

# Hybrid smiles made fast

Risk management of equity-linked structured notes requires consistent modelling of both stocks' smiles and stochastic interest rates. Existing approaches require costly computations to capture highly curved smiles – especially at long-dated maturities. *Messaoud Chibane and Dikman Law* show how a quadratic parameterisation of volatility, with some analytic approximations, can be much quicker

Since the late 1990s, equity-linked structured notes have become increasingly popular with retail investors, particularly in Japan. These products, such as enmansai and uridashi, provide some controlled exposure to stocks or indexes by embedding derivatives payouts with a floating rate in coupon payments. Given the dependence of option prices on forwards – and of the latter on rates – the interaction between the implied volatility smile of the underlying equities and the interest rate dynamics is a crucial aspect of hybrids' modelling. However, existing approaches fail to fit highly curved market smiles effectively, especially in the long-dated limit, without considerable computational effort.

Although extending the usual local and stochastic volatility models from deterministic drifts to a Hull-White diffusion for yield curve dynamics seems like an obvious first step, finding efficient calibration routines for such complex models is not straightforward. This problem has been solved for the case of hybrid Hull-White/Heston models by Andreasen (2006), but only under the strong constraint of zero correlation between rates and stock volatility. For a more general correlation structure, Van Haastrecht *et al* (2005) derive semi-analytical expressions for European options under the Hull-White/Schöbel-Zhu model. Other approaches are used in the related area of forex hybrids, such as using a constant elasticity of variance (CEV) parameterisation to control long-dated skew, but with poor fitting to the wings. A hybrid model with a general local volatility for the stock, that can fit smiles accurately without additional computational complexity, is lacking. This means a computationally intensive brute force method has to be used in the general case.

In this article, we use a quadratic parameterisation for the stock volatility, together with a Markovian projection to simplify the hybrid forward price dynamics to capture most smile shapes. The result is a quicker calibration with low error compared with the brute force approach. First, we introduce the hybrid model with local volatility for the stock and Hull-White dynamics for the rates. We then detail a quadratic parameterisation for the stock volatility and use Markovian projection and analytical techniques to simplify the forward price dynamics, which allows simple and fast calibration to the smile. Calibration to Nikkei data reduces computation time by orders of magnitude compared with a simple finite difference solution of the Dupire equation, and the error in pricing long-dated enmansai products is shown to be of the order of single-digit basis points for a range of up-and-out barrier levels in the underlying derivative. The method is adaptable to hybrids based on other asset classes, such as foreign exchange.

## The model

We consider the following dynamics for the underlying stock price:

$$\frac{dS(t)}{S(t)} = (r(t) - d(t))dt + \sigma_s(t, S(t))dW_s(t)$$

where  $W_s$  is a Brownian motion under the domestic risk-neutral measure. The volatility  $\sigma_s$  is a function of time and the prevailing spot price, the short rate  $r(t)$  is stochastic, while the dividend yield is deterministic.

We use the classic Hull-White model with Gaussian short rates, (see, for instance, Andersen & Piterbarg, 2010). The short rate and the discount bonds solve the following stochastic differential equations (SDEs):

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma_r(t)dW_r(t)$$

$$\langle dW_r(t), dW_s(t) \rangle = \rho(t)dt$$

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \Sigma(t, T)dW_r(t)$$

$$\Sigma(t, T) = \sigma_r(t) \int_t^T \exp(-\kappa(s-t))ds$$

where  $W_r$  is a Brownian motion under the risk-neutral measure,  $r$  represents the short-rate process and  $P(\cdot, T)$  is the price process of a discount bond maturing at time  $T$ .  $\kappa$  is a constant mean-reversion parameter,  $\sigma_r$  is the time-dependent volatility, while the time-dependent parameter  $\theta$  is calibrated to the initial term structure. The rates-equity instantaneous correlation  $\rho$  is a time-dependent function. The model is popular as the bond price admits a simple affine formula.

