Accurate Pricing of Continuous Barrier Options With Local Volatility

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Abstract
In this paper we develop accurate techniques to price continuous barrier options using a one-dimensional finite difference scheme, allowing for time-dependent drift as well as time- and state-dependent volatilities. We provide numerical examples which demonstrate the smoothness and accuracy of such methods in the Black–Scholes context as well as in the Dupire local volatility model.

Keywords
barrier option, finite difference, local volatility, smoothing

Introduction
The pricing of barrier options has always been of the utmost interest to academics and practitioners. Analytical formulas were developed in the early works of Merton (1973) and Reiner and Rubinstein (2006), in the context of constant drift and volatility. The problem with these approaches is that although they provide an intuitive answer to the various market data sensitivities of barrier options, they fail to capture realistic features, starting from the most trivial term structure of interest rates and dividends to the more involved term structure of volatility smiles. However, in more general contexts analytical tractability is lost and one has to resort to numerical methods. The favored solution is generally a finite difference scheme. This approach also has its own shortcomings, including approximating the continuous barrier structure with discrete barriers, as well as the lack of smoothness of the discrete value function which can create a substantial lack of accuracy for Greek calculation purposes. In the following, we show how to address both of these issues in a robust and easily generalizable way.

This paper is organized as follows: we first introduce our general pricing framework and notations. We then present our barrier option terminology and recall the classic barrier option results under the constant volatility and drift assumptions. In the third section we show how to approximate a continuous barrier option price using a finite difference scheme and a simple adjusted barrier structure. In the subsequent section we demonstrate how to remove the barrier option finite difference pricing inaccuracy due to the lack of smoothness of the approximate value function by jointly using smooth pasting and accurate analytical corrections based on constant drift and volatility barrier option analytical formulas. Then we provide some numerical examples and show how our approximations perform in the situation where analytical formulas are known and in the local volatility case. Lastly we conclude with suggested improvements and extensions to this work.

Notations and pricing framework
We postulate the dynamics of some underlying spot price under the risk-neutral measure as:

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma(t, S(t))\,dW(t) \tag{1}$$

where:

- S is the spot price of the traded asset at time t;
- W is a one-dimensional Brownian motion under the risk-neutral measure;
- μ is the instantaneous drift function, which is assumed to be time-dependent only;
- σ is the instantaneous volatility function, which we assume to be strictly positive and time- and level-dependent.

Furthermore, we assume that the volatility function σ verifies the following assumptions:

A1 σ is C^1

A2 $\exists K | \forall (t, S)\ \sigma(t, x) < K, \left|\frac{\partial \sigma(t,x)}{\partial x}\right| < K$

We also introduce the short rate $r(t)$, which is assumed to be a deterministic function of time, and we will denote the discount bond price at time t and for maturity T by $P(t, T)$. Under the deterministic rate assumption the two quantities relate as follows:

$$P(t, T) = \exp\left(-\int_t^T r(s)\, ds\right)$$

Furthermore, in the case where the underlying price is a stock price (or foreign exchange rate), the drift will be defined by:

$$\mu(t) = r(t) - d(t)$$

where $d(t)$ denotes the instantaneous dividend yield (or foreign short rate function) at time t, which we assume to be a deterministic function of time.

We assume that the instantaneous rate/dividend functions have already been calibrated to a term structure of market instruments using standard bootstrapping methods. An exhaustive and rigorous account of these methods can be found in Chibane and Sheldon (2009). Similarly, we expect the instantaneous volatility surface to be pre-calibrated according to the celebrated Dupire formula, as described in Dupire (1994):

$$\sigma(T, K)^2 = 2\frac{\frac{\partial C(T,K)}{\partial T} + \mu(T) K \frac{\partial C(T,K)}{\partial K} + d(T) C(T, K)}{K^2 \frac{\partial^2 C(T,K)}{\partial K^2}} \tag{2}$$

where $C(T, K)$ is the price of a European call option with expiry T and strike K. This formula assumes call option prices are available for a continuum of expiries and strikes.

Now we need to present a generic framework for pricing exotics. We first consider a European security with maturity T, with no intermediate cash flows, and with terminal payoff function denoted by g. We denote the value of this security at time t by $V(t, S(t))$. By standard non-arbitrage arguments one can prove that:

$$D(t)\, V(t, S(t)) = E_t\left[D(T)\, g(S(T))\right] \tag{3}$$

where:

- E_t denotes the risk-neutral expectation operator conditional on the information available at time t;
- $D(t) = \exp(-\int_0^t r(s)\, ds)$ is the stochastic discount factor (inverted money market account) at time t.
- Using the deterministic rate assumption, equation (3) yields:

$$V(t, S(t)) = P(t, T)\, E_t\left[g(S(T))\right]$$

From the Feynman–Kac theorem we know that V solves the following parabolic PDE:

$$\frac{\partial V}{\partial t} + \mu(t) S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(t, S) S^2 \frac{\partial^2 V}{\partial S^2} = r(t) V \qquad V(T, S) = g(S) \tag{4}$$

where we abuse the notation by using V instead of $V(t, S)$. It is often more convenient to work with the log spot variable defined by $x(t) = \ln(S(t))$.

