Putting Smiles Back to the Futures

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Abstract

This article reports a practical approach to extend the classical Gabillon model to allow explicit modeling of commodity futures smiles. The original Gabillon model is first extended with a deterministic shift to fit the term structure of futures prices. The smile information of individual futures is extracted from the futures options markets in terms of the implied marginal distributions. An algorithm based on the copula technique is then developed to reconstruct the joint distribution of the underlying futures prices that is consistent with both the term structure of volatilities and the marginal distributions of individual futures. Examples are provided to illustrate the calibration procedure and options pricing.

Keywords

Futures curve model, Volatility Smile, Gabillon model, Copula function, Commodity options

1 Introduction

Gabillon (1991) introduces a two-factor model for the term structure of futures curve in crude oil markets. The first factor is the spot price $S(t)$, which is assumed to follow a log-normal diffusion that mean-reverts to the second stochastic factor, the long-term price $L(t)$, which itself follows a log-normal diffusion. This leads to the following dynamics for the spot and long-term price

$$\frac{dS(t)}{S(t)} = \kappa \ln \frac{L(t)}{S(t)} dt + \sigma_S dW_S(t),$$

$$\frac{dL(t)}{L(t)} = \mu_L dt + \sigma_L dW_L(t),$$

where W_S and W_L are Brownian motions with instantaneous correlation ρ_{SL} defined as $dW_S(t)dW_L(t) = \rho_{SL}dt$. The model parameters μ_L, $\sigma_S > 0$, $\sigma_L > 0$, $\kappa > 0$ and $\rho_{SL} \in [-1,1]$ are assumed to be constants. Gabillon's analysis leads to the following formula for the forward price

$$F(t,T) = A(t,T)S(t)^{B(t,T)}L(t)^{1-B(t,T)},$$

where

$$A(t,T) = \exp\left\{\frac{1}{4\kappa}\left(\sigma_S^2 + \sigma_L^2 - 2\rho_{SL}\sigma_S\sigma_L\right) B(t,T)[1 - B(t,T)]\right\}$$

$$B(t,T) = e^{-\kappa(T-t)}.$$

In commodity markets, the futures price in the front-end of the futures curve is usually more volatile, while the back-end of the curve is less so. By introducing two factors to model both the short- and long-term effects, the Gabillon model can effectively capture the characteristics of futures price movements. No explicit modeling on the convenience yield is needed.[1] In practice, the model can offer excellent performance in modeling the various shapes of futures curve (e.g. contango/backwardation) and the term structure of volatilities (e.g. Samuelson effects) observed in a variety of commodity markets.

However, the Gabillon model ignores the effects of volatility smiles that are commonly observed in the options markets. Unlike the equity and forex markets, the commodity spot is usually not a liquid asset. The payoff of many commodity derivatives depends on multiple points on the futures curve. As a result, the value of commodity derivatives can be sensitive to the volatility smiles of the underlying futures prices. To be able to price and manage the risk of these products, we need a model that incorporates smile information into the correlation and term structure modeling of the futures curve.

Here we document a practical approach to extend the classical Gabillon model to allow explicit modeling of volatility smiles. The idea is to reconstruct the joint distribution of the underlying futures prices that is consistent with both the classical correlation structure and the market-implied marginal distributions of individual futures. Our approach exploits the copula technique, which is an industry standard often used to deal with basket derivatives. The approach can be easily applied to other multi-factor log-normal models.

We briefly review the Gabillon model (with deterministic-shift extension) and derive the forward-price representation in Section 2. To incorporate the market smile information, we first recover the marginal distributions of the underlying futures prices from market volatility smiles. We then apply the copula technique to "twist" the log-normal prices generated from the Gabillon model, such that their margins match the implied marginal distributions of the underlying futures. These techniques are addressed in

Section 3. In Section 4, we address the practical issues in calibration and surface construction. Section 5 provides examples to illustrate the pricing of path-dependent options. We conclude in Section 6.

2 Gabillon Model with Extension

In this section we consider a two-factor log-normal model for the commodity spot price, which can be viewed as an extension to the classical Gabillon (1991) model. We assume that the spot price process under the risk-neutral measure $\mathbb{Q}$ is defined by

$$S(t) = \exp\{f(t) + X(t)\}, \quad S(0) = S_0, \tag{1}$$

where $f(t)$ is a deterministic function and $X(t)$ is an Ornstein-Uhlenbeck process with a stochastic long-term mean $Y(t)$. The dynamics of $X(t)$ and $Y(t)$ are given, respectively, by

$$dX(t) = \kappa(t)[Y(t) - X(t)]dt + \sigma_S(t)dW_S(t), \quad X(0) = 0, \tag{2}$$

$$dY(t) = \sigma_L(t)dW_L(t), \quad Y(0) = 0, \tag{3}$$

where (W_s, W_l) is a two-dimensional Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ with a natural filtration and a risk-neutral measure $\mathbb{Q}$. The instantaneous correlation between W_s and W_L is assumed to be $\rho_{SL}(t)$ defined as

$$dW_S(t)dW_L(t) = \rho_{SL}(t)dt. \tag{4}$$

In the above model, the function $f(t)$ is deterministic and defined in a time interval $[0, T_{max}]$ with T_{max} a given time horizon sufficiently large for our interest. The other model parameters are in general time dependent, and $\kappa(t) \geq 0$, $\sigma_S(t) \geq 0$ and $\sigma_L(t) \geq 0$ and $\rho_{SL}(t) \in [-1, 1]$ for all $t \in [0, T_{max}]$.

Introducing the function f can provide flexility to model the seasonality of commodity price. Here we use f as a deterministic shift to exactly reproduce the market observed futures curve, which can be difficult to achieve in the original Gabillon model. This technique has been widely used in interest rate term structure modeling; see for example Brigo and Mercurio (2006) and references therein. It can be shown that, with constant parameters and a proper choice of $f(t)$,[2] the above model will reduce to the classical Gabillon model.