## A quadratic local volatility parameterisation

We use a quadratic function of the log in the moneyness to parameterise the stock instantaneous volatility function, thereby adding a degree of freedom over CEV approaches, which will be used to fit the smile curvature. The dynamics of the forward stock price process are:

$$\frac{dF(t, T)}{F(t, T)} = \sigma_s(t)dW_s(t) + \Sigma(t, T)dW_d(t)$$

$$\sigma_s(t) = a(t)X^2(t) + b(t)X(t) + c(t) \quad (1)$$

$$X(t) = \ln\left(\frac{S(t)}{F(0, t)}\right)$$

where:

$$F(t, T) = S(t) \frac{P_d(t, T)}{P(t, T)}$$

$$P_d(t, T) = e^{-\int_0^t d(s) ds}$$

and the discount bond  $P(t, T)$  has a simple analytic expression (see Andersen & Piterberg, 2010). It should be intuitively clear that while constant  $c$  controls the at-the-money volatility level, coefficients  $a$  and  $b$  will respectively control the curvature and the slope of the smile around the at-the-money forward. We can rewrite the SDE (4) in a simpler form as:

$$\frac{dF(t, T)}{F(t, T)} = \lambda(t, T) dW_T(t)$$

$$\lambda^2(t, T) = \sigma_S^2(t) + 2\rho(t) \sigma_S(t) \Sigma(t, T) + \Sigma^2(t, T) \quad (2)$$

$$dW_T = \frac{\sigma_S(t) dW_S(t) + \Sigma(t, T) dW_d(t)}{\lambda(t, T)}$$

However, because of the dependency of the forward volatility  $\lambda$  on the spot price, this does not allow for closed-form expressions for vanilla option prices, which would help calibration. To circumvent this technical hurdle, we use a Markovian projection technique to obtain a local volatility model approximation for the original dynamics. Therefore we use results derived in Gyöngy (1986) to come up with a related process defined by:

$$\frac{d\hat{F}(t, T)}{\hat{F}(t, T)} = \hat{\lambda}(t, T) dW_T(t)$$

$$\hat{F}(0, T) = F(0, T) \quad (3)$$

$$\hat{\lambda}^2(t, T) = E[\lambda^2(t, T) | F(t, T)]$$

The projected process defined by (3) has the same marginal distributions as the original forward process  $F(\cdot, T)$  for any time  $t \in [0, T]$ , and so gives the same European option prices, that is:

$$\forall K: E^T[(S(T) - K)^+] = E^T[(\hat{F}(T, T) - K)^+]$$

$$x^+ = \max(x, 0)$$

Furthermore, the volatility  $\hat{\lambda}$  depends only on calendar time and forward level, which makes (3) a pure local volatility model. So the smile fitting problem becomes one of pricing European vanilla options on the final forward value  $\hat{F}(T, T)$ .

By using Gaussian approximations similar to those introduced in Piterberg (2006), we first obtain:

$$E[X(t) | F(t, T)] = \ln\left(\frac{F(0, T) Q_0(t, T)}{F(0, t)}\right)$$

$$+ \ln\left(\frac{F(t, T)}{F(0, T)}\right) \left(1 + \frac{C_{Q,F}(t, T)}{C_F(t, T)}\right) + \frac{1}{2}(C_{Q,F}(t, T) + C_Q(t, T)) \quad (4)$$

with:

$$C_{Q,F}(t, T) = \int_0^t (\Sigma(s, T) - \Sigma(s, t)) (\Sigma(s, T) + \rho(s) c(s)) ds$$

$$C_F(t, T) = \int_0^t (\Sigma^2(s, T) + c^2(s) + 2\rho(s) \Sigma(s, T) c(s)) ds$$

$$C_Q(t, T) = \int_0^t (\Sigma(s, T) - \Sigma(s, t))^2 ds$$

For higher-order terms, similar expressions could be obtained in the Gaussian framework, but for the sake of simplicity we neglect convexity and make the following crude approximations:

$$E[X^2(t) | F(t, T)] = E[X(t) | F(t, T)]^2$$

$$E[\sigma_S^2(t) | F(t, T)] = E[\sigma_S(t) | F(t, T)]^2$$

In conclusion, we approximate the exact Markovian projection defined in (3) by:

$$\frac{d\bar{F}(t, T)}{\bar{F}(t, T)} = \bar{\lambda}(t, T, F(t, T)) dW_T(t)$$

$$\bar{F}(0, T) = F(0, T)$$

$$\bar{\lambda}^2(t, T, x) = \bar{\sigma}_S^2(t, T, x) + 2\rho(t) \Sigma(t, T) \bar{\sigma}_S(t, T, x) + \Sigma^2(t, T)$$