Using Itô's lemma we readily obtain the SDE governing the dynamics of x:

$$dx(t) = \left(\mu(t) - \frac{1}{2}\eta^2(t, x)\right) + \eta(t, x)\, dW(t)$$

$$\eta(t, x) = \sigma(t, \exp(x))$$

Then, the value function $\hat{V}$ defined by $\hat{V}(t, x) = V(t, \exp(x))$ is a solution to the following PDE:

$$\frac{\partial \hat{V}}{\partial t} + \left(\mu(t) - \frac{1}{2}\eta^2(t, x)\right)\frac{\partial \hat{V}}{\partial x} + \left(\frac{1}{2}\eta^2(t, x)\right)\frac{\partial^2 \hat{V}}{\partial x^2} = r(t)\hat{V} \qquad \hat{V}(T, x) = g(\exp(x)) \tag{5}$$

This PDE can be solved efficiently using a Crank–Nicholson scheme, as described in Morton and Mayer (2005). This scheme has quadratic accuracy in time and space. Later, we show how to extend this approach to the pricing of continuous barrier options. First, we introduce some barrier option terminology and recall the standard analytical formulas.

Barrier option pricing framework

Firstly we define a general knock-out continuous barrier option as a security which pays off at maturity a general function of the prevailing spot price provided that the spot price process *has not breached* a given barrier level H.[1] Otherwise the option is automatically worth 0. A corresponding knock-in option pays the same final payoff at maturity only if the spot process *has breached* the barrier level. Clearly the prices of the knock-in and knock-out option with the same terminal payoff are governed by the parity rule:

Knock in + Knock out = underlying europeian option

Furthermore, if the barrier is hit from below (above), the knock-out option will be referred to as 'up and out' ('down and out'). The corresponding knock-in options will be referred to as 'up and in' and 'down and in.'

We will denote the running maximum and running minimum between two dates respectively as follows:

$$M(s, t) = \sup(S(u);\ u \in [s, t])$$

$$m(s, t) = \inf(S(u);\ u \in [s, t])$$

The final payoff function conditional on the barrier condition being verified will be denoted by g. We denote respectively the value at time t of 'up and out,' 'up and in,' 'down and out,' and 'down and in' options by $V^{UO}(t, S(t))$, $V^{UI}(t, S(t))$, $V^{DO}(t, S(t))$, $V^{DI}(t, S(t))$.

Using non-arbitrage arguments it is easy to establish that conditional on not breaching the barrier condition prior to time t:

$$V^{UO}(t, S(t)) = P(t, T)\, E_t\left[1_{M(t,T)<H}\, g(S(T))\right]$$

$$V^{UI}(t, S(t)) = P(t, T)\, E_t\left[1_{M(t,T)>H}\, g(S(T))\right]$$

$$V^{DO}(t, S(t)) = P(t, T)\, E_t\left[1_{m(t,T)>H}\, g(S(T))\right]$$

$$V^{DI}(t, S(t)) = P(t, T)\, E_t\left[1_{m(t,T)<H}\, g(S(T))\right]$$

The only assumptions we make about the payoff function g is that it is continuous on $[0, +\infty[$.

From now on, for the sake of concreteness and without losing generality, we will focus our analysis on the 'up and out' option. In this paragraph we recall the option pricing formula in the context of constant drift, constant volatility, and constant barrier referred to as constant Black–Scholes (from now on CBS). As proved in Poulsen (2006), using the reflection principle, the value of an 'up and out' option with arbitrary terminal payoff g is:

$$
\begin{aligned}
V^{UO}(t, S(t)) &= f(t, S(t)) - \left(\frac{S(t)}{H}\right)^{p} f\left(t, \frac{H^2}{S(t)}\right) \\
f(t, S(t)) &= P(t,T)\, E_t\left[h(S(T))\right] \\
h(S(T)) &= 1_{S(T)<H}\, g(S(T)) \\
p &= \frac{\sigma^2 - 2\mu}{\sigma^2}
\end{aligned}
$$

In the particular case where the payoff is one of a call option, we recover the classic formula:

$$
\begin{aligned}
f^{UO,Call}(t, S(t)) &= P(t,T)(S(t)\exp(\mu(T-t))(N(d_H^*) - N(d_K^*)) \\
&\quad - K(N(d_H) - N(d_K))) \\
d_x^* &= \frac{\ln\left(\frac{x}{F(t,T)}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
V^{UO,Call}(t, S(t)) &= f^{UO,Call}(t, S(t)) - \left(\frac{S(t)}{H}\right)^{p} f^{UO,Call}\left(t, \frac{H^2}{S(t)}\right) \\
&= P(t,T)C^{UO}(S(t), \sigma, \mu, H, K, T-t)
\end{aligned}
$$

where N is the standard cumulative normal distribution function.