Valuation of futures contracts. We next proceed to value futures (forward) contracts. Let $F(t, T)$ denote the price at time t of a forward contract with maturity T. In the risk-neutral valuation framework, the forward price is equal to the conditional expectation of the future spot price under the risk-neutral measure. Formally we have

$$F(t, T) = \mathbb{E}_t^{\mathbb{Q}}[S(T)] = \mathbb{E}_t^{\mathbb{Q}}[\exp\{f(T) + X(T)\}]. \tag{5}$$

Since we assume deterministic interest rates, the futures price is equal to the forward price[3] see, for example, Duffie (1996). To evaluate the expectation, we first solve the equations (2) and (3). The solutions can be written as

$$X(T) = b(t, T)X(t) + [1 - b(t, T)]Y(t) + \int_t^T b(s, T)\sigma_S(s)dW_S(s) + \int_t^T [1 - b(s, T)]\sigma_L(s)dW_L(s),$$

$$Y(T) = Y(t) + \int_t^T \sigma_L(s)dW_L(s),$$

where

$$b(t, T) = \exp\left\{-\int_t^T \kappa(s)ds\right\}. \tag{6}$$

We see that $X(T)$ conditional on $(X(t), Y(t))$ is normally distributed with conditional expectation and variance:

$$\mathbb{E}_t^{\mathbb{Q}}[X(T)] = b(t, T)X(t) + [1 - b(t, T)]Y(t),$$

$$\mathrm{Var}_t^{\mathbb{Q}}[X(T)] = \int_t^T \sigma^2(s, T)ds,$$

where $\sigma(t, T)$ is the instantaneous volatility determined by

$$\sigma^2(t, T) = \sigma_S^2(t)b^2(t, T) + 2\rho_{SL}(t)\sigma_S(t)\sigma_L(t)b(t, T)[1 - b(t, T)] + \sigma_L^2(t)[1 - b(t, T)]^2. \tag{7}$$

Therefore, by equation (5), the forward price $F(t,T)$ follows a log-normal distribution and we can write

$$F(t, T) = \exp\left\{f(T) + b(t, T)X(t) + [1 - b(t, T)]Y(t) + \frac{1}{2}\int_t^T \sigma^2(s, T)ds\right\}. \tag{8}$$

Moreover, by assuming $X(0) = Y(0) = 0$ (without loss of generality), we can write the model implied initial forward curve as $F(0, T) = \exp\left\{f(T) + \frac{1}{2}\int_0^T \sigma^2(t, T)dt\right\}$. Now let $F_{mkt}(T)$ denote the current term structure of futures prices observed from the market. It is easy to verify that the above model can reproduce exactly the current futures curve $F_{mkt}(T)$ if we set the deterministic-shift function

$$f(T) = \ln(F_{mkt}(T)) - \frac{1}{2}\int_0^T \sigma^2(t, T)dt.$$

Dynamics of Forward Prices. To facilitate further analysis, it is convenient to represent the model in terms of the forward-price dynamics. Applying Ito's formula to (8) leads to the following dynamics

$$\frac{dF(t, T)}{F(t, T)} = b(t, T)\sigma_S(t)dW_S(t) + [1 - b(t, T)]\sigma_L(t)dW_L(t), \tag{9}$$

which confirms the martingale property of $F(t,T)$. In terms of the instantaneous volatility $\sigma(t,T)$ given in (7), the dynamics of $F(t,T)$ can be written as

$$\frac{dF(t, T)}{F(t, T)} = \sigma(t, T)dW(t),$$

where $W(t)$ is a standard Wiener process defined by

$$dW(t) = \sigma_S(t)\frac{b(t, T)}{\sigma(t, T)}dW_S(t) + \sigma_L(t)\frac{1 - b(t, T)}{\sigma(t, T)}dW_L(t).$$

Integrating (9) leads to the following representation for the forward price:

$$F(t, T) = F(0, T)\exp\left\{-\frac{1}{2}\int_0^t \sigma^2(s, T)ds + \int_0^t \sigma(s, T)dW(s)\right\}$$

with $F(0, T) = F_{mkt}(T)$ being the current forward curve. Moreover, we can write

$$F(t, T) = F(0, T)\exp\{\mu(t, T) + \nu(t, T)Z(t)\}, \tag{10}$$

where $Z(t) \sim N(0,1)$ is a normal random variable, and

$$\mu(t,T) = -\frac{1}{2}\int_0^t \sigma^2(s,T)ds \quad \text{and} \quad \nu(t,T) = \left(\int_0^t \sigma^2(s,T)ds\right)^{1/2}. \tag{11}$$

Conversely, we note that, for a given forward price $F(t,T)$ generated by the above model (1)–(4), the random variable

$$Z(t) := \frac{\ln(F(t,T)/F(0,T)) - \mu(t,T)}{\nu(t,T)} \sim N(0,1) \tag{12}$$

follows a standard normal distribution.

3 Futures Model with Smiles

The model discussed in the previous section generates futures prices that follow a log-normal distribution. This is in line with the classical multi-factor models, such as Schwartz (1997), Gibson and Schwartz (1990) and Schwartz and Smith (2000). In this section we describe a practical approach to model the effects of volatility smiles that are commonly observed in the markets.

3.1 Term Structure of Marginal Distributions

We first focus on the marginal distributions of the underlying forward (or futures) prices, which can be recovered from the market prices of vanilla options.

Let us consider a European call option on a forward contract with maturity T. The option is assumed to expire on $\tau \leq T$. The payoff of the option at expiration τ is $(\tilde{F}(\tau,T) - K)^+$, where we write $\tilde{F}$ instead of F to emphasize that we are dealing with a forward price with a market-implied distribution (which is not necessarily log-normal). In risk-neutral valuation, the option price (without discount) can be represented by

$$C(\tau,K) = \mathbb{E}_0^{\mathbb{Q}}[(\tilde{F}(\tau,T) - K)^+] = \int_K^{+\infty} (y-K)f(\tau,y)dy,$$

where $f(\tau,y)$ stands for the risk-neutral density of the underlying forward price $\tilde{F}(\tau,T)$ at expiration τ. Simple differentiation yields

$$\frac{\partial}{\partial K}C(\tau,K) = \int_{-\infty}^{K} f(\tau,y)dy - 1,$$

which leads to the following probability distribution function for the forward price

$$\psi(\tau,K) := \mathbb{P}(\tilde{F}(\tau,T) < K) = 1 + \frac{\partial}{\partial K}C(\tau,K). \tag{13}$$

This formula provides us a way to extract the marginal distribution of forward price from the non-discounted option prices contingent on the forward price.