$$\bar{\sigma}_S(t, T, x) = a(t) f^2(t, T, x) + b(t) f(t, T, x) + c(t) \quad (5)$$

$$f(t, T, x) = \ln\left(\frac{F(0, T) Q_0(t, T)}{F(0, t)}\right)$$

$$+ \ln\left(\frac{x}{F(0, T)}\right) \left(1 + \frac{C_{Q,F}(t, T)}{C_F(t, T)}\right) + \frac{1}{2}(C_{Q,F}(t, T) + C_Q(t, T))$$

Now we project the approximate local variance function  $\lambda^2(t, T)$ , which is currently a fourth-order polynomial in the log moneyness, on to a quadratic function of spot, as opposed to the shifted lognormal dynamics proposed in Piterberg (2006). This strategy preserves tractability and speed while improving accuracy, as we will show later.

This is done by first matching the derivatives of the two volatility functions up to order two. In other words, we need to find functions of time  $\alpha_i, \beta_i, \zeta_i$  such that:

$$d\bar{F}(t, T_i) \approx \tilde{\lambda}(t, T_i) dW_T(t)$$

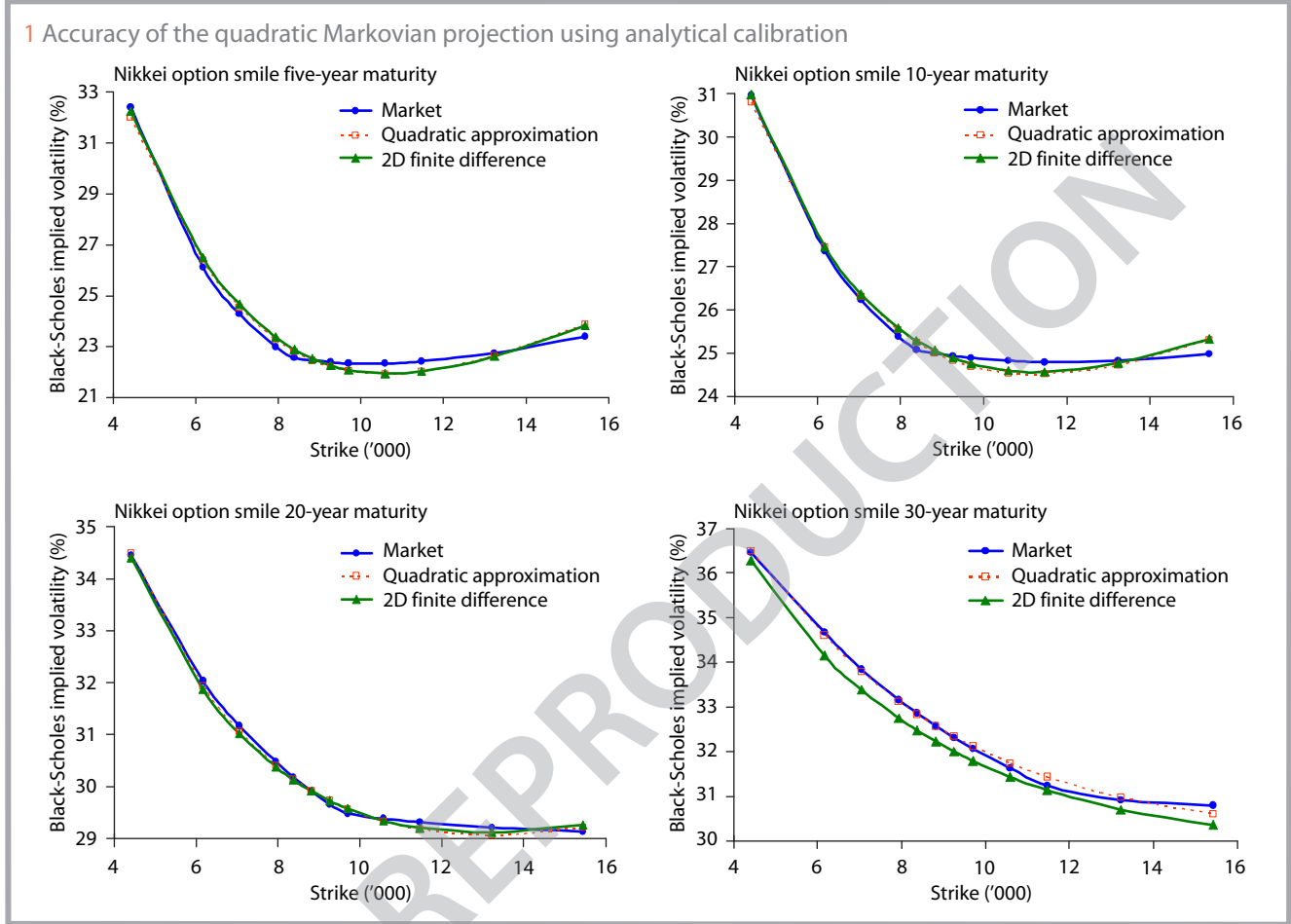
$$\tilde{\lambda}(t, T_i) = \alpha_i(t) (\beta_i(t) \bar{F}(t, T_i) + (1 - B_i(t)) F(0, T_i) + \zeta_i(t) (\bar{F}(t, T_i) - F(0, T_i))^2)$$