The digital version of the 'up and out' call which pays one unit of domestic currency provided that the barrier condition is not breached and if the terminal spot price is above the strike K can be obtained by simple differentiation with respect to strike as follows:

$$
C^{UO,digital}(S(t), \sigma, \mu, H, K, T-t) = -\frac{\partial C^{UO}(S(t), \sigma, \mu, H, K, T-t)}{\partial K}
$$

However, these formulas are useless in practical cases since real market data will always imply non-constant drift and volatility functions. In this situation one needs to resort to numerical schemes. We show how to do that in the next section.

Pricing barrier options in a finite difference scheme

We now identify the current pricing date with calendar time $t = 0$. Conditional on not having already knocked out, the initial price of the 'up and out' option is a solution to the following pricing PDE subject to terminal and boundary conditions:

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \left(\mu(t) - \frac{1}{2}\eta^2(t,x)\right)\frac{\partial V}{\partial x} + \frac{1}{2}\eta^2(t,x)\frac{\partial^2 V}{\partial x^2} = r(t)\,V \\
&V(T,x) = 1_{x<h}\, g(\exp(x)) \\
&\forall t \in [0,T]\ x > h\ V(t,x) = 0 \\
&h = \ln(H)
\end{aligned} \tag{6}
$$

where 1_A is the indicative function relative to event A.

One way to find an approximate solution for (6) on a Crank–Nicholson finite difference scheme is to make the approximation of replacing the continuous-time barrier by the discrete-time barrier acting at each of the times in the finite difference grid. In this case, PDE (6) is approximated through the following finite difference scheme:

$$
\begin{aligned}
&\forall j \in [1, N] && \hat{V}_{M,j} = 1_{x<h}\, g(x_i) \\
&\tilde{V}_i = \left(\tilde{V}_{ij}\right)_{1\le j\le N} && \\
&\forall i \in [0,\ldots,M-1] && \tilde{V}_i = \hat{E}_i[\hat{V}_{i+1}] \\
&\forall j \in [0,\ldots,N-1] && \textit{if } x_j > h: \quad \hat{V}_{ij} = 0 \\
& && \textit{else}: \quad \hat{V}_{ij} = \tilde{V}_{ij} \\
&\hat{E}_i[X] = (I - \theta\Delta t_i L_i)^{-1}(I + (1-\theta)\Delta t_i L_i)X && \\
&X \in \mathbb{R}^N, L_i \in \mathbb{R}^{N\times N} &&
\end{aligned}
$$

where:

- $(\hat{V}_{i,j})_{0\le i\le M, 1\le j\le N}$ is the finite difference approximation to the true value function V;
- coefficients of the matrix L_i are detailed in Appendix A;
- the operator $\hat{E}_i$ can be interpreted as a discrete expectation operator to the true conditional expectation $E_{t_i}[]$.

Using the true barrier as a discrete barrier obviously underestimates the effect of the barrier on the price, since it neglects the possibility of the barrier condition being breached between two monitoring dates. It is well known that for this kind of discretized algorithm to yield accurate enough values, we need an impractically high number of time steps. This issue has been covered extensively under the CBS assumptions as in Kou (2003), where the author develops some adjusted barrier approximations to be used in a discretized numerical scheme. Generalizations to more complex instantaneous volatility structures have been studied in Gobet (2009).[2] We will adopt the technique of using a barrier adjusted for continuity, but do not dwell on this feature as it is not the core issue of this work. The discrete adjusted barrier is given by:

$$
\begin{aligned}
H_{ij} &= H\exp\left(-\beta\sigma\left(t_i, S_j\right)\sqrt{\Delta t_i}\right) \\
\beta &= -\zeta\left(\frac{1}{2}\right)/\sqrt{2\pi} \simeq 0.5826 \\
\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s}
\end{aligned} \tag{7}
$$

Under the continuity adjustment, the discrete approximation to PDE (6) becomes:

$$
\begin{aligned}
&\forall j \in [1, N] && \hat{V}_{M,j} = 1_{x<h}\, g(x_i) \\
&\tilde{V}_i = \left(\tilde{V}_{ij}\right)_{1\le j\le N} && \\
&\forall i \in [0,\ldots,M-1] && \tilde{V}_i = \hat{E}_i[\hat{V}_{i+1}] \\
&\forall j \in [0,\ldots,N-1] && \textit{if } x_j > h_{ij}: \quad \hat{V}_{ij} = 0 \\
& && \textit{else}: \quad \hat{V}_{ij} = \tilde{V}_{ij} \\
&h_{ij} = \ln H_{ij} &&
\end{aligned}
$$

We note that equation (7) implies a floating barrier condition tied to level $H_{i,j}$, since the volatility depends on both the prevailing state and time. This can have severe implications from the point of view of computational efficiency, since obtaining accurate numbers from a finite difference scheme generally requires computing all these floating barriers. We have found, however, that this effect can be avoided by noting that the barrier condition needs to be checked with high accuracy only near the barrier. We suggest replacing the state- and time-dependent adjusted barrier with a simpler time-dependent only adjusted barrier defined by the volatility at the barrier:

$$H_i = H \exp\left(-\beta\sigma\,(t_i, H)\,\sqrt{\Delta t_i}\right)$$
$$\Leftrightarrow$$
$$h_i = h - \beta\eta\,(t_i, h)\,\sqrt{\Delta t_i} \tag{8}$$

In the following, we show that h_i can be used as an accurate effective barrier.