Moreover, a direct application of the Black (1976) model yields that, in terms of implied volatility, the European call price (without discount) is given by

$$C(\tau,K) = F(0,T)\Phi(d_+) - K\Phi(d_-), \tag{14}$$

where $F(0,T)$ is the initial forward price, $\Phi(\cdot)$ is the cumulative normal distribution function, and

$$d_\pm(\tau,K) = \frac{\ln(F(0,T)/K)}{w(\tau,K)} \pm \frac{1}{2}w(\tau,K) \tag{15}$$

with $w(\tau,K) = \sqrt{\tau}\sigma_{imp}(\tau,K)$ being the (implied) total volatility for expiration τ and strike K. Combining (13), (14) and (15) we obtain the following representation for the marginal distribution of forward prices

$$\psi(\tau,K) = 1 - \Phi(d_-) + K\varphi(d_-)\frac{\partial}{\partial K}w(\tau,K), \tag{16}$$

where $\varphi(\cdot)$ is the standard normal density function.

Therefore, the term structure of marginal distributions for the underlying futures prices can be characterized by the surface $\psi(\tau,K)$, which can be recovered from the implied volatility surface $\sigma_{imp}(\tau,K)$ by using (15) and (16). In practice, it is convenient to represent the surface in terms of the forward-log moneyness $x(K) := \ln(K/F(0,T))$ instead of the strike K. In this case, the term structure can be defined as

$$\chi(\tau,x) := \mathbb{P}(\ln(\tilde{F}(\tau,T)/F(0,T)) < x) = \psi(\tau,F(0,T)e^x). \tag{17}$$

In order to price a derivative with a payoff depending on the entire futures curve, we need to obtain the joint distribution of the underlying forward prices. However, knowing the marginal distributions of its components is not sufficient to determine such a joint distribution. We need to model the co-dependence structure across different maturities. In the following section we discuss how copula technique can be used for this purpose.

3.2 Joint Distribution: A Copula Approach

We first briefly introduce the copula function. The idea is to transform random variables by their marginal distribution function to obtain uniform random variables that contain the same information. Precisely, let $g(x_1,\ldots,x_n)$ be an n-dimensional distribution function with margins $g_1(x_1),\ldots,g_n(x_n)$. Then a copula function $c(x_1,\ldots,x_n)$ can be defined by

$$c(x_1,\ldots,x_n) := g(g_1^{-1}(x_1),\ldots,g_n^{-1}(x_n)).$$

Moreover, for a given copula function $c(x_1,\ldots,x_n)$, we can define a joint distribution $g(x_1,\ldots,x_n)$ via

$$g(x_1,\ldots,x_n) := c(g_1(x_1),\ldots,g_n(x_n)),$$

such that its margins are given by $g_1(x_1),\ldots,g_n(x_n)$.

The copula function allows us to model the dependence of random variables by concentrating on the corresponding uniform variables. With a copula function, we can easily specify the joint distribution for given marginal distributions. This technique has been widely used to model basket credit derivatives; see, for example, Li (2000). It is also applied to construct basket volatility surface in Qu (2005). In what follows, we apply the copula technique to extract the information of co-dependence across different maturities from the classical log-normal framework, which is further incorporated with the implied marginal distributions to model futures smiles.

In the pricing engine, we first simulate the Monte Carlo paths from a log-normal model, such as the one described in Section 2. Let $F(t,T_1),\ldots,F(t,T_n)$ denote the simulated forward prices at time spot t. Using equation (12), we can transform them to the following normal random variables:

$$(Z_1(t), \ldots, Z_n(t)) \sim N(0, \Sigma)$$

with some correlation matrix Σ implicitly defined by the log-normal model. These normal random variables $Z_i(t)$ can be further transformed to the uniform random variables $U_i(t)$ by

$$U_i(t) = \Phi(Z_i(t)), \quad (i = 1, \ldots, n)$$

with $\Phi(\cdot)$ being the standard normal cumulative distribution function. It is easy to verify that the distribution function of the random vector $(U_1(t), \ldots, U_n(t))$ defines a Gaussian copula:

$$c(u_1, \ldots, u_n) := F_{U_1(t),\ldots,U_n(t)}(u_1, \ldots, u_n). \quad (18)$$

In our approach, this copula function will be used to specify the co-dependence of the forward prices $\tilde{F}(t, T_1), \ldots, \tilde{F}(t, T_n)$. Now fix t and consider the random variables

$$\tilde{Z}_i(t) = \frac{\ln(\tilde{F}(t, T_i)/F(0, T_i)) - \mu(t, T_i)}{\nu(t, T_i)}, \quad (i = 1, \ldots, n),$$

where $\mu(t, T)$ and $\nu(t, T)$ are defined in (11). Let us assume that the random variable $\tilde{Z}_i(t)$ has a distribution function given by $\tilde{\Phi}_i(z) := \mathbb{P}(\tilde{Z}_i(t) < z)$. Then, we can define the random vector $(\tilde{Z}_1(t), \ldots, \tilde{Z}_n(t))$ through the following transform

$$\tilde{Z}_i(t) = \tilde{\Phi}_i^{-1}(U_i(t)) = \tilde{\Phi}_i^{-1}(\Phi(Z_i(t))), \quad (i = 1, \ldots, n),$$

where one can verify that it has marginal distributions $\tilde{\Phi}_1(z), \ldots, \tilde{\Phi}_n(z)$ and shares the same copula function as $(Z_1(t), \ldots, Z_n(t))$ given by (18). Moreover, the "skewed" forward prices can be recovered by the following equation

$$\tilde{F}(t, T_i) = F(0, T_i) \exp\left\{\mu(t, T_i) + \nu(t, T_i)\tilde{Z}_i(t)\right\}, \quad (19)$$

whose margins match those implied from the market.

