Negative Probabilities in Financial Modeling

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Abstract
We first define and derive general properties of negative probabilities. We then show how negative probabilities can be applied to modeling financial options such as Caps and Floors. In trading practice, these options are typically valued in a Black–Scholes–Merton framework, assuming a lognormal distribution for the underlying interest rate. However, in some cases, such as the 2008/2009 financial crisis, interest rates can get negative. Then the lognormal distribution is inapplicable. We show how negative probabilities associated with negative interest rates can be applied to value interest rate options. A model in VBA, which prices Caps and Floors with negative probabilities, is available upon request. A follow up paper will address bigger than unity probabilities in financial modeling.

Keywords
negative probabilities, negative interest rates, lognormal distribution, Caps, Floors

JEL Classification: C10

1. Introduction
The classical probability theory is applied in most sciences and in many of the humanities. In particular, it is successfully used in physics and finance. However, physicists found that they need a more general approach than the classical probability theory. The first was Eugene Wigner (1932), who introduced a function, which looked like a conventional probability distribution and has later been better known as the Wigner quasi-probability distribution because in contrast to conventional probability distributions, it took negative values, which could not be eliminated or made nonnegative. The importance of Wigner's discovery for foundational problems was not recognized until much later. Another outstanding physicist, Nobel Laureate Dirac (1942) not only supported Wigner's approach but also introduced the physical concept of negative energy. He wrote: "*Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money*" (referring to debt).

After this, negative probabilities slowly but steadily have become more popular techniques in physics. Maurice Bartlett (1945) worked out the mathematical and logical consistency of negative probabilities. However, he did not establish a rigorous foundation for negative probability utilization. Andrei Khrennikov published numerous papers going back to 1982 related to negative probabilities, some endorsed by Andrey Kolmogorov. In a 2007 interview Khrennikov discusses the *p*-adic model and finds that negative probabilities 'appear quite naturally' in this framework. In 2009 Khrennikov provides the first rigorous mathematical theory of negative probabilities in the setting of *p*-adic analysis.

Negative probabilities are also used in mathematical finance. The concept of risk-neutral or pseudo probabilities is a popular concept and has been numerously applied, for example, in credit modeling by Jarrow and Turnbull (1995), and Duffie and Singleton (1999). Espen Haug (2004) is the first author to explicitly support negative probabilities in financial modeling. He shows that negative probabilities can naturally occur, for example in the binomial CRR model, which is a discrete representation of the Black–Scholes–Merton model discussed below. Haug argues that rather than disregarding the whole model or transforming the negative probabilities to positive ones, negative probabilities can be a valuable mathematical tool in financial models to add flexibility.

Negative probabilities are able to model a random process if that process might become negative in rare events. In this paper we demonstrate this using negative interest rates. More generally, negative probabilities might also be applied to any random process where limited liability doesn't eliminate losses in all states. For example, suppose an enterpreneur owns companies x and y. If company x defaults and has net liabilities, assets of company y might have to be liquidated to cover the liabilities of x. Hence we can argue that the price of x can de facto become negative. In this paper we will concentrate on negative interest rates and show how they can be modeled with negative probabilities.

The remaining paper is organized as follows. In Section 2, we resolve the mathematical issue of the negative probability problem. We build a mathematical theory of extended probability as a probability function, which is defined for real numbers and can take both positive and negative values.

Thus, extended probabilities include negative probabilities. Different properties of extended probabilities are found. In Section 3, we give examples of negative nominal interest rates in financial practice and show problems of current financial modeling of negative interest rates. In Section 4, we build a mathematical model for Caps and Floors integrating extended probabilities into the pricing model to allow for negative interest rates. Conclusions are given in Section 5. A follow up paper will specify >1 probabilities and apply them to financial options.

2. Mathematical theory of extended probability

In 2009, Andrei Khrennikov derived a rigorous mathematical theory of negative probabilities in his textbook in the framework of *p*-adic analysis. Our derivation of negative probabilities is in the conventional setting of real numbers.

Extended probabilities generalize the standard definition of a probability function. At first, we define extended probabilities in an axiomatic way and then develop application of extended probabilities to finance.

To define extended probability, EP, we need some concepts and constructions, which are described below. Some of them are well-known, such as, for example, set algebra, while others, such as, for example, random antievents, are new.

We **remind** that if X is a set, then $|X|$ is the number of elements in (cardinality of) X (Kuratowski and Mostowski, 1967). If $A \subseteq X$, then the complement of A in X is defined as $C_X A = X \backslash A$.

A system $\mathbf{B}$ of sets is called a *set ring* (Kolmogorov and Fomin, 1989) if it satisfies conditions (R1) and (R2):

(R1) $A, B \in \mathbf{B}$ implies $A \cap B \in \mathbf{B}$.