Proposition 1 *Existence of an effective barrier state*: there exists a sufficiently small time step and a value of the state variable y such that:

$$\forall x_j\; x_j < y \Leftrightarrow x_j < h_{ij}$$

Proof:

Let us consider the function f defined by:

$$f(x) = x - h + \beta\eta\,(t_i, x)\,\sqrt{\Delta t_i}$$

The barrier condition being breached translates into $f(x) > 0$.

f is differentiable, and its first-order derivative is:

$$f'(x) = 1 + \beta\frac{\partial\eta\,(t_i, x)}{\partial x}\sqrt{\Delta t_i} \tag{9}$$

Because $\frac{\partial\eta(t_i, x)}{\partial x}$ is bounded, there exists an α such that:

$$\forall\Delta t_i < \alpha : 0 < f'(x) = 1 + \beta\frac{\partial\eta\,(t_i, x)}{\partial x}\sqrt{\Delta t_i}$$

Furthermore, we have $f(0) < 0$ and since the instantaneous volatility is bounded, $\lim_{x\to+\infty} f(x) = +\infty$. Therefore, for Δt_i sufficiently small, there exists a point y_i such that $f(y_i) = 0$. By definition of y_i we have proved the proposition:

$$\begin{aligned} &\forall x < y_i : \quad f(x) < 0 \\ &\Leftrightarrow \forall y < y_i : \quad x < h - \beta\eta\,(t_i, x)\,\sqrt{\Delta t_i} \end{aligned}$$

Therefore, y_i can be used as an effective barrier. It could be computed by finding the root of equation (9). Instead, we propose the following approximate effective barrier:

$$h_i = h - \beta\eta\,(t_i, h)\,\sqrt{\Delta t_i}$$

Proposition 2 Assuming Δt_i is *small enough*, we have $h_i = y_i + O(\Delta t_i)$.

Proof:

By definition $f(y_i) = 0$, therefore:

$$\begin{aligned} y_i &= h + \beta\eta\,(t, y_i)\,\sqrt{\Delta t_i} \\ &= h - \beta\left(\eta\,(t, h) + \frac{\partial\eta\,(t, h)}{\partial x}(y_i - h) + o\left((y_i - h)^2\right)\right)\sqrt{\Delta t_i} \end{aligned}$$

Since η is bounded it is clear that $y_i - h = O(\sqrt{\Delta t_i})$. It follows that:

$$y_i = h - \beta\sigma\,(t, h)\,\sqrt{\Delta t_i} + O\,(\Delta t_i) = h_i + O\,(\Delta t_i)$$

We note that in the CBS framework, $y_i \equiv h_i$. From now on we use h_i at each time t_i as our effective adjusted barrier. Later we demonstrate the accuracy of this last scheme compared with the non-adjusted scheme with a high number of time steps.

Although the above-described algorithm is accurate enough for valuation purposes, it still exhibits instabilities in the first derivative close to the barrier. This effect stems from the fact that the terminal payoff is actually discontinuous at the barrier. Furthermore, the discrete barrier approximation (described above) results in the fresh introduction of a discontinuity at the barrier at each time step. The finite difference scheme, being a discrete approximation, does not properly handle these discontinuities.

Practitioners commonly address the terminal payoff discontinuity by performing a number of fully implicit finite difference rollbacks ($\theta = 1$) and then switching to the Crank–Nicholson scheme ($\theta = 0.5$) for the remaining integration. This can work, but in practice it requires a lot of fine-tuning, and can not be made general enough.

In the following section we show how to remedy this problem in a systematic manner by applying convenient and robust smoothing techniques which combine the CBS analytical framework and the finite difference scheme presented in the current section.

Smoothing techniques for barrier option pricing

In this section, for the sake of clarity and intuition, we use spot as the state variable but the following logic can equally be applied to $x(t) = \ln(S(t))$.

Let us look at a general 'up and out' barrier option as introduced previously. We recall that its price at any time t can be written as:

$$\begin{aligned} V^{UO}(t, S(t)) &= P(t,T)\,E_t\left[1_{M(t,T)<H}\,g(S(T))\right] = P(t,T)\,E_t\left[1_{M(t,T)<H}\,h(S(T))\right] \\ h(S) &= 1_{S(T)<H}\,g(S(T)) \end{aligned} \tag{10}$$

It is clear that the function h exhibits a discontinuity on state H. However, the left derivative of the final payoff exists at point H and is defined by:

$$\frac{\partial h\left(H^-\right)}{\partial S} = \lim_{S\to H^-}\frac{\partial h(S)}{\partial S}$$

Without loss of generality, we can rewrite (10) as:

$$\begin{aligned} \frac{V^{UO}(t, S(t))}{P(t,T)} &= E_t\left[1_{M(t,T)<H}\,\tilde{h}(S(T))\right] - \frac{\partial h(H^-)}{\partial S}E_t\left[1_{M(t,T)>H}\,(S(T) - H)^+\right] \\ &\quad - g(H^-)E_t\left[1_{M(t,T)>H}\right] \end{aligned} \tag{11}$$

$$\tilde{h}(S) = h(S) + \frac{\partial h(H^-)}{\partial S}(S - H)^+ + g(H^-)1_{S>H}$$