It remains to specify the distributions function $\tilde{\Phi}_1(z), \ldots, \tilde{\Phi}_n(z)$. Using the technique laid out in Section 3.1, we can recover the market-implied term structure of marginal distributions $\psi(\tau, K)$ (or $\chi(\tau, x)$ in terms of forward-log moneyness) for the underlying futures prices. Obviously, if the sampling time t happens to be the option expiration τ_i, the distribution function $\tilde{\Phi}_i(z)$ can be calculated directly from $\psi(\tau_i, K)$ (or $\chi(\tau_i, x)$) as

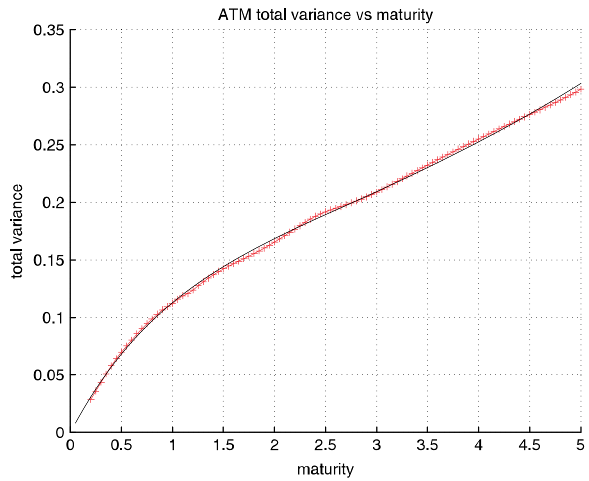
$$\begin{aligned}\tilde{\Phi}_i(z) &= \mathbb{P}\left(\tilde{F}(\tau_i, T_i) < F(0, T_i)e^{\mu(\tau_i, T_i) + z\nu(\tau_i, T_i)}\right) \\ &= \psi\left(\tau_i, F(0, T_i)e^{\mu(\tau_i, T_i) + z\nu(\tau_i, T_i)}\right) \\ &= \chi(\tau_i, \mu(\tau_i, T_i) + z\nu(\tau_i, T_i))\end{aligned}$$

with $\psi(\tau, K)$ and $\chi(\tau, x)$ being defined, respectively, in (16) and (17).

More often, the sampling time t in $\tilde{F}(t, T_i)$ is not equal to its option expiration τ_i. In this case, an interpolation procedure is usually required to fill in the gaps. We need to make sure that the interpolated distributions are consistent with the term structure of volatilities (implied by the log-normal model) and the terminal distributions (implied by the futures smiles). To this end, we observe that the term structure of total variance is a monotonic increasing function of time to maturity (see e.g. Figure 1). This enables us to use the total variance as as a new "clock" to locate a specific time slice inside the marginal distribution surface $\psi(\tau, K)$ (or $\chi(\tau, x)$).

We recall that, in the log-normal model, the term structure of total variance as a function of time to maturity τ is defined as[4]

Figure 1: Term structure of ATM total variance.



$$v(\tau) := \nu^2(\tau, T(\tau)) = \int_0^\tau \sigma^2(t, T(\tau))dt, \quad (20)$$

where $\sigma^2(t, T)$ is the instantaneous variance of $F(t, T)$ given by (7). The function $v(\tau)$ can be calculated directly from the calibrated volatility function $\sigma(t, T)$. In practice, it can be defined by interpolating the market-implied term structure of ATM total variances given by pairs $\{(\tau_i, v(\tau_i))\}$.[5]

On the other hand, at the sampling time t, the total variance of $F(t, T_i)$ is given by

$$\nu^2(t, T_i) = \int_0^t \sigma^2(s, T_i)ds,$$

where T_i is fixed for a given futures. In order to match the variance $\nu^2(t, T_i)$ to the term structure of total variance $v(\tau)$, we introduce the following transform to time t

$$\tau(t) = v^{-1}(\nu^2(t, T_i)), \quad (21)$$

and define the distribution function $\tilde{\Phi}_i(z)$ as

$$\tilde{\Phi}_i(z) = \psi\left(\tau(t), F(0, T_i)e^{\mu(t, T_i) + z\nu(t, T_i)}\right) = \chi(\tau(t), \mu(t, T_i) + z\nu(t, T_i)). \quad (22)$$

With these distributions $\tilde{\Phi}_1(z), \ldots, \tilde{\Phi}_n(z)$ defined, we are ready to apply (19) to simulate the "skewed" forward prices for derivatives pricing and risk management.

This practical approach allows us to incorporate the market smile information into the multi-factor model without complicating the calibration procedure. In fact, the calibration to ATM volatilities and to market smiles can be done separately.

4 Model Calibration and Surface Construction

In this section we address the calibration of the model to the real market data and the construction of marginal distribution surface. As an example, we use the market-implied volatilities of crude-oil futures options from NYMEX on December 20, 2011 to illustrate the calibration procedure.

4.1 Calibration of The Gabillon Model

We recall that the model presented in Section 2 reproduces exactly the initial term structure forward prices.[6] What we need is to calibrate to the term structure of volatilities, so as to match the model-implied volatilities to the market-implied volatilities.

Let $\{\sigma_{mkt}(j)\}_{j=1,\ldots,k}$ denote the at-the-money implied volatilities for an option contract with expiration τ_j on an a futures contract with maturity T_j. To calculate the model-implied volatilities, we recall the instantaneous variance $\sigma^2(t,T)$ defined in (7) and write the model-implied variance $\sigma^2_{mod}(\tau)$ for option expiration τ as

$$\sigma^2_{mod}(\tau,T) = \frac{1}{\tau}\int_0^\tau \sigma^2(t,T)dt$$
$$= \frac{1}{\tau}\int_0^\tau \left[(\sigma_L^2 + 2\sigma_L(\rho_{SL}\sigma_S - \sigma_L)b(t,T) + (\sigma_S^2 - 2\rho_{SL}\sigma_S\sigma_L + \sigma_L^2)b^2(t,T))\right]dt,$$

where $b(t,T)$ is given in (6). If we restrict our attention to constant parameters κ, ρ_{SL}, σ_S and σ_L, then we have

$$\sigma^2_{mod}(\tau,T) = \sigma_L^2 + 2\sigma_L(\rho_{SL}\sigma_S - \sigma_L)\frac{e^{-\kappa(T-\tau)} - e^{-\kappa T}}{\kappa\tau} + \left(\sigma_S^2 - 2\rho_{SL}\sigma_S\sigma_L + \sigma_L^2\right)\frac{e^{-2\kappa(T-\tau)} - e^{-2\kappa T}}{2\kappa\tau}.$$