(R2) $A, B \in \mathbf{B}$ implies $A \,\Delta\, B \in \mathbf{B}$ where $A \,\Delta\, B = (A \backslash B) \cup (B \backslash A)$.

For any set ring $\mathbf{B}$, we have $\varnothing \in \mathbf{B}$ and $A, B \in \mathbf{B}$ implies $A \cup B, A \backslash B \in \mathbf{B}$.

Indeed, if $A \in \mathbf{B}$, then by R1, $A \backslash A = \varnothing \in \mathbf{B}$. If $A, B \in \mathbf{B}$, then $A \backslash B = ((A \backslash B) \cup (B \backslash A)) \cap A \in \mathbf{B}$. If $A, B \in \mathbf{B}$ and $A \cap B = \emptyset$, then $A \,\Delta\, B = A \cup B \in \mathbf{B}$. It implies that $A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B) \in \mathbf{B}$. Thus, a system $\mathbf{B}$ of sets is a set ring if and only if it is closed with respect union, intersection and set difference.

Example 2.1. The set **CI** of all closed intervals $[a, b]$ in the real line $\boldsymbol{R}$ is a set ring.

Example 2.2. The set **OI** of all open intervals (a, b) in the real line $\boldsymbol{R}$ is a set ring.

A set ring $\mathbf{B}$ with a unit element, i.e., an element E from $\mathbf{B}$ such that for any A from $\mathbf{B}$, we have $A \cap E = A$, is called a *set algebra* (Kolmogorov and Fomin, 1989).

Example 2.3. The set **BCI** of all closed subintervals of the interval $[a, b]$ is a set algebra.

Example 2.4. The set **BOI** of all open subintervals of the interval $[a, b]$ is a set algebra.

A set algebra $\mathbf{B}$ closed with respect to complement is called a *set field*.

Let us consider a set Ω, which consists of two irreducible parts (subsets) Ω^+ and Ω^-, i.e., neither of these parts is equal to its proper subset, a set $\mathbf{F}$ of subsets of Ω, and a function P from $\mathbf{F}$ to the set $\boldsymbol{R}$ of real numbers.

Elements from $\mathbf{F}$, i.e., subsets of Ω that belong to $\mathbf{F}$, are called *random events*.

Elements from $\mathbf{F}^+ = \{X \in \mathbf{F};\, X \subseteq \Omega^+\}$ are called *positive random events*.

Elements from Ω^+ that belong to $\mathbf{F}^+$ are called *elementary positive random events* or simply, *elementary positive random events*.

If $w \in \Omega^+$, then $-w$ is called the *antievent* of w.

Elements from Ω^- that belong to $\mathbf{F}^-$ are called *elementary negative random events* or *elementary random antievents*.

For any set $X \subseteq \Omega^+$, we define

$$X^+ = X \cap \Omega^+,$$
$$X^- = X \cap \Omega^-,$$
$$-X = \{-w;\, w \in X\}$$

and

$$\mathbf{F}^- = \{-A;\, A \in \mathbf{F}^+\}$$

If $A \in \mathbf{F}^+$, then $-A$ is called the *antievent* of A.

Elements from $\mathbf{F}^-$ are called *negative random events* or *random antievents*.

Definition 1. The function P from $\mathbf{F}$ to the set $\boldsymbol{R}$ of real numbers is called a *probability function*, if it satisfies the following axioms:

EP 1 (*Order structure*). There is a graded involution $\alpha: \Omega \to \Omega$, i.e., a mapping such that α^2 is an identity mapping on Ω with the following properties: $\alpha(w) = -w$ for any element w from Ω, $\alpha(\Omega^+) \supseteq \Omega^-$, and if $w \in \Omega^+$, then $\alpha(w) \notin \Omega^+$.

EP 2 (*Algebraic structure*). $\mathbf{F}^+ \equiv \{X \in \mathbf{F};\, X \subseteq \Omega^+\}$ is a set algebra that has Ω^+ as a member.

EP 3 (*Normalization*). $P(\Omega^+) = 1$.

EP 4 (*Composition*) $\mathbf{F} \equiv \{X;\, X^+ \subseteq \mathbf{F}^+ \;\&\; X^- \subseteq \mathbf{F}^- \;\&\; X^+ \cap -X^- \equiv \varnothing \;\&\; X^- \cap -X^+ \equiv \varnothing\}$.

EP 5 (*Finite additivity*)
$P(A \cup B) = P(A) + P(B)$
for all sets $A, B \in \mathbf{F}$ such that
$A \cap B \equiv \emptyset$

EP 6 (*Annihilation*). $\{v_i, w, -w;\, v_i, w \in \Omega \;\&\; i \in I\} = \{v_i;\, v_i \in \Omega \;\&\; i \in I\}$ for any element w from Ω.