It is clear that the payoff function $\tilde{h}$ is C^1 on $[0, +\infty[$. Initially, let us assume that the CBS assumptions apply on $[0, T]$ with constant drift μ and volatility σ then, using the analytic formulas already presented, the second and third expectation of equation (11) can be computed exactly as shown below:

$$\begin{aligned} E_t\left[1_{M(t,T)>H}\,(S(T) - H)^+\right] &= C^{UI}\,(S(t), \sigma, \mu, H, H, T - t) \\ E_t\left[1_{M(t,T)>H}\right] &= C^{UI,\text{digital}}\,(S(t), \sigma, \mu, H, H, T - t) \end{aligned}$$

The first expectation can be computed numerically using the finite difference approach combined with the adjusted discrete barrier approximation already presented. The advantage of this decomposition is

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that the part which is computed numerically has been corrected for the lack of payoff continuity in value and first derivative, therefore yielding a much smoother integrated profile.

However, in the local volatility framework the CBS assumptions cease to exist, but we can still reuse the previous idea as demonstrated below. First, let us focus our analysis on the interval $[t_i, t_{i+1}]$. Over this interval drift is constant by construction. Assume now that we know the discrete approximation $\hat{V}_{i+1}$ of the true value function $V(t_{i+1})$ at time t_{i+1}. We denote the index of point H in the space axis by j_0.

We begin with the true value function:

$$V\left(t_i, S_j\right) = P\left(t_i, t_{i+1}\right) E_{t_i}\left[1_{M\left(t_i, t_{i+1}\right)<H} V\left(t_{i+1}, S\left(t_{i+1}\right)\right)\right]$$

Following the technique above, we introduce a transformation of V to decompose the expectation as follows:

$$\begin{aligned} & E_{t_i}\left[1_{M\left(t_i, t_{i+1}\right)<H} V\left(t_{i+1}, S\left(t_{i+1}\right)\right)\right] \\ & = E_{t_i}\left[1_{M\left(t_i, t_{i+1}\right)<H} \tilde{V}\left(t_{i+1}, S\left(t_{i+1}\right)\right)\right] \\ & \quad - \frac{\partial V\left(t_{i+1}, H^-\right)}{\partial S} E_{t_i}\left[1_{M\left(t_i, t_{i+1}\right)<H}\left(S\left(t_{i+1}\right) - K\right)^+\right] \\ & \quad - V\left(t_{i+1}, H^-\right) E_{t_i}\left[1_{M\left(t_i, t_{i+1}\right)<H}\right] \end{aligned} \tag{12}$$

$$\tilde{V}\left(t_{i+1}, S\left(t_{i+1}\right)\right) = V\left(t_{i+1}, S\left(t_{i+1}\right)\right) + \frac{\partial V\left(t_{i+1}, H^-\right)}{\partial S}\left(S\left(t_{i+1}\right) - K\right)^+ + V\left(t_{i+1}, H^-\right)$$

We will refer to $\tilde{V}$ as the *shifted value function*.

Our finite difference approximation relies on taking $\sigma(t_i, S_j)$ to be the volatility prevailing over $[t_i, t_{i+1}]$ conditional on state S_j. Therefore, our discrete time approximation enables us to write the expectations in the second and third terms of (12) as:

$$E_{t_i}\left[1_{M(t_i,t_{i+1})>H}(S(t_{i+1}) - H)^+\right] \simeq C^{UI}(S(t_i), \sigma(t_i, S(t_i)), S(t_i), \mu(t_i), H, H, t_{i+1} - t_i)$$

$$E_t\left[1_{M(t_i,t_{i+1})>H}\right] \simeq C^{UI,digital}(S(t_i), \sigma(t_i, S(t_i)), S(t_i), \mu(t_i), H, H, t_{i+1} - t_i)$$

Furthermore, since the current discrete estimation of $\hat{V}_{i+1}$ is available, the factors multiplying those expectations can be approximated as:[3]

$$V(t_i, H^-) \simeq \hat{V}_{ij_0}$$

$$\frac{\partial V(t_{i+1}, H^-)}{\partial S} \simeq \frac{\hat{V}_{i+1j_0} - \hat{V}_{i+1j_0-1}}{S_{j_0} - S_{j_0-1}} = \delta_{j_0}(\hat{V}_{i+1})$$

We will need to evaluate the first term on the right-hand side of (12) numerically, so we need a discrete version of the shifted value function at t_{i+1} and use the following scheme:

$$\tilde{V}_{i+1j} = \tilde{V}(t_{i+1}, S_j) = \hat{V}_{i+1_j} + \delta_{j_0}(\hat{V}_{i+1})(S_j - K)^+ + \hat{V}_{ij}$$

$$\tilde{V}_{i+1} = (\tilde{V}_{ij})_{1\leq j\leq N}$$

It is clear that $\delta_{j_0}(\tilde{V}_{i+1}) = \delta_{j_0-1}(\tilde{V}_{i+1})$, therefore the discrete shifted value function $\tilde{V}_{i+1}$ is regular around the barrier level H and is well suited for a finite difference integration.

