The SABR Model

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The SABR model is used to model a forward Libor rate, a forward swap rate, a forward index price, or any other forward rate. It is an extension of Black's model and of the CEV model. The model is not a pure option pricing model—it is a stochastic volatility model. But unlike other stochastic volatility models such as the Heston model, the model does not produce option prices directly. Rather, it produces an estimate of the implied volatility curve, which is subsequently used as an input in Black's model to price swaptions, caps, and other interest rate derivatives.

1 Process for the Forward Rate

The SABR model of Hagan *et al.* [2] is described by the following 3 equations

$$
\begin{aligned}
df_t &= \alpha_t f_t^{\beta} dW_t^1 \\
d\alpha_t &= v\alpha_t dW_t^2 \\
E\left[dW_t^1 dW_t^2\right] &= \rho dt
\end{aligned}
\tag{1}
$$

with initial values f_0 and $\alpha = \alpha_0$. In these equations, f_t is the forward rate, α_t is the volatility, and W_t^1 and W_t^2 are correlated Brownian motions, with correlation ρ. The parameters are

- α the initial variance

- v the volatility of variance

- β the exponent for the forward rate

- ρ the correlation between the Brownian motions.

The case $\beta = 0$ produces the stochastic normal model, $\beta = 1$ produces the stochastic lognormal model, and $\beta = \frac{1}{2}$ produces the stochastic CIR model.

2 SABR Implied Volatility and Option Prices

The prices of European call options in the SABR model are given by Black's model. For a current forward rate f, strike K, and implied volatility σ_B the price of a European call option with maturity T is

$$
C_B(f, K, \sigma_B, T) = e^{-rT}\left[fN(d_1) - KN(d_2)\right]
\tag{2}
$$

with

$$d_{1,2} = \frac{\ln f/K \pm \frac{1}{2}\sigma_B^2 T}{\sigma_B \sqrt{T}}$$

and analogously for a European put. The volatility parameter σ_B is provided by the SABR model. With estimates of $\alpha, \beta, v,$ and ρ, the implied volatility is

$$\sigma_B(K, f) = \frac{\alpha \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\rho\beta v\alpha}{(FK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} v^2 \right] T \right\}}{(fK)^{(1-\beta)/2} \left[1 + \frac{(1-\beta)^2}{24} \ln^2 \frac{f}{K} + \frac{(1-\beta)^4}{1920} \ln^4 \frac{f}{K} \right]} \quad (3)$$

$$\times \frac{z}{\chi(z)}$$

$$z = \frac{v}{\alpha} (fK)^{(1-\beta)/2} \ln \frac{f}{K}$$

$$\chi(z) = \ln \left[\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right].$$

Once the parameters $\alpha, \beta, \rho,$ and v are estimated, the implied volatility σ_B is a function only of the forward price f and the strike K. Since the SABR model produces implied volatilities for a single maturity, the dependence of σ_B on T is not reflected in the notation $\sigma_B(K, f)$.

3 Estimating Parameters

The β parameter is estimated first, and is not very important in the model because the choice of β does not greatly affect the shape of the volatility curve. With β estimated, there are two possible choices for estimating the remaining parameters

- Estimate $\alpha, \rho,$ and v directly, or

- Estimate ρ and v directly, and infer α from $\rho, v,$ and the at-the-money volatility, σ_{ATM}.

3.1 Estimating β

From equation (3), the at-the-money volatility σ_{ATM} is obtained by setting $f = K$ in equation (3), which produces

$$\sigma_{ATM} = \sigma_B(f, f) = \frac{\alpha \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho\beta v\alpha}{f^{1-\beta}} + \frac{2-3\rho^2}{24} v^2 \right] T \right\}}{f^{1-\beta}}. \quad (4)$$

Taking logs produces

$$\ln \sigma_{ATM} \approx \ln \alpha - (1 - \beta) \ln f.$$

Hence, β can be estimated by a linear regression on a time series of logs of ATM volatilities and logs of forward rates. Alternatively, β can be chosen from prior beliefs about which model (stochastic normal, lognormal, or CIR) is appropriate. In practice, the choice of β has little effect on the resulting shape of the volatility curve produced by the SABR model, so the choice of β is not crucial. The choice of β, however, can affect the Greeks. Barlett [1] provides more accurate Greeks and shows that they are less sensitive to the choice of β. This is described in Section 5.3.