Axiom EP6 shows that if w and $-w$ are taken (come) into one set, they annihilate one another. Having this in mind, we use two equality symbols $=$ and $\equiv$. The second symbol means equality of elements of sets. The second symbol also means equality of sets, when two sets are equal when they have exactly the same elements (Kuratowski and Mostowski, 1967). The equality symbol $=$ is used to denote equality of two sets with annihilation, for example, $\{w, -w\} = \emptyset$. Note that for sets, equality $\equiv$ implies equality $=$.

For equality of numbers, we, as it is customary, use symbol $=$.

EP 7. (*Adequacy*) $A = B$ implies $P(A) = P(B)$ for all sets $A, B \in \mathbf{F}$.

For instance, $P(\{w, -w\}) = P(\emptyset) = 0$.

EP 8. (*Non-negativity*) $P(A) \geq 0$, for all $A \in \mathbf{F}^+$.

It is known that for any set algebra $\mathbf{A}$, the empty set $\emptyset$ belongs to $\mathbf{A}$ and for any set field $\mathbf{B}$ in Ω, the set Ω belongs to $\mathbf{A}$ (Kolmogorov and Fomin, 1989).

Definition 2. The triad $(\Omega, \mathbf{F}, P)$ is called an *extended probability space*.

Definition 3. If $A \in \mathbf{F}$, then the number $P(A)$ is called the *extended probability* of the event A.

Let us obtain some properties of the introduced constructions.

Lemma 1. $\alpha(\Omega^+) \equiv -\Omega^+ \equiv \Omega^-$ and $\alpha(\Omega^-) \equiv -\Omega^- \equiv \Omega^+$.

Proof. By Axiom EP1, $\alpha(\Omega^+) \equiv -\Omega^+$ and $\alpha(\Omega^+) \supseteq \Omega^-$. As $\Omega \equiv \Omega^+ \cup \Omega^-$, Axiom EP1 also implies $\alpha(\Omega^+) \subseteq \Omega^-$. Thus, we have $\alpha(\Omega^+) \equiv \Omega^-$. The first part is proved.

The second part is proved in a similar way.

Thus, if $\Omega^+ = \{w_i ; i \in I\}$, then $\Omega^- = \{-w_i ; i \in I\}$.

As α is an involution of the whole space, we have the following result.

Proposition 1. α is a one-to-one mapping and $|\Omega^+| = |\Omega^-|$.

Corollary 1. (*Domain symmetry*) $w \in \Omega^+$ if and only if $-w \in \Omega^-$.

Corollary 2. (*Element symmetry*) $-(-w) = w$ for any element w from Ω.

Corollary 3. (*Event symmetry*) $-(-X) \equiv X$ for any event X from Ω.

Lemma 2. $\alpha(w) \neq w$ for any element w from Ω.

Indeed, this is true because if $w \in \Omega^+$, then by Axiom EP1, $\alpha(w) \notin \Omega^+$ and thus, $\alpha(w) \neq w$. If $w \in \Omega^-$, then we may assume that $\alpha(w) = w$. However, in this case, $\alpha(v) = w$ for some element v from Ω^+ because by Axiom EP1, α is a projection of Ω^+ onto Ω^-. Consequently, we have

$$\alpha(\alpha(v)) = \alpha(w) = w$$

However, α is an involution, and we have $\alpha(\alpha(v)) = v$. This results in the equality

$$v = w$$

Consequently, we have $\alpha(v) = v$. This contradicts Axiom EP1 because $v \in \Omega^+$. Thus, lemma is proved by contradiction.

Proposition 2. $\Omega^+ \cap \Omega^- \equiv \emptyset$.

Proposition 3. $\mathbf{F}^+ \subseteq \mathbf{F}$, $\mathbf{F}^- \subseteq \mathbf{F}$ and $\mathbf{F} \subseteq \mathbf{F}^+ \cup \mathbf{F}^-$.

Corollary 1 implies the following result.

Proposition 4. $X \subseteq \Omega^+$ if and only if $-X \subseteq \Omega^-$.

Proposition 5. $\mathbf{F}^- \equiv \{X \in \mathbf{F}; X \subseteq \Omega^-\} = \mathbf{F} \cap \Omega^-$.

Corollary 4. $\mathbf{F}^+ \cap \mathbf{F}^- \equiv \emptyset$.

Axioms EP6 implies the following result.

Lemma 3. $X \cup -X = \emptyset$ for any subset X of Ω.

Indeed, for any w from the set X, there is $-w$ in the set X, which annihilates w.

Let us define the union with annihilation of two subsets X and Y of Ω by the following formula:

$$X + Y \equiv (X \cup Y) \backslash [(X \cap -Y) \cup (-X \cap Y)]$$

Here the set-theoretical operation $\backslash$ represents annihilation, while sets $X \cap -Y$ and $X \cap -Y$ depict annihilating entities.