Of course, we still rely on the continuous barrier adjustment to compensate for the time discreteness of the finite difference scheme. To make this approximation explicit we write:

$$E_{t_i}\left[1_{M(t_i,t_{i+1})<H}\tilde{V}(t_{i+1}, S(t_{i+1}))\right] \simeq \hat{E}_i\left[B_{i+1}\tilde{V}_{i+1}\right]$$

$$B_{i+1} = diag(\mathbf{1}_{S_j<H_{i+1}}), \quad H_{i+1} = H\exp\left(-\beta\sigma(t_i, H)\sqrt{\Delta t_{i+1}}\right)$$

Wrapping everything up, we get our new discrete expectation operator:

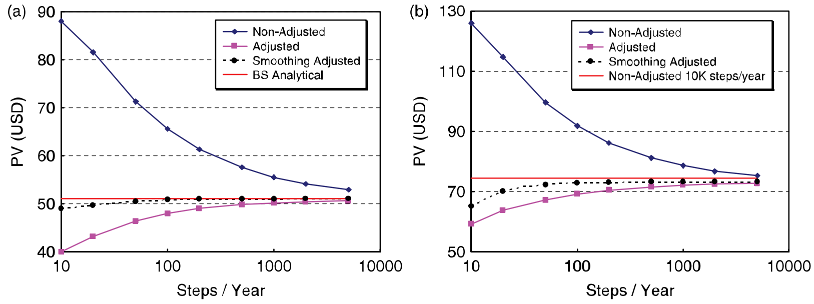
$$\begin{aligned} \hat{V}_i & \simeq \hat{E}_i\left[B_{i+1}\tilde{V}_{i+1}\right] - \delta_{j_0}\left(\hat{V}_{i+1}\right)C_i^{UI} - \hat{V}_{i+1j_0}C_i^{UI,Digital} \\ C_i^{UI} & = \left(C^{UI}(S_j, \sigma(t_i, S_j), S_j, \mu(t_i), H, H, t_{i+1} - t_i)\right)_{1\leq j\leq N} \\ C_i^{UI,Digital} & = \left(C^{UI,Digital}(S_j, \sigma(t_i, S_j), S_j, \mu(t_i), H, H, t_{i+1} - t_i)\right)_{1\leq j\leq N} \end{aligned}$$

This operator can easily be applied iteratively from time index N to 0.

Numerical results

In this section we present numerical results demonstrating the accuracy and smoothness of our improved barrier option pricing scheme. As a benchmark instrument we take an AUD/USD 'up and out' call barrier option with a notional of 10,000USD, time to maturity $T = 0.5$ *year*, strike $K = 0.85$, and barrier level $H = 0.9475$. The market data used in pricing was based on close

Figure 1: Barrier option convergence w.r.t. number of steps per year for (a) CBS (σ_{ATM}=14.75%) compared with analytical formula ('BS Analytical'); (b) 'Local Vol' compared with non-adjusted discrete barrier with 10,000 steps per year. We use 1000 states in our finite difference scheme.



of business September 1, 2010. The spot price is $S_0 = 0.89955$. For simplicity we have assumed that the interest rate markets implied flat rates of r_{USD} = 0.153% and r_{AUD} = 2.005%. These zero rates were implied from the respective 1Y swap rates on that market date.

Figure 1 compares the convergence properties of different algorithms as the number of time steps per year is increased. As expected, simply replacing the continuous barrier with discrete barriers ('Non-Adjusted') is slow to converge. Introducing the adjusted barrier approximation ('Adjusted,' cf. equation (8)) is an improvement, and in combination with the smoothing technique ("Smoothing Adjusted," cf. earlier) is even better.

Figures 2 and 3 show a spot ladder for the option price – price variation with spot level – in order to compare the numerical smoothness of the algorithms under the CBS assumptions, while Figures 4 and 5 show the same for local volatility ('Local Vol'). Both spot ladders are generated using a finite difference resolution of 1000 states and 200 steps per year. It is clear that the barrier adjustment and smoothing drastically improves smoothness and accuracy – especially close to the barrier.

Figure 2: FX spot ladder of PV (CBS).

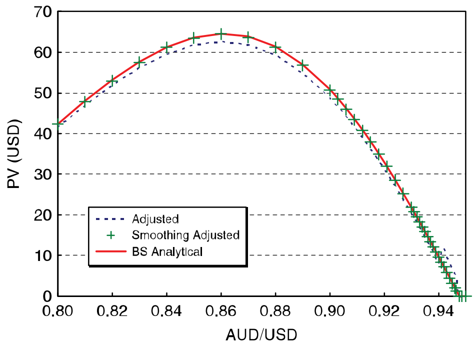


Figure 3: Details of Figure 2 close to the barrier.