Moreover, assuming $\tau = T$, we can write

$$\sigma^2_{mod}(T) = \sigma_L^2 + 2\sigma_L(\rho_{SL}\sigma_S - \sigma_L)\frac{1 - e^{-\kappa T}}{\kappa T} + \left(\sigma_S^2 - 2\rho_{SL}\sigma_S\sigma_L + \sigma_L^2\right)\frac{1 - e^{-2\kappa T}}{2\kappa T}$$

By using the above formulae, the calibration problem can be formalized as the following optimization problem:

$$\min_{\kappa>0,\sigma_S>0,\sigma_L>0,\rho_{SL}\in[-1,1]} \sum_{j=1}^k w_j\left[\sigma^2_{mod}(\tau_j,T_j) - \sigma^2_{mkt}(j)\right]^2 \tag{23}$$

where w_j is the weight of average. This is a standard nonlinear least-squares problem. We apply a procedure based on the Levenberg-Marquardt algorithm to search for a local minimizer for this problem. The initial guess and the calibrated results are shown in Table 1, where we simply set $w_j = 1$ for all j, although other weights can be used to slightly improve the fit. The calibration procedure is very efficient and typically takes less than 0.5 seconds to converge in a PC with Intel(R) Core(TM) 2 CPU@ 2.40 GHz. In this data set, the RMSE of implied volatility $\sqrt{\frac{1}{k}\sum_j^k[\sigma_{mod}(\tau_j,T_j) - \sigma_{mkt}(j)]^2} = 0.26\%$ against the RMS of the volatility $\sqrt{\frac{1}{k}\sum_j^k \sigma^2_{mkt}(j)} = 29.11\%$. In practice, one can use multiple initial guesses across the parameter domain to reduce the risk of being trapped inside a local minimum.

Table 1: Calibration to ATM implied volatility.

	κ	ρ_{SL}	σ_S	σ_L
Initial Guess	1.0	−0.5	$\sigma_{mkt}(1)$	$\sigma_{mkt}(k)$
Calib Results	0.366432	−0.29271	0.409231	0.290985

To illustrate the goodness of fit, we plot the term structure of market-implied total variances and that of the calibrated model in Figure 1. The red cross represents the market data and black line is for the results of the calibrated model.

4.2 Construction of Implied Volatility Surface

As shown in Section 3.1, the marginal distributions of futures prices can be recovered from the market implied volatility surface (IVS). One essential step requires differentiating the volatility smiles as in (16). In reality, the implied volatilities are only quoted for some discrete strike levels, so that volatility modeling is necessary for extrapolating and smoothing the volatility smiles. In what follows, we apply the popular SVI model to illustrate the construction of an arbitrage-free IVS in the crude-oil market, although other parameterizations could be used.

The SVI model. The Stochastic Volatility Inspired (SVI) model is a parametric form on the implied variance suggested by Gatheral (2004). For a fixed maturity, it is given by

$$\sigma^2_{svi}(x) = f(x) := a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\right) \tag{24}$$

where x is the forward log moneyness and a, b, m, ρ and σ are constants. Although it is designed based on practical experience, this model can be viewed as the large-time asymptotic implied volatility of the Heston model, as pointed out by Gatheral and Jacquier (2009). It turns out that this simple parameterization can provide an outstanding fit to volatility smiles in equity and FX markets; see Deryabin (2011) for its application in energy markets.

Using the SVI model, we can fit the market volatility smiles slice by slice. This can be done either forward or backward in time providing appropriate constraints for absence of arbitrage. For a fixed maturity, we aim to match the market smiles in terms of implied variances $\{\sigma^2_{mkt}(x_j)\}_{j=1,\ldots,k}$ for a set of forward log moneyness $\{x_j\}_{j=1,\ldots,k}$. Formally, the calibration can be described as a least-squares optimization problem on the SVI parameters:

$$\min_{a,b,m,\rho,\sigma}\sum_{j=1}^k w_j\left[\sigma^2_{svi}(x_j;\{a,b,m,\rho,\sigma\}) - \sigma^2_{mkt}(x_j)\right]^2 \tag{25}$$

subject to a certain boundary and no-arbitrage constraints. In general the parameters a, b, m, ρ and σ can depend on the time to maturity τ. We first impose the following basic constraints to the calibration:

$$b \geq 0, \quad |\rho| \leq 1, \quad \sigma > 0 \tag{26}$$

No-Arbitrage Constraints. Absence of arbitrage is a fundamental requirement for constructing a volatility surface. Practically the arbitrages to exclude are negative vertical spreads, negative butterflies and negative calendar spreads. For a fixed maturity slice, it turns out that a necessary condition is a constraint on the slopes of the total variance $V(\tau,x) := \tau\sigma^2_{imp}(\tau,x)$, which states

$$\left|\frac{\partial}{\partial x}V(\tau,x)\right| \leq 4,$$

for all forward log moneyness x; see Rogers and Tehranchi (2010). As pointed out by Gatheral (2004), this translates to the following constraint on the SVI parameters b and ρ:

$$b(1+|\rho|) \leq \frac{4}{\tau}. \tag{27}$$

Clearly this condition applies to options on futures as well as options on spot.

To avoid arbitrage between maturity slices, we need additional conditions to ensure that option prices (equivalently, the implied total variances) are non-decreasing in time to maturity for an arbitrary forward log moneyness x. For the SVI model, this is given by

$$\tau_2\sigma_{svi}^2(x;\, a_2, b_2, m_2, \rho_2, \sigma_2) \geq \tau_1\sigma_{svi}^2(x;\, a_1, b_1, m_1, \rho_1, \sigma_1), \tag{28}$$

for time to maturity $\tau_2 \geq \tau_1$. This turns out to be a sufficient condition to ensure absence of calendar spread arbitrage for options on futures, while it is sufficient and necessary for options on spot; see Deryabin (2011).