Some properties of the new set operation + are the same as properties of the union $\cup$, while other properties are different. For instance, there is no distributivity between operations + and $\cap$.

Lemma 4. a) $X + X \equiv X$ for any subset X of Ω;
b) $X + Y \equiv X + Y$ for any subsets X and Y of Ω;
c) $X + \emptyset \equiv X$ for any subset X of Ω;
d) $X + (Y + Z) \equiv (X + Y) + Z$ for any subsets X, Y and Z of Ω;
e) $X + Y \equiv X \cup Y$ for any subsets X and Y of Ω^+ (of Ω^-);

Lemma 5. a) $Z \cap (X + Y) \neq Z \cap X + Z \cap Y$;
b) $X + (Y \cap Z) \neq (X \cap Y) + (X \cap Z)$.

Lemma 6. $A \cap B \equiv (A^+ \cap B^+) + (A^- \cap B^-)$ for any subsets A and B of Ω.

Indeed, as $A \equiv A^+ \cup A^-$ and $B \equiv B^+ \cup B^-$, we have

$$A \cap B \equiv (A^+ \cup A^-) \cap (B^+ \cap B^-) \equiv$$
$$(A^+ \cap B^+) \cup (A^+ \cap B^-) \cup (A^- \cap B^+) \cup (A^- \cap B^-) \equiv$$
$$(A^+ \cap B^+) + (A^- \cap B^-)$$

because $(A^+ \cap B^-) \equiv \emptyset$ and $(A^- \cap B^+) \equiv \emptyset$.

In a similar way, we prove the following results.

Lemma 7. $A \backslash B \equiv (A^+ \backslash B^+) + (A^- \backslash B^-)$ for any subsets A and B of Ω.

Lemma 8. $X \equiv X^+ + X^- = X^+ \cup X^-$ for any set X from $\mathbf{F}$.

Lemma 9. $A + B \equiv (A^+ + B^+) + (A^- + B^-)$ for any sets X and Y from $\mathbf{F}$.

Axioms EP6 and EP7 imply the following result.

Proposition 6. $P(X + Y) = P(X \cup Y)$ for any two events X and Y from Ω.

Lemma 10. $P(\emptyset) = 0$.

Properties of the structure $\mathbf{F}^+$ are inherited by the structure $\mathbf{F}$.

Theorem 1. (*Algebra symmetry*) If $\mathbf{F}^+$ is a set algebra (or set field), then $\mathbf{F}$ is a set field (or set algebra) with respect to operations + and $\cap$.

Proof. At first, we prove that $\mathbf{F}^-$ is a set algebra (or set field).

Let us assume that $\mathbf{F}^+$ is a set algebra and take two negative random events H and K from $\mathbf{F}^-$. By the definition of $\mathbf{F}^-$, $H = -A$ and $K = -B$ for some positive random events A and B from $\mathbf{F}^+$. Then we have

$$H \cap K = (-A) \cap (-B) = -(A \cap B)$$

As $\mathbf{F}^+$ is a set algebra, $A \cap B \in \mathbf{F}^+$. Thus, $H \cap K \in \mathbf{F}^-$.
In a similar way, we have

$$H \cup K = (-A) \cup (-B) = -(A \cup B)$$

As $\mathbf{F}^+$ is a set algebra, $A \cup B \in \mathbf{F}^+$. Thus, $H \cup K \in \mathbf{F}^-$.

By the same token, we have $H \backslash K \in \mathbf{F}^-$.

Besides, if $\mathbf{F}^+$ has a unit element E, then $-E$ is a unit element in $\mathbf{F}^-$.

Thus, $\mathbf{F}^-$ is a set algebra.

Now let us assume that $\mathbf{F}^+$ is a set field and $H \in \mathbf{F}^-$. Then by the definition of $\mathbf{F}^-$, $H = -A$ for a positive random event A from $\mathbf{F}^+$. It means that $C_{\Omega+}A = \Omega^+ \backslash A \in \mathbf{F}^+$. At the same time,

$$C_{\Omega-}H = \Omega^- \backslash H = (-\Omega^+) \backslash (-A) = -(\Omega^+ \backslash A) = -C_{\Omega+}A$$

As $C_{\Omega+}A$ belongs to $\mathbf{F}^+$, the complement $C_{\Omega-}H$ of H belongs to $\mathbf{F}^-$. Consequently, $\mathbf{F}^-$ is a set field.