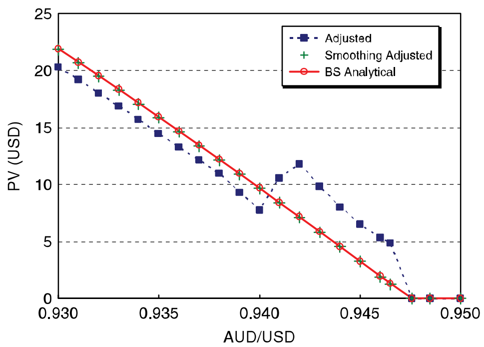


Figure 4: FX spot ladder of PV ('Local Vol'). Analytical formula with constant volatility is also listed to show the smile effects.

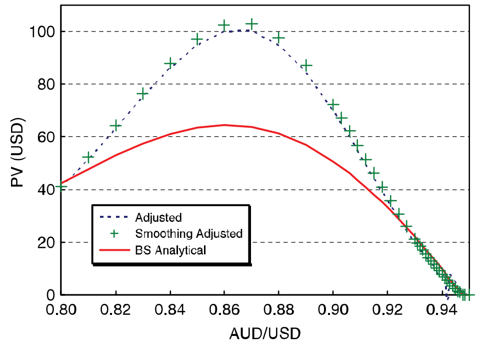


Figure 5: Details of Figure 4 close to the barrier.

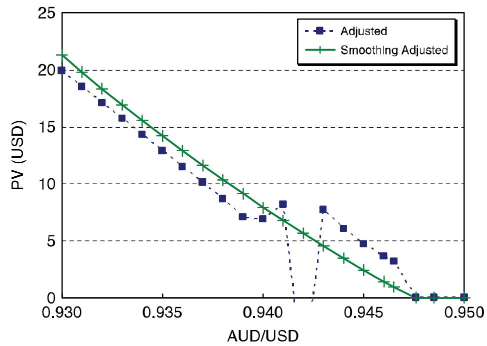


Figure 6: FX spot ladder of delta (CBS).

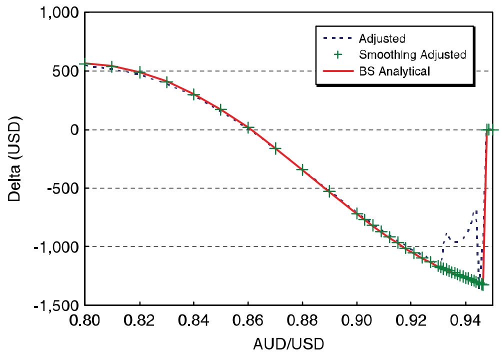


Figure 7: FX spot ladder of delta ('Local Vol').

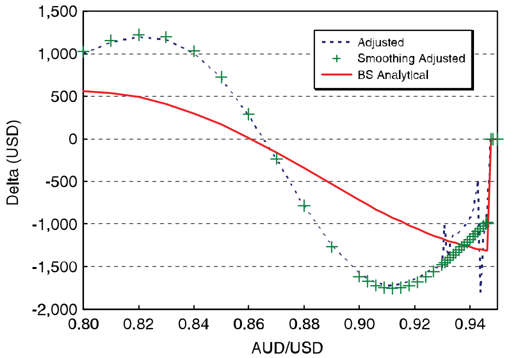
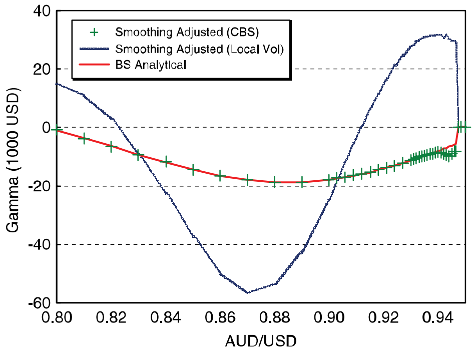


Figure 8: FX spot ladder for gamma.



Smoothness is more clearly demonstrated by the spot ladder for delta and gamma shown for both CBS and local volatility assumptions in Figures 6–8. It is clear that the delta has been made very smooth by our adjustment and that for all practical purposes, gamma is sufficiently accurate.

Conclusion

We have presented a novel approximation which to our knowledge has never been treated in the existing literature. The concept behind these approximations is very similar to that of the control variate in a Monte Carlo framework and can easily be extended to other modeling frameworks, like non-deterministic drift or stochastic volatility models. One has to bear in mind that the additional CBS computations involved by smoothing can cause substantial computational overhead, but this can be mitigated by using quick approximations for the normal distribution function and using quick interpolation of the local volatility function.