3.2   First Parameterization–Estimating $\alpha, \rho,$ and v

Once $\hat{\beta}$ is set, it remains to estimate $\alpha, \rho,$ and v. This can be accomplished by minimizing the errors between the model and market volatilities $\left\{\sigma_i^{\mathrm{mkt}}\right\}$ (from interest rate derivatives, for example) with identical maturity T. Hence, for example, we can use SSE, which produces

$$(\hat{\alpha}, \hat{\rho}, \hat{v}) = \underset{\alpha,\rho,v}{\arg\min} \sum_i \left\{\sigma_i^{\mathrm{mkt}} - \sigma_B(f_i, K_i; \alpha, \rho, v)\right\}^2. \qquad (5)$$

We then use α, β, ρ, v in equation (3) to obtain σ_B and plug σ_B into Black's formula (2) to get the call price. Other objective functions are of course possible, such as the one by West [5] that uses vega as weights. A free Matlab program for estimating the SABR parameters in this fashion is available at **www.Volopta.com**.

3.3   Second Parameterization–Estimating ρ and v

We can reduce the number of parameters to be estimated by using σ_{ATM} to obtain $\hat{\alpha}$ via equation (4), rather than estimating α directly. This means that we only need to estimate ρ and v, and obtain an estimate of α by inverting equation (4) and noting that α is the root of the cubic equation

$$\left[\frac{(1-\beta)^2 T}{24 f^{2-2\beta}}\right]\alpha^3 + \left[\frac{\rho\beta v T}{4 f^{1-\beta}}\right]\alpha^2 + \left[1 + \frac{2-3\rho^2}{24}v^2 T\right]\alpha - \sigma_{ATM}f^{1-\beta} = 0. \quad (6)$$

West [5] notes that it is possible for this cubic to have more than a single real root, and suggests selecting the smallest positive root in this case. It is relatively straightforward to estimate the parameters using this second parameterization. In our minimization algorithm, at every iteration we find α in terms of ρ and v by solving equation (6) for $\alpha = \alpha(\rho, v)$. Hence, for example, SSE from equation (5) becomes

$$(\hat{\alpha}, \hat{\rho}, \hat{v}) = \underset{\alpha,\rho,v}{\arg\min} \sum_i \left\{\sigma_i^{\mathrm{mkt}} - \sigma_B(f_i, K_i; \alpha(\rho, v), \rho, v)\right\}^2. \qquad (7)$$

This estimation will take more time to converge. Indeed, at every iteration step, the minimization algorithm produces ρ and v, but it must use a root-finding algorithm to obtain α from equation (6) that uses the parameters β, ρ, v

as inputs along with f, K, σ_{ATM}, and T. The three parameter values ρ, v, and $\alpha = \alpha(\rho, v)$ are then plugged into equation (3) to produce σ_B, which is used in the objective function (7). The value of the objective function is compared to the tolerance level (or other convergence criterion) and the algorithm moves to the next iteration. A free Matlab program for estimating the SABR parameters under this parameterization scheme is available at **www.Volopta.com**.

4   Illustration

We illustrate the SABR model under both parameterizations by reproducing Figure 3.3 of Hagan *et al* [2]. We use $\beta = 0.5$ and fit the SABR model using both estimation approaches. This appears in Figure 1 below.
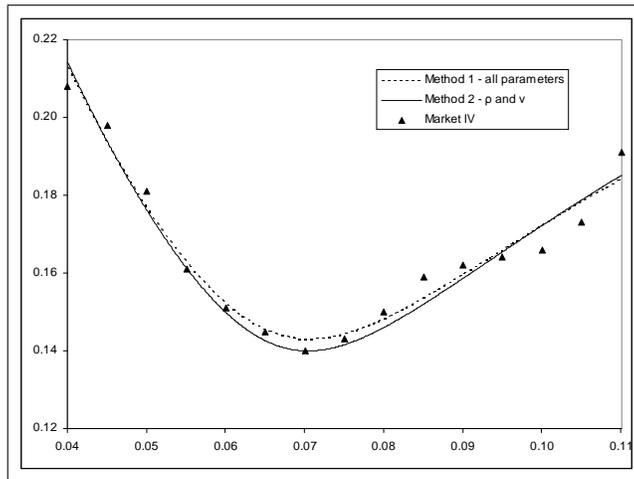