Let us once more assume that $\mathbf{F}^+$ is a set algebra and take two random events A and B from $\mathbf{F}$. Then by Theorem 1, $\mathbf{F}^-$ is a set algebra. By Lemma 8, $A \equiv A^+ + A^-$ and $B \equiv B^+ + B^-$. By Axiom EP4, $A^+, B^+ \in \mathbf{F}^+$, $A^-, B^- \in \mathbf{F}^-$, while by Proposition 2, $A^+ \cap A^- \equiv \emptyset$, $B^+ \cap B^- \equiv \emptyset$, $A \equiv A^+ \cup A^-$, and $B \equiv B^+ \cup B^-$.

By Lemma 6, $A \cap B \equiv (A^+ \cap B^+) + (A^- \cap B^-)$. Thus, $(A \cap B)^+ \equiv A^+ \cap B^+$ and $(A \cap B)^- \equiv A^- \cap B^-$. As $\mathbf{F}^+$ is a set algebra, $(A \cap B)^+ \equiv A^+ \cap B^+ \in \mathbf{F}^+$. As it is proved that $\mathbf{F}^-$ is a set algebra, $(A \cap B)^- \equiv A^- \cap B^- \in \mathbf{F}^-$. Consequently, $A \cap B \in \mathbf{F}$.

By Lemma 7, $A \backslash B \equiv (A^+ \backslash B^+) + (A^- \backslash B^-)$. Thus, $(A \backslash B)^+ \equiv A^+ \backslash B^+$ and $(A \backslash B)^- \equiv A^- \backslash B^-$. As $\mathbf{F}^+$ is a set algebra, $(A \backslash B)^+ \equiv A^+ \backslash B^+ \in \mathbf{F}^+$. As it is proved that $\mathbf{F}^-$ is a set algebra, $(A \backslash B)^- \equiv A^- \backslash B^- \in \mathbf{F}^-$. Consequently, $A \backslash B \in \mathbf{F}$.

By Lemma 9, $A + B \equiv (A^+ + B^+) + (A^- + B^-)$. Thus, $(A + B)^+ \equiv A^+ + B^+$ and $(A + B)^- \equiv A^- + B^-$. As $\mathbf{F}^+$ is a set algebra, $(A + B)^+ \equiv A^+ + B^+ \equiv A^+ \cup B^+ \in \mathbf{F}^+$. As it is proved that $\mathbf{F}^-$ is a set algebra, $(A + B)^- \equiv A^- + B^- \equiv A^- \cup B^- \in \mathbf{F}^-$. Consequently, $A + B \in \mathbf{F}$.

Besides, if $\mathbf{F}^+$ has a unit element E, then $-E$ is a unit element in $\mathbf{F}^-$ and $E \cup -E$ is a unit element in $\mathbf{F}$.

Thus, $\mathbf{F}$ is a set algebra.

Now let us assume that $\mathbf{F}^+$ is a set field and $A \in \mathbf{F}$. Then as it is demonstrated above, $\mathbf{F}^-$ is a set field. By Lemma 8, $A \equiv A^+ + A^-$. By Proposition 2, $\Omega^+ \cap \Omega^- = \emptyset$, we have

$$C_\Omega A = C_{\Omega+}A + C_{\Omega-}A$$

Then $C_{\Omega+}A$ belongs to $\mathbf{F}^+$ as $\mathbf{F}^+$ is a set field and as it is proved in Theorem 1, $C_{\Omega-}A$ belongs to $\mathbf{F}^-$. Consequently, $C_{\Omega}A$ belongs to $\mathbf{F}^-$ and $\mathbf{F}^-$ is a set field.

Theorem is proved.

3. Negative interest rates and the problem of their modeling

Negative probabilities can help to model interest rates and interest rate derivatives. To show this, let us start with the equation

$$\text{Real interest rate} = \text{Nominal interest rate} - \text{Inflation rate} \tag{1}$$

where the nominal interest rate is the de facto rate, which is received by the lender and paid by the borrower in a financial contract. For example, the nominal interest rate is the rate, which the lender receives on a saving account or the coupon of a bond. From equation (1) we see that a real interest rate can easily be negative and in reality often is. For example, if the nominal interest rate on a savings account is 1% and the inflation rate is 3%, naturally, the real interest rate, i.e. the inflation adjusted rate of return for the lender is −2%.

3. 1. Examples of negative nominal interest rates

However, in rare cases, also the nominal interest rate can be negative. An example of this would be that the lender gives money to a bank, and additionally pays the bank an interest rate. This happened in the 1970s in Switzerland. The lender had several motives

a) Switzerland is considered an extremely safe country to place capital
b) Investors were speculating on an increase of the Swiss franc
c) Some investors avoided paying taxes in their home country

Another example of negative nominal interest rates occurred in Japan in 2003. Banks lent Japanese Yen and were willing to receive a lower Yen amount back several days later. This means de facto a negative nominal interest rate. The reason for this unusual practice was that banks were eager to reduce their exposure to Japanese Yen, since confidence in the Japanese economy was low and the Yen was assumed to devalue.