Appendix A

Here we propose to solve (6) on a finite difference grid defined by a time axis $\delta = (t_i)_{0\le i\le N}$ and a space axis $X = (x_j)_{0\le j\le N}$ such that $t_0 = 0$, $t_N = T$. We assume that both drift and rate functions can be formulated in terms of piecewise flat curves on this time axis. Also, for improved accuracy the space grid should contain the barrier level and all relevant strikes defining the payoff. We also define the maximum sizes of time and space grids as:

$$\Delta_t = \max_{1\le i\le M}(t_i - t_{i-1})$$

$$\Delta_x = \max_{1\le j\le N}\left(x_j - x_{j-1}\right)$$

In that setting we define the discrete approximation of the true value function V obtained through a one-factor finite difference θscheme by $\hat{V} = (V_{i,j})_{1\le i\le M,\,1\le j\le N}$, where $\theta$is the degree of implicitness of our scheme. Imposing $\theta \in [\frac{1}{2}, 1]$ guarantees the stability of the scheme. Maximum accuracy in time is obtained for $\theta = \frac{1}{2}$, and this is the value we use in practice. For ease of notation we introduce the slice vector defined by $\hat{V}_i = (\hat{V}_{i,j})_{1\le j\le N}$:

$$\begin{aligned}
&\forall j \in [1,N] && \hat{V}_{M,j} = \mathbf{1}_{x<h}\, g(x_i) \\
&\tilde{V}_i = \left(\tilde{V}_{ij}\right)_{1\le j\le N} && \\
&\forall i \in [0,\ldots,M-1] && (I - \theta\,\Delta t_i L_i)\,\tilde{V}_i = (I + (1-\theta)\,\Delta t_i L_i)\,\hat{V}_{i+1} \\
&\forall j \in [0,\ldots,N-1] && \textit{if } x_j > h: \quad \hat{V}_{ij} = 0 \\
& && \textit{else}: \quad \hat{V}_{ij} = \tilde{V}_{ij} \\
&j = N && L_{iNN} = \frac{\mu(t_i)}{\Delta x_{N-1}(\frac{1}{2} - \Delta x_N)} \\
& && L_{iNN} = -\frac{\mu(t_i)}{\Delta x_{N-1}(\frac{1}{2} - \Delta x_N)} + r(t_i) \\
&\Delta t_i = t_{i+1} - t_i && \\
&\Delta x_j = x_{j+1} - x_i && \\
&L_i = (L_{ijk})_{1\le j,k\le N} && \\
&2 \le j \le N-1 && L_{ijj-1} = \frac{\left(\mu(t_i) - \frac{1}{2}\eta^2(t_i,x_j)\right)}{\Delta x_j + \Delta x_{j-1}} - \frac{\eta^2(t_i,x_j)}{2\Delta x_j \Delta x_{j-1}} \\
& && L_{ijj} = r(t_i) + \frac{\eta^2(t_i,x_j)}{2\Delta x_{j-1}}\left(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j-1}}\right) \\
& && L_{ijj+1} = -\frac{(\mu(t_i) - \frac{1}{2}\eta^2(t_i,x_j))}{\Delta x_j + \Delta x_{j-1}} - \frac{\eta^2(t_i,x_j)}{2\Delta x_j \Delta x_{j-1}} \\
& && k \notin \{j-1, j, j+1\} \quad L_{ijj+1} = 0 \\
&j = 1 && L_{i11} = \frac{\mu(t_i)}{\Delta x_1\left(1 + \frac{\Delta x_1}{2}\right)} + r(t_i) \\
& && L_{i12} = -\frac{\mu(t_i)}{\Delta x_1\left(1 + \frac{\Delta x_1}{2}\right)} \\
& && k \notin \{1,2\} \quad L_{ijj+1} = 0
\end{aligned} \tag{A-1}$$

where:

- the boundary coefficients of matrix L are obtained by applying the discretized version of the loglinear boundary condition on the bottom and top points of the space axis as characterized below;
- I is the identity matrix of dimension N.

$$\frac{\partial V}{\partial x} = \frac{\partial^2 V}{\partial x^2}$$

We now denote the space made up of real-valued matrices of dimension $M \times N$ by Γ.

We introduce the following $\|\cdot\|_\infty$ on this space, defined by:

$$\|x\| = \max_{1\le i\le M, 1\le j\le N} \left|x_{ij}\right|$$
$$x = \left(x_{ij}\right)_{1\le i\le M, 1\le j\le N}$$

Furthermore, we define the approximate discrete conditional expectation operator $\hat{E}_i[]$ defined on R^N and with values R^N by:

$$\hat{E}_i : \quad \mathbb{R}^N \to \mathbb{R}^N$$
$$V \mapsto \hat{E}_i[V] = (I - \theta\Delta t_i L_i)^{-1}(I + (1-\theta)\Delta t_i L_i)V$$

Introducing the following error matrix:

$$e = \left(e_{ij}\right)_{1\le i\le M, 1\le j\le N}$$
$$e_{ij} = \hat{V}_{ij} - V\left(t_i, x_j\right)$$

It is clear that:

$$\lim_{\Delta_x \Delta_t \to 0} \|e\| = 0$$

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ENDNOTES

1. The barrier level can be a time-dependent function, but we will assume without loss of generality that it is constant.
2. Gobet (2009) also treats Brownian bridge techniques to adjust for the overshooting probability.
3. We note that by continuity of the option price, for any time t except the maturity time $t_N = T$ we have: $V(t_i, H^-) = V(t_i, H^+) = 0$ and $\frac{\partial V(t_{i+1}, H^-)}{\partial S} = \frac{\partial V(t_{i+1}, H^+)}{\partial S}$.

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