Efficient Calibration. Various techniques, such as dimension reduction, can be applied to improve the efficiency of SVI calibration; see, for example, Zeliade (2009). Consider the following change of variable

$$y = \frac{x-m}{\sigma},$$

by which we can rewrite the total variance $V = \tau\sigma_{svi}^2$ as

$$V(y;\, \{\alpha, \beta, \gamma\}) = \alpha + \beta\sqrt{y^2+1} + \gamma y,$$

where

$$\alpha = a\tau, \quad \beta = b\sigma\tau, \quad \gamma = b\rho\sigma\tau. \tag{29}$$

This enables us to consider the following sub-problem for given m and σ:

$$\min_{\alpha,\beta,\gamma} \sum_{j=1}^{k} w_j \left[V(y_j;\, \{\alpha,\beta,\gamma\}) - V_{mkt}(j)\right]^2 \tag{30}$$

where $V_{mkt}(j) = \tau\sigma_{mkt}^2(x_j)$ are the market total variances and $y_j = (x_j - m)/\sigma$. We impose the following constraints

$$\begin{cases} \alpha \leq \max_j(V_{mkt}(j)) \\ 0 \leq \beta \leq 4\sigma \\ -\beta \leq \gamma \leq \beta \\ -(4\sigma-\beta) \leq \gamma \leq 4\sigma - \beta \\ (\tau_0 - \tau)(\alpha + \beta\sqrt{\bar{y}^2+1} + \gamma\bar{y} - V_0^*(\bar{y})) \leq 0, \quad \bar{y} \in \mathcal{Y} \end{cases}$$

where $V_0^*(y) = \alpha_0^* + \beta_0^*\sqrt{y^2+1} + \gamma_0^* y$ is the total variance of the previous slice with time to maturity τ_0 and $\mathcal{Y}$ contains the grid points on which no-arbitrage constraint is satisfied.

These constraints can be derived from the original constraints on a, b and ρ. First of all, the upper bound on α is obvious. For β, since $b \geq 0$, $\sigma > 0$ and $b(1+|\rho|) \leq 4/\tau$, we see that $\beta = b\sigma\tau \geq 0$ and $\beta = b\sigma\tau \leq b\sigma\tau(1+|\rho|) \leq 4\sigma$. For γ, since $|\rho| \leq 1$ and $b(1+|\rho|) \leq 4/\tau$, we have $|\gamma| = \beta|\rho| \leq \beta$ and $\beta(1+|\rho|) \leq 4\sigma$ that implies $|\gamma| = \beta|\rho| \leq 4\sigma - \beta$. The last constraint is implied by the no-arbitrage condition (28).

Now let $\{\alpha^*(m,\sigma), \beta^*(m,\sigma), \gamma^*(m,\sigma)\}$ denote an optimal solution to the problem (30) for given m and σ. Then, by (29), we can easily identify the triplet $\{a^*(m,\sigma), b^*(m,\sigma), \rho^*(m,\sigma)\}$ and the original calibration problem (25) can be restored as

$$\min_{m,\sigma} \sum_{j=1}^{k} w_j \left[\sigma_{svi}^2(x_j;\, \{m, \sigma, a^*, b^*, \rho^*\}) - \sigma_{mkt}^2(x_j)\right]^2 \tag{31}$$

subject to the boundary constraints

$$\begin{cases} m_{min} \leq m \leq m_{max} \\ \sigma_{min} \leq \sigma \leq \sigma_{max} \end{cases}$$

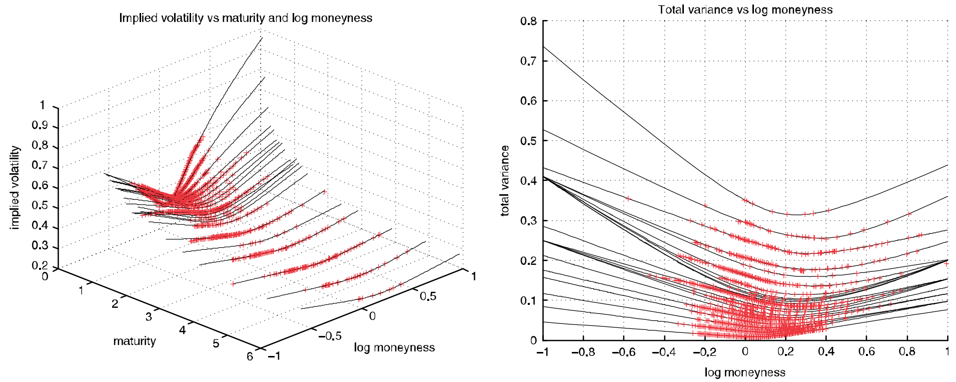
This becomes a two-dimensional nonlinear least-squares problem. The advantage to splitting the original calibration is that the sub-problem (30) is convex with a linear program and linear constraints. Such a program admits a global solution that can be solved explicitly. In fact, the global minimum is either at the interior that makes gradient zero or on the boundary of its domain that can be solved using Lagrange multipliers. Once the sub-problem is solved, we can apply some iterative scheme, such as the Levenberg-Marquardt algorithm, to search for a local minimizer for the reduced problem (31).

Table 2 shows the calibrated results for the NYMEX crude-oil data, where we simply set $w_j = 1$ for all j, $m_{min} = 3\min_j(x_j)$, $m_{max} = 3\max_j(x_j)$, $\sigma_{min} = 1\%$ and $\sigma_{max} = 100\%$. To avoid arbitrage between maturity slices, we impose the no-arbitrage condition to an equispaced grid of 100 points (forward log-moneyness) from −1.0 to 1.0, namely, set $\mathcal{Y}$={−1.00, −0.99, . . . ,0.99.1.00}. The RMSE of implied volatility for the whole surface is 0.19%, while the RMS of the volatility surface is 35.60%. The calibration procedure usually takes a few seconds to converge in a PC with Intel(R) Core(TM) 2 CPU @ 2.40 GHz.

Table 2: The calibrated SVI parameters.