Similarly, in the US, from August to November 2003, 'repos', i.e. repurchase agreements traded at negative interest rates. A repo is just a collateralized loan, i.e. the borrower of money gives collateral, for example a Treasury bond, to the lender for the time of the loan. When the loan is paid back, the lender returns the collateral. However, in 2003 in the US, settlement problems when returning the collateral occurred. Hence the borrower was only willing to take the risk of not having the collateral returned if he could pay back a lower amount than originally borrowed. This constituted a negative nominal interest rate.

A further example of the market expecting the possibility of negative nominal interest rates occurred in the worldwide 2008/2009 financial crisis, when strikes on options on Eurodollars Futures contracts were quoted above 100. A Eurodollar is a dollar invested at commercial banks outside the US. A Eurodollar futures price reflects the anticipated future interest rate. The rate is calculated by subtracting the Futures price from 100. For example, if the 3 month March Eurodollar future price is 98.5, the expected interest rate from March to June is 100 − 98.5 = 1.5, which is quoted in per cent, so 1.5%. In March 2009, option strikes on Eurodollar future contracts were quoted above 100 on the CME, Chicago Mercantile Exchange. This means that market participants could buy the right to pay a negative nominal interest on US dollars in the future if desired. The reason for this unusual behavior is that investors wanted to invest in the safe haven currency US dollar even if they had to pay for it.

3.2. Modeling interest rates

In finance, interest rates are typically modeled with a geometric Brownian motion,

$$\frac{dr}{r} = \mu_r dt + \sigma_r \varepsilon \sqrt{dt} \tag{2}$$

dr: change in the interest rate r

μ_r: drift rate, which is the expected growth rate of r, assumed non-stochastic and constant

dt: infinitely short time period

σ_r: expected volatility of rate r, assumed non-stochastic and constant

ε: random drawing from a standardized normal distribution. All drawings at times t are iid.

In equation (2), the first term on the right hand side gives the average expected growth rate of r. The second term on the ride side adds stochasticity to the process via ε, i.e. provides the distribution around the average growth rate. Importantly, from equation (1) we can observe that the relative change dr/r is normally distributed, since ε is normally distributed. If the relative change of a variable is normally distributed, it follows that the variable itself is lognormally distributed with a pdf

$$\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} \tag{3}$$

In the equation (3), μ and σ are the mean and standard deviation of ln(x) respectively. Figure 1 shows a lognormal distribution.

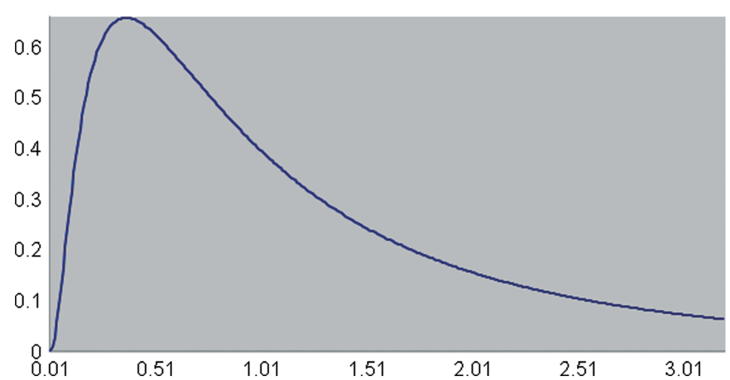
The logarithm of a negative number is not defined, hence with the pdf equation (3), negative values of interest rates cannot be modeled. However, as discussed above, negative interest rates do exist in the real financial world. Here negative probabilities come into play. We will explain this with options on interest rates.

4. How negative probabilities allow more adequate interest rate modeling

4.1. Modeling interest rate options

The two main types of options are call options and put options. A call option is the right but not the obligation to pay a strike price and receive an

Figure 1: Lognormal distribution with μ = 0 and σ=1.



underlying asset. A put option is the right but not the obligation to receive a strike price and deliver an underlying asset.

In the interest rate market, the main types of options are Caps and Floors, Bond options, and Swap options. A Cap is a series of Caplets and a Floor is a series of Floorlets. We will discuss Caplets and Floorlets in this paper.

Caplets and Floorlets are typically valued in the Black–Scholes–Merton framework:

$$\text{Caplet}_{t_s,t_l} = m\,PA\,e^{-r_l t_l}\,\{r_f N(d_1) - r_k N(d_2)\} \tag{4}$$

$$\text{Floorlet}_{t_s,t_l} = m\,PA\,e^{-r_l t_l}\,\{r_k N(-d_2) - r_f N(-d_1)\} \tag{5}$$

where $d_1 = \frac{\ln(\frac{r_f}{r_k}) + \frac{1}{2}\sigma^2 t_x}{\sigma\sqrt{t_x}}$ and $d_2 = d_1 - \sigma\sqrt{t_x}$

Caplet $_{t_s,t_l}$: option on an interest rate from time t_s to time t_l, $t_l > t_s$, i.e. the right but not the obligation to pay the rate r_K at time t_l.