Figure 1.   Fitted SABR volatilities under both estimation
methods, $\beta = 0.5$

The figure illustrates that the choice of estimation has little effect, and that both methods produce a set of implied volatilities that fit the market volatilities reasonably well. The error sum of squares (SSE) from the first method is $SSE_1 = 2.33 \times 10^{-4}$, which is slightly larger than that from the second method, $SSE_2 = 2.74 \times 10^{-4}$. The parameter estimates obtained under both methods are presented in Table 1. The sets of parameters are very similar.

Table 1.   Parameter Estimates

Parameter	Method 1	Method 2
α	0.037561	0.036698
β	0.5	0.5
ρ	0.100044	0.098252
v	0.573296	0.599714

Free Matlab code for parameter estimation under both methods is available at **www.Volopta.com**.

4.1 The Backbone

Given values of α, β, ρ and v, we can vary the value of f and trace out the ATM volatility $\sigma_B(f, f)$ from Equation (4) to obtain the *backbone*. For a fixed value of f, if we plot the SABR volatilities $\sigma_B(K, f)$ from Equation (3), this will trace out the *smile* and *skew*. This is illustrated in Figure 2, which reproduces Figure 3.1 in Hagan *et al* [2], using the parameters in Table 1 estimated under Method 1 and using a maturity $T = 1$ year.
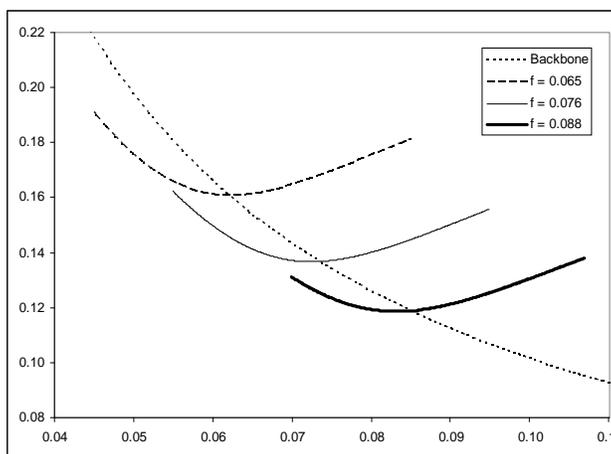
Figure 2. The backbone with its smiles and skews when
$$\beta = 0$$

Figure 3 plots the backbone and its smiles and skews, but using $\beta = 1$. This reproduces Figure 3.2 of Hagan *et al* [2]. Since the value of β is different, the parameter estimates in Table 1 are no longer valid. We must re-estimate the parameters wtih $\beta = 1$ instead of $\beta = 0$. The updated parameters, estimated using Method 1, are $\alpha = 0.13927$, $\rho = -0.06867$, and $v = 0.5778$.
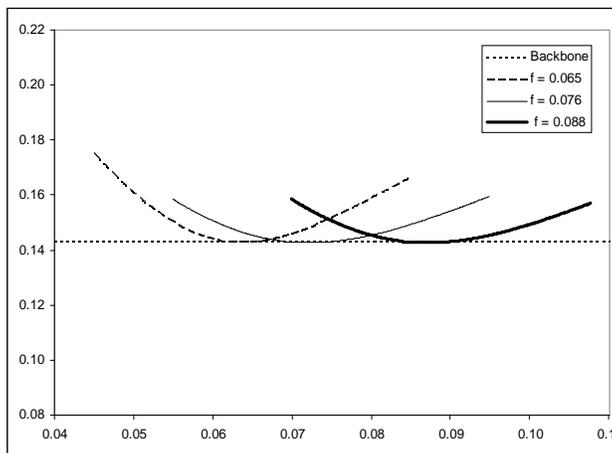
Figure 3. The backbone with its smiles and skews when
$\beta = 1$

5 Option Sensitivities

The Greeks from the SABR model resemble those from Black's model, but contain additional terms to reflect the fact that σ_B is not constant. This is explained by Hagan et al. [2], Lesniewski [3], and Barlett [1].