τ	a	b	m	ρ	σ
.0822	0.0468	0.7187	0.0818	0.3496	0.0820
.1616	0.0726	0.4992	0.1214	0.2043	0.1107
.2466	0.0819	0.3515	0.1288	0.0161	0.1234
.3315	0.0673	0.3055	0.1440	−0.1536	0.1845
.4192	0.0747	0.2664	0.1790	−0.0878	0.1687
.4986	0.0676	0.2485	0.1655	−0.2225	0.1934
.5808	0.0461	0.2659	0.1807	−0.2014	0.2368
.6685	0.0206	0.2647	0.2511	−0.0323	0.3229
.7507	0.0469	0.2113	0.2518	−0.0583	0.2568
.9068	0.0653	0.1509	0.2719	−0.0826	0.1830
.9945	0.0564	0.1525	0.2039	−0.2379	0.2458
.0904	−0.0374	0.2794	0.1667	−0.1925	0.4615
.1699	−0.0432	0.2640	0.1633	−0.1935	0.4988
.2466	−0.0011	0.2206	0.1744	−0.2190	0.3932
.4164	−0.0182	0.2017	0.1946	−0.1933	0.4995
.9178	0.0094	0.1280	0.1887	−0.2587	0.4997
.4137	0.0506	0.0678	0.1902	−0.4543	0.2552
.9178	0.0336	0.0711	0.2944	−0.1193	0.3828
.9178	0.0496	0.0377	0.2757	−0.2567	0.1586
.9260	0.0330	0.0542	0.3664	0.0314	0.3493
.9233	0.0384	0.0559	0.1726	−0.2807	0.2729

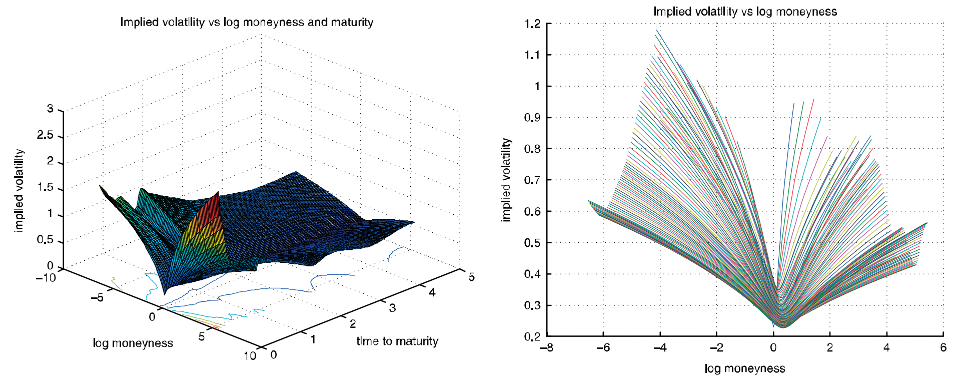
Figure 2: Calibration to implied volatility.



(a) Implied Volatility

(b) Total Variance

Figure 3: Calibrated implied volatility surface.



(a) Volatility Surface

(b) Volatility Smiles

To illustrate the goodness of fit, we plot the implied volatilities (left panel) and total variances (right panel) in Figure 2. In both cases, the red cross represents the market data and black line is for the results of the calibrated model. Figure 3 shows the interpolated volatility surface, where a Stineman interpolation (Stineman, 1980) is applied to the time dimension.

4.3 Construction of The Marginal Distribution Surface

Once we have a smooth and arbitrage-free IVS, we are ready to construct the marginal distribution surface $\chi(\tau, x)$ (or $\psi(\tau, K)$ in terms of strike). In terms of the SVI model (24), the total volatility is given by

$$w(\tau, K) = \sqrt{\tau f(x(K))}, \tag{32}$$

with $x(K) = \ln(K/F(0,T))$, and its derivative is given by

$$\frac{\partial}{\partial K} w(\tau, K) = \frac{\tau f'(x(K))}{2K w(\tau, x(K))}, \tag{33}$$

where

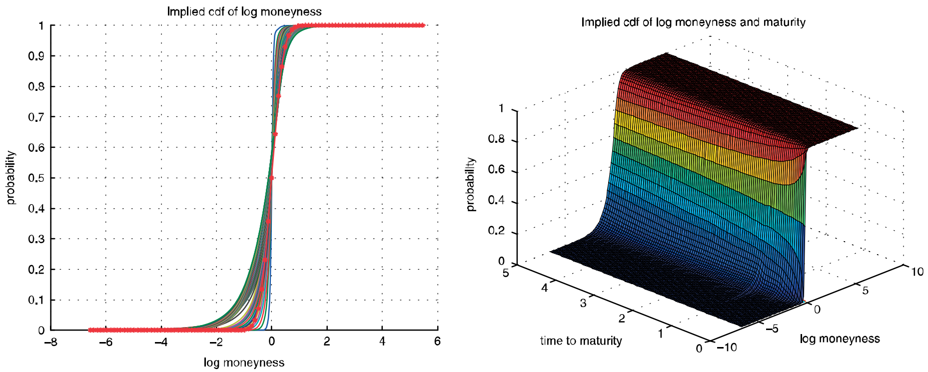
$$f'(x) := \frac{\partial}{\partial x}\sigma^2_{svi}(x) = b\left(\rho + \frac{x-m}{\sqrt{(x-m)^2+\sigma^2}}\right)$$

is the derivative of the SVI model.

Therefore, by (15)–(17) and (32)–(33), we derive the following formula for the term structure of marginal distributions

$$\chi(\tau, x) = 1 - \Phi(d_-) + \frac{\tau f'(x)}{2w}\varphi(d_-), \tag{34}$$

Figure 4: Implied surface of marginal distributions.



(a) Implied Margins (b) Interpolated Surface

where

$$w = \sqrt{\tau f(x)} \quad \text{and} \quad d_- = -\frac{x}{w} - \frac{w}{2}.$$

Using the formula (34) together with the calibrated SVI parameters in Table 2, we construct the surface of marginal distributions $\chi(\tau, x)$ as shown in Figure 4. The left panel shows the term structure of marginal distributions implied from the market. The right one demonstrates the interpolated surface of marginal distributions, where a Stineman interpolation (Stineman 1980) is applied to the time dimension. This surface serves as an input to the pricing routine, where the copula technique will be utilized to incorporate smile information to the log-normal model.

5 Valuation Algorithm

In this section, we illustrate the application of the model to price options. In general, the options payoff may depend on the intermediate realizations of the underlying futures prices. The following summarizes an algorithm to simulate the "skewed" prices $\tilde{F}(t, T_i)$, $(i = 1, \ldots, n)$, at the intermediate time step t.