Floorlet $_{t_s,t_l}$: option on an interest rate from time t_s to time t_l, $t_l > t_s$, i.e. the right but not the obligation to receive the rate r_K at time t_l.

m: time between t_s and t_l, expressed in years

PA: principal amount

t_x: option maturity, $t_x \le t_s < t_l$

r_f: forward interest rate, derived as $r_{f_{t_s,t_l}} = \left(\frac{df_{t_s}}{df_{t_l}} - 1\right)\left(\frac{1}{t_l - t_s}\right)$ where df is a discount factor, i.e. $df_{t_y} = 1/(1+r_y)$.

r_K: strike rate i.e. the interest rate that the Caplet buyer may pay and the Floorlet buyer may receive from time t_s to time t_l.

Equations (4) and (5) give the arbitrage-free tradable price of a Caplet and a Floorlet in the Black–Scholes–Merton framework. Partially differentiating equations (4) and (5) results in the risk parameters, which underlie hedging the risks of Caplets and Floorlets. See www.dersoft.com/greeks.doc for details.

The functions (4) and (5) satisfy the famous Black–Scholes–Merton PDE

$$D = \frac{\partial D}{\partial t}\frac{1}{i} + \frac{\partial D}{\partial S} S + \frac{1}{2}\frac{\partial^2 D}{\partial S^2}\frac{1}{i}\sigma^2 S^2 \tag{6}$$

D: financial derivatives as for example a Caplet or a Floorlet

i: discount rate

S: modeled variable

σ: volatility of S

showing that the equations (4) and (5) are arbitrage free[1].

4.2. Applying negative probabilities to Caplets and Floorlets

Our original problem is that the market applied lognormal distribution, which is underlying the valuation of interest rate derivatives, cannot model negative interest rates. Several solutions to this problem are possible.

1) We can model interest rates with an entirely different distribution as for example the normal distribution, which allows negative interest rates. This is done by Vasicek (1977), Ho and Lee (1986), and Hull and White (1990). However, empirical data shows that interest rate distribution is far from normal, see for example Chen and Scott 2002. Thus, the suggested solutions do not correctly reflect the reality. One could further argue that the use of the empirical interest rate distribution should underlie the option valuation process, as for example applied in the Omega function (see Keating and Shadwick 2002). However, there are two drawbacks of this approach. a) The empirical distribution might not be a good indication of future interest rate distributions. b) Numerical methods as Monte Carlo are necessary to value the option. Our approach outlined in point 3) below results in a convenient closed form for solution.
2) We can add a location parameter to the lognormal distribution. Hence equation (3) $\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2}$ becomes $\frac{1}{(x-\alpha)\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x-\alpha)-\mu}{\sigma}\right)^2}$, where α is the location parameter. For $\alpha > 0$, the lognormal distribution is shifted to the left. As a result, the probability distribution allows negative interest rates with a positive probability while maintaining the skew and kurtosis of the lognormal distribution. The problem is that shifting the entire probability distribution means also shifting the initial current value of the interest rate r. Hence the modeled starting value of r will be different from the current market interest rate r. This is inconsistent, especially for American style options, which determine early exercise opportunities.
3) A consistent way to model options on negative interest rates is to apply negative probabilities to equations (4) and (5). We add a parameter β to equations (4) and (5) and for simplicity we set m = 1 and PA =1. Hence we derive

$$\text{Caplet}_{t_s,t_l} = e^{-r_l t_l}\,\{r_f[N(d_1)-\beta] - r_k[N(d_2)-\beta]\}\ \beta \in \Re \tag{7}$$

$$\text{Floorlet}_{t_s,t_l} = e^{-r_l t_l}\,\{r_k[N(-d_2)-\beta] - r_f[N(-d_1)-\beta]\} \tag{8}$$

where $d_1 = \frac{\ln\left(\frac{r_f}{r_k}\right) + \frac{1}{2}\sigma^2 t_x}{\sigma\sqrt{t_x}}$ and $d_2 = d_1 - \sigma\sqrt{t_l}$

This brings us to negative probabilities which imply negative interest rates. Let's show this. From basic option theory we know that the value of a Caplet is divided into intrinsic value IV and time value TV:

$$\text{Caplet}_{t_s,t_l} = IV_{Caplet} + TV_{Caplet} \ge 0 \tag{9}$$

The intrinsic value is defined

$$IV_{Caplet} = \max(r - r_K, 0) \tag{10}$$

where r is the current value of the underlying interest rate.