$$\forall k = 0, 1, 2: \left. \frac{\partial^k \tilde{\lambda} F}{\partial F^k} \right|_{F=F(0, T_i)} = \left. \frac{\partial^k \bar{\lambda}}{\partial F^k} \right|_{F=F(0, T_i)}$$

Once this is done, we can use the averaging formulas for pricing options within a quadratic local volatility model as detailed in Andersen (2010) and Chibane (2011) and transform any time dependent quadratic volatility model on to a constant coefficient one to get the equivalent model:

$$d\bar{F}(t, T_i) = \bar{\lambda}(t, T_i) dW_T(t)$$

with:



$$\bar{\lambda}(t, T_i) = \bar{\alpha}_i \left( \bar{\beta}_i \bar{F}(t, T_i) + (1 - \bar{\beta}_i) F(0, T_i) + \bar{\zeta}_i (\bar{F}(t, T_i) - F(0, T_i))^2 \right)$$

Calibration via computation of the constant coefficients  $\bar{\alpha}_i$ ,  $\bar{\beta}_i$ ,  $\bar{\zeta}_i$  is simple (see Chibane & Law, 2013).

**Calibration comparison**

Calibration of the approximation process is compared with a finite difference solution of the generalised Dupire partial differential equation:

$$\frac{\partial C}{\partial T} + d(t) \left( C - K \frac{\partial C}{\partial K} \right) - \frac{1}{2} \sigma^2(T, K) K^2 \frac{\partial^2 C}{\partial K^2} - KE \left[ D(T) r(T) 1_{S(T) > K} \right] = 0 \quad (6)$$

where  $d$  is the dividend rate and  $D$  the discount factor.

The calibration is run on Nikkei equity option smiles, up to 30-year maturities, as of August 31, 2012 (see figure 1). The quadratic approximation is compared with a two-dimensional finite difference scheme on a 20 steps per year by 20 rates, and 400 equity, states grid, both run on a 3.46 GHz Xeon central processing unit. It calibrates in 0.2 seconds, compared with 25 for the finite difference scheme, with comparable accuracy. The

maximum error at-the-money was found at the 30-year expiry and amounted to 35 basis points of Black-Scholes implied volatility. At the long-end of the smile, the fit is arguably more accurate.