Simulation Algorithm.

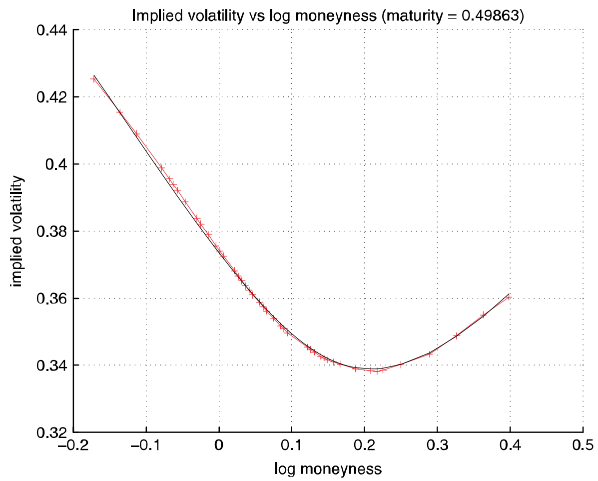
Input: the marginal distribution surface $\chi(\tau, x)$
Output: the "skewed" forward price $\tilde{F}(t, T_i)$

1. Generate a log-normal forward price $F(t, T_i)$ following the log-normal model (9)
2. Calculate the accumulative drift $\mu(t, T_i)$ and volatility $\nu(t, T_i)$ according to (11)
3. Transform the log-normal price $F(t, T_i)$ to a Gaussian variable $Z_i(t)$ using (12)
4. Compute the probability $U_i(t) = \Phi(Z_i(t))$ with $\Phi(z)$ being the standard normal CDF
5. Transform the current time t to $\tau(t) = \upsilon^{-1}(\nu^2(\tau, T_i))$, where $\upsilon(\tau)$ is the term structure of total variance defined in (20)
6. Compute the "skewed" random variable $\tilde{Z}_i(t) = \tilde{\Phi}_i^{-1}(U_i(t))$, where $\tilde{\Phi}_i(z)$ is the "skewed" marginal distribution defined in (22)
7. Calculate the "skewed" forward price $\tilde{F}(t, T_i)$ according to the formula (19). In terms of $\chi(\tau, x)$, we write

$$\tilde{F}(t, T_i) = F(0, T_i)\exp\{\chi^{-1}_{\tau(t)}(U_i(t))\}.$$

Note that the steps 5 and 6 require the inverse of the functions $\upsilon(\tau)$ and $\tilde{\Phi}_i(z)$. This can be time consuming if a root finding scheme is directly applied to search for the inverse. However, thanks to their monotonicity, a more efficient technique is to discretize the image of the functions and, subsequently, to interpolate the image to obtain the inverse image.

Figure 5: Implied volatility smile: model vs market.



We next apply the calibrated model to price vanilla options. Consider a maturity slice with expiration equal to, for example, 6 month ($T = 0.4986$). We develop a Monte Carlo routine based on the above algorithm. We price all options in this maturity slice and convert the options price to Black implied volatilities. The results are shown in Figure 5, which demonstrates that the model has an excellent fit to the market volatility. The simulation uses 52 time steps and 10^5 paths. The RMSE of implied volatility $\sqrt{\frac{1}{k}\sum_j^k[\sigma_{mod}(j) - \sigma_{mkt}(j)]^2} = 8$ bp against the RMS of the volatility $\sqrt{\frac{1}{k}\sum_j^k \sigma_{mkt}^2(j)} = 36.42\%$. The results clearly demonstrate the model's ability to recover the market implied volatility smiles. The same technique can be directly applied to price path-dependent options.

6 Conclusions

We have shown a practical method to extend the classical log-normal Gabillon model to incorporate volatility smiles and discussed its application to the NYMEX crude oil market. The method can be applied to other multi-factor log-normal models, such as Gibson and Schwartz (1990) and Schwartz and Smith (2000). In this modeling framework, different models may be built upon different intuitions for fitting the term structure of prices and volatilities. However, thanks to the martingale property, the differences are restricted to the volatility function in the forward-price representation. The important feature of our approach is its ability to "twist" the underlying log-normal distribution to capture the market smile information. In addition, the method can be easily extended to handle a general multi-asset multi-factor commodity model, which can be applied to price the fast growing market on commodity baskets.

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ENDNOTES

1. In fact, the convenience yield is implicitly determined by the interaction between the short- and long-term price movements.
2. Precisely, let parameters κ, μ_L, σ_S, σ_L and ρ_{SL} be constants and $f(t) = \ln(S_0) + (\mu_L - \frac{1}{2}\sigma_L^2)t$.
3. This conclusion holds as long as the interest rates are independent of the spot prices.
4. Here we write function $T(\tau)$ to emphasis that the futures maturity T is tightly coupled with the option expiration τ.
5. In this case, the transform (21) can be replaced by an appropriate interpolation routine.
6. As a matter of fact, the current forward curve can be used as an input to the model and the forward curve is matched automatically.

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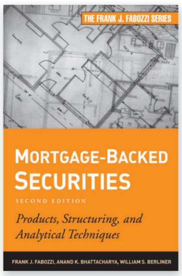

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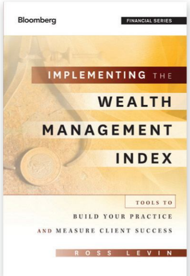

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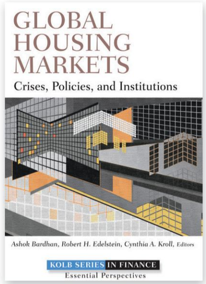

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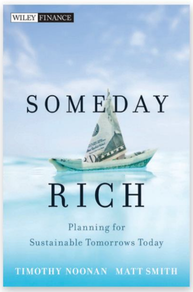

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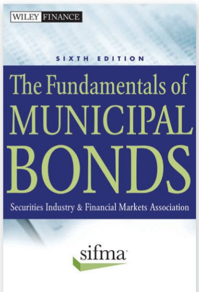

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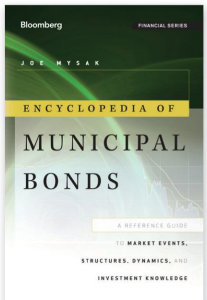

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