Let's investigate the case of the Caplet being in-the-money and the Floorlet being out-of the money, i.e. $r > r_K$.[2] Hence equation (10) changes to

$$IV_{Caplet} = r - r_K \tag{11}$$

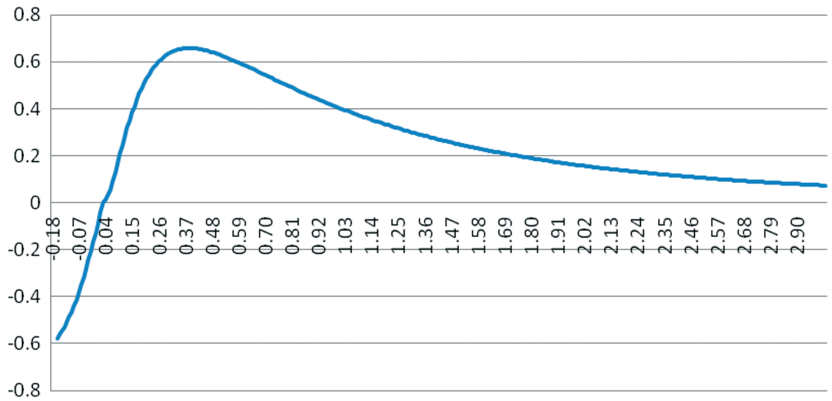
Since a Caplet does not pay a return, the time value of a Caplet is bigger than 0,

$$TV_{Caplet} \ge 0 \tag{12}$$

With a positive β, negative probabilities may emerge for certain input parameter constellations of in equations (7) and (8). I.e. $N(d_1)-\beta$, $N(d_2)-\beta$, $N(-d_2)-\beta$ and $N(-d_1)-\beta$ may become negative. In this case, from equations (7) and (8), the resulting Caplet price can become, especially for low implied volatility, smaller than the intrinsic value, i.e. $\text{Caplet}_{t_s,t_l} < IV_{Caplet}$. From equations (9), (11), and (12), this implies that $r < 0$ for small r_K. Hence we have an extension of the lognormal distribution, with negative probabilities associated with negative values for r (see Figure 2).

The higher the value of β, the more likely it is that negative probabilities with associated negative interest rates will emerge. This lowers Caplet prices and increases the Floorlet price, which is a desired result, since it adjusts the Caplet and Floorlet prices for the possibility of negative interest rates. The magnitude of the parameter β, that a trader applies, reflects a trader's opinion on the possibility of negative rates. A trader will use more extreme β-values if he/she believes strongly in the possibility of negative interest rates, vice versa.

Figure 2: Extended distribution with negative probabilities associated with negative interest rates.



5. Concluding Summary

We have defined extended probabilities, which include negative probabilities, and derived their general properties. Then we have applied extended probabilities to financial modeling. We have shown that negative nominal interest rates have occurred several times in the past in financial practice, as in the 2008/2009 global financial crisis. This is inconsistent with the conventional theoretical models of interest rates, which typically apply a lognormal distribution. In particular, when Caps and Floors are valued in a lognormal Black–Scholes–Merton framework, then the probability of negative interest rates is zero. Here negative probabilities come into play. We have shown that integrating negative probabilities in the Black–Scholes–Merton framework allows to consistently model negative nominal interest rates, which exist in financial practice.

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After a lectureship in mathematics and statistics at the Economic Academy Kiel, **Gunter Meissner** joined Deutsche Bank in 1990, trading interest rate futures, swaps, and options in Frankfurt and New York. He became Head of Product Development in 1994, responsible for originating algorithms for new derivatives products, which at the time were Lookback Options, Multi-asset Options, Quanto Options, Average Options, Index Amortizing Swaps, and Bermuda Swaptions. In 1995/1996 Gunter Meissner was Head of Options at Deutsche Bank Tokyo. From 1997 to 2007, Gunter was Professor of Finance at Hawaii Pacific University. Currently, he is President of Derivatives Software (www.dersoft.com), Adjunct Professor of Mathematical Finance at NYU (www.cims.nyu.edu) and Director of the Master in Financial Engineering program at the Shidler College of Business at the University of Hawaii (www.mfe.shidler.hawaii.edu). He has published numerous papers on derivatives and is a frequent speaker on conferences and seminars. He is author of 4 books, including his 2008 edition of "The Definitive Guide to CDOs – Application, Pricing, and Risk Management".

ENDNOTES

1. For a derivation of equation (6) see www.dersoft.com/BSMPDEgeneration.pptx. For a proof that the functions (4) and (5) satisfy the PDE (6), see www.dersoft.com/bspdeproof.doc
2. In a follow up paper we will discuss the case $r < r_K$ and bigger than unity probabilities.

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