5.1 Vega

Vega, the sensitivity of the option price to volatility, α, is obtained by applying the chain rule on the call price from equation (2) and using equation (3)

$$\text{Vega} = \frac{\partial C_B}{\partial \sigma_B} \cdot \frac{\partial \sigma_B}{\partial \alpha} \tag{8}$$

In practice, finite differences are used to evaluate the derivative $\frac{\partial \sigma_B}{\partial \alpha}$, rather than obtaining this derivative analytically from equation (3).

5.2 Delta

Delta, the sensitivity of the option price to the forward rate, is dependent on the parameterization used. If the first parameterization is used then delta is the total derivative

$$\text{Delta} = \frac{\partial C_B}{\partial f} + \frac{\partial C_B}{\partial \sigma_B} \cdot \frac{\partial \sigma_B}{\partial f} \tag{9}$$

If, on the other hand, the second parameterization is used then delta is

$$\text{Delta} = \frac{\partial C_B}{\partial f} + \frac{\partial C_B}{\partial \sigma_B} \cdot \left(\frac{\partial \sigma_B}{\partial f} + \frac{\partial \alpha}{\partial f} \right)$$

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to reflect the fact that α is a function of f.

5.3 Barlett Updated Greeks

Bartlett [1] has proposed refinements of the Greeks in equations (8) and (9). In this section we explain the development of these updated Greeks.

5.3.1 Updated Delta

The SABR Delta in equation (9) is obtained by assuming a shift in the forward rate while keeping the value of α constant

$$
\begin{aligned}
f &\rightarrow f + \Delta f \\
\alpha &\rightarrow \alpha.
\end{aligned}
$$

Bartlett [1] explains that since α and f are correlated, a shift in f will likely be accompanied by a shift in α. Hence a more realistic scenario is

$$
\begin{aligned}
f &\rightarrow f + \Delta f \\
\alpha &\rightarrow \alpha + \delta_f \alpha.
\end{aligned}
$$

To calculate $\delta_f \alpha$ we use the the well-know result that the two correlated Brownian motions W_t^1 and W_t^2 from equation (1) can be expressed in terms of two *independent* Brownian motions W_t and Z_t by setting, for example, $dW_t^1 = dW_t$ and $dW_t^2 = \rho dW_t + \sqrt{1 - \rho^2} dZ_t$. Hence we can write the SABR model from equation (1) as

$$
\begin{aligned}
df_t &= \alpha_t f_t^\beta dW_t \tag{10} \\
d\alpha_t &= v\alpha_t \left(\rho dW_t + \sqrt{1 - \rho^2} dZ_t \right) \\
E\left[dW_t \left(\rho dW_t + \sqrt{1 - \rho^2} dZ_t \right) \right] &= \rho dt.
\end{aligned}
$$

This implies that the volatility process from equation (10) can be written as

$$
d\alpha_t = \frac{\rho v}{f_t^\beta} df_t + v\alpha_t \sqrt{1 - \rho^2} dZ_t.
$$

The instantaneous change in volatility, $d\alpha_t$, can now be expressed in two terms (1) the instantaneous change in the forward, df_t, and (2) the level of the volatility, α_t. The change in volatility due to a change in the forward is the first term

$$
\frac{d\alpha_t}{df_t} = \frac{\rho v}{f_t^\beta}.
$$

The SABR delta is updated by including the change in σ_B brought on by changes in α

$$
\begin{aligned}
\text{Updated Delta} &= \frac{\partial C_B}{\partial f} + \frac{\partial C_B}{\partial \sigma_B} \cdot \left(\frac{\partial \sigma_B}{\partial f} + \frac{\partial \sigma_B}{\partial \alpha} \frac{\partial \alpha}{\partial f} \right) \\
&= \frac{\partial C_B}{\partial f} + \frac{\partial C_B}{\partial \sigma_B} \cdot \left(\frac{\partial \sigma_B}{\partial f} + \frac{\partial \sigma_B}{\partial \alpha} \frac{\rho v}{f^\beta} \right).
\end{aligned}
$$

