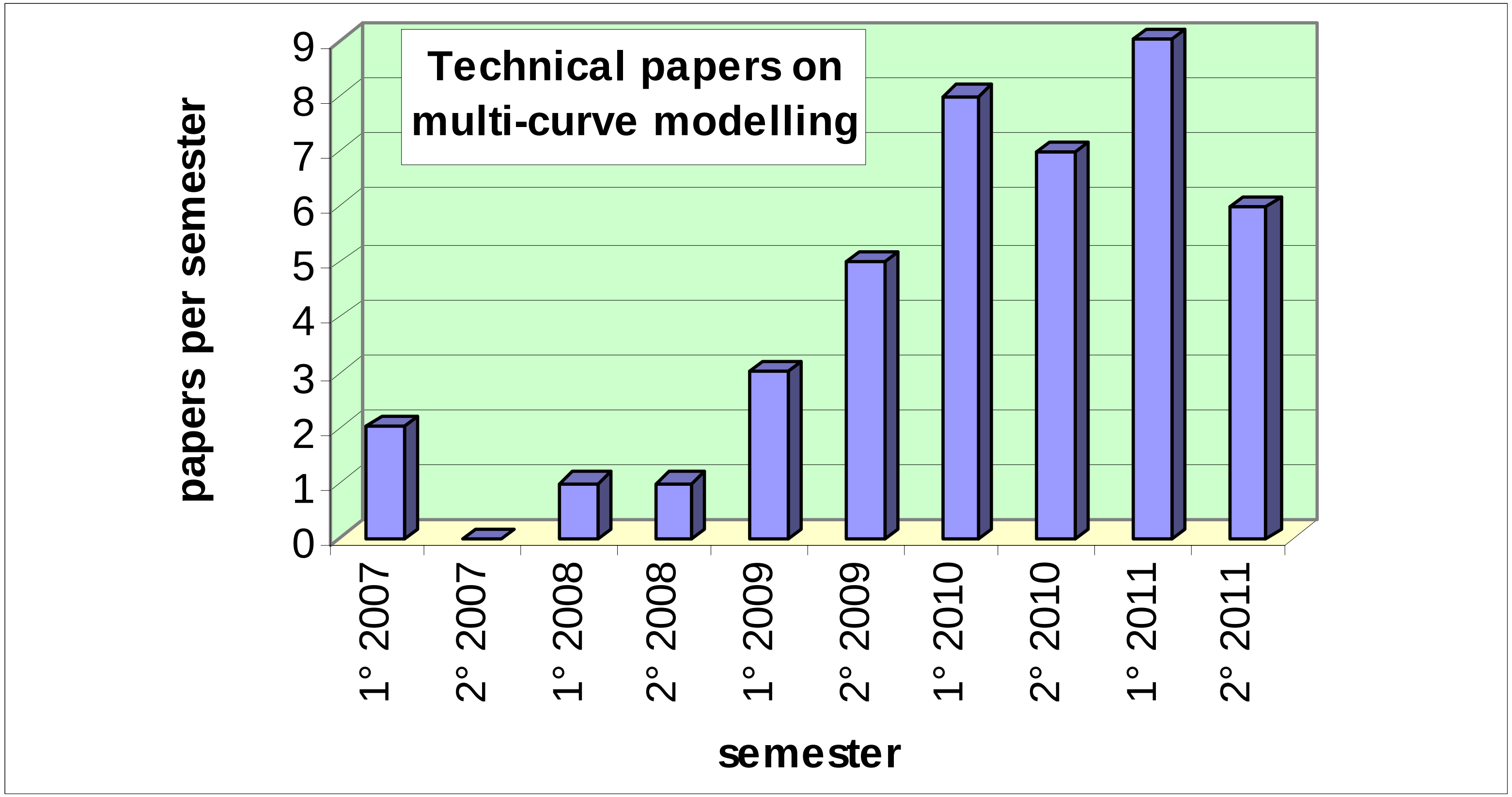

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