5.3.2 Updated Vega

Analogously to the SABR Delta, the SABR Vega in equation (8) is updated by assuming a shift in the volatility while keeping the value of f constant

$$
\begin{aligned}
f &\rightarrow f \\
\alpha &\rightarrow \alpha + \Delta\alpha.
\end{aligned}
$$

Bartlett [1] explains that a more realistic scenario is

$$
\begin{aligned}
f &\rightarrow f + \delta_\alpha f \\
\alpha &\rightarrow \alpha + \Delta\alpha.
\end{aligned}
$$

Turning to equation (1) again, the forward process can be written

$$
\begin{aligned}
df_t &= \alpha_t f_t^\beta \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t \right) \quad (11) \\
d\alpha_t &= v\alpha_t dW_t \\
E\left[dW_t \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t \right) \right] &= \rho dt.
\end{aligned}
$$

This implies that the forward process from equation (11) can be written as

$$
df_t = \frac{\rho f_t^\beta}{v} d\alpha_t + f_t^\beta \alpha_t \sqrt{1-\rho^2} dZ_t.
$$

The instantaneous change in volatility, df_t, can be expressed in two terms (1) the instantaneous change in the forward, $d\alpha_t$, and (2) the level of the volatility, α_t. The change in the forward due to a change in volatility is the first term

$$
\frac{df_t}{d\alpha_t} = \frac{\rho f_t^\beta}{v}.
$$

The SABR delta is updated by including the change in σ_B brought on by changes in α

$$
\text{Vega} = \frac{\partial C_B}{\partial \sigma_B} \cdot \left(\frac{\partial \sigma_B}{\partial \alpha} + \frac{\partial \sigma_B}{\partial f} \frac{\partial f}{\partial \alpha} \right).
$$

A free Matlab program for the updated Greeks is available at **www.Volopta.com**.

6 SABR Refinements

The original formula by Hagan *et al.* [2] in Equation (3) has been shown to break down when the strike is small and the maturity is long. In response, a number of researchers have sought to refine the implied volatility. One such refinement is summarized by Jan Obłój [4], so we state his results here. The implied volatility surface $\sigma(x, T)$ for log-moneyness $x = \log(F/K)$ and maturity T can be approximated as

$$
\sigma_B(x, T) \approx I_B^0(x) \left(1 + I_H^1(x) T \right). \tag{12}
$$

In this expression, we have

$$I_H^1(x) = \frac{(1-\beta)^2 \alpha^2}{24(fK)^{1-\beta}} + \frac{\rho v \alpha \beta}{4(fK)^{(1-\beta)/2}} + \frac{(2-3\rho^2)v^2}{24},$$

and four cases for $I_B^0(x)$.

Case 1: $x = 0$.

$$I^0(0) = \alpha K^{\beta-1}.$$

Case 2: $v = 0$.

$$I^0(x) = \frac{x\alpha(1-\beta)}{f^{1-\beta} - K^{1-\beta}}.$$

Case 3: $\beta = 1$.

$$I^0(x) = \frac{vx}{\ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)}$$

where $z = \frac{vx}{\alpha}$.

Case 4: $\beta < 1$.

$$I^0(x) = \frac{vx}{\ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)}$$

where $z = \frac{v(f^{1-\beta} - K^{1-\beta})}{\alpha(1-\beta)}$. As before, the SABR implied volatility $\sigma_B(x,T)$ is plugged into Black's formula in Equation (2), and the price of the call is obtained. A free Matlab program for estimating the SABR parameters under this refined scheme is available at **www.Volopta.com**.

References

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[5] West, G. (2005). "Calibration of the SABR Model in Illiquid Markets," *Applied Mathematical Finance*, Vol. 12, No. 4, pp. 371-385.