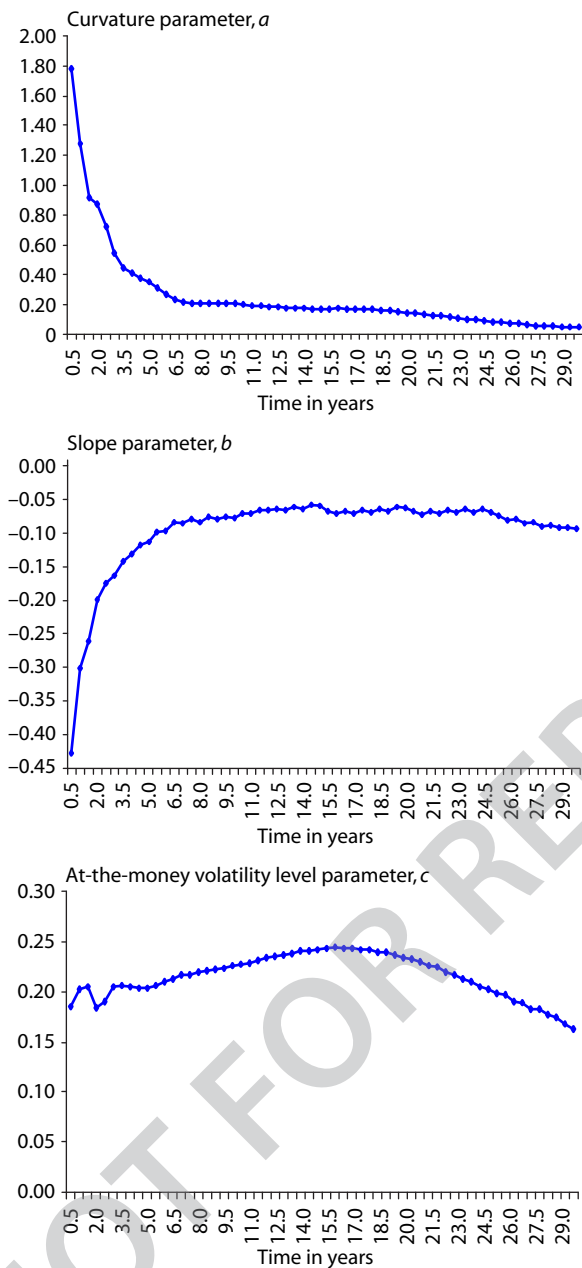
It is always interesting to look at the term structure of model parameters (see figure 2). The curvature is very pronounced at short, under five-year, maturities before dropping off as the smile flattens. The slope is large and negative at first but flattens out above six-and-a-half years, indicating declining skew and a roughly log-normal smile in the limit. The at-the-money volatility is relatively stable, rising slightly to middle-length maturities before dropping at the long-end.

**Impact on pricing enmansai**

To investigate the impact of Gaussian approximations and the quadratic volatility averaging formulas on its accuracy for pricing long-dated structures, we apply the analytical calibration method to the pricing of equity-linked conditional trigger swaps, known as enmansai in the Japanese market. Basically, these are swap contracts where the funding leg is paying or receiving Libor coupons while the structured leg is receiving or paying equity-linked structured coupons. These could be digital options on the equity index (for example, the Nikkei), for instance, which pays a high coupon if the index is above a strike level and a low coupon otherwise. The structure often has a knock-out feature, so that when the equity index is above the barrier level the structure ceases to exist.

The same set of market data as presented in the previous section

## 2 Term structure of model parameters



is used. The strike is fixed at 11,000 and the model prices are calculated using the analytical calibration method for different choices of maturities and up-and-out barrier levels. The swap pays a semiannual digital coupon, with the low and high rates as described above set at 0.1% and 2% respectively, and receives a Libor six-month coupon.

The model prices are compared with an exact calibration based on a brute force calibration method using a two-dimensional finite difference scheme. The error due to the Gaussian approximation and the quadratic volatility averaging formulas is given in table A.

Clearly, errors are quite small between types of calibrations, that is, less than 3bp for all maturities and barrier levels. It suggests

## A. Enmansai pricing error (percentage points of notional)

Maturity	Barrier level					
	13,000	14,000	15,000	16,000	17,000	18,000
5-year	-0.0094	-0.0112	-0.0112	-0.0115	-0.0119	-0.0119
10-year	-0.0235	-0.0236	-0.0235	-0.0233	-0.0249	-0.0244
15-year	-0.0309	-0.0299	-0.0287	-0.0274	-0.0263	-0.0254
20-year	-0.0247	-0.0229	-0.0211	-0.0194	-0.0180	-0.0167
25-year	-0.0153	-0.0131	-0.0113	-0.0099	-0.0087	-0.0078
30-year	-0.0063	-0.0037	-0.0022	-0.0013	-0.0010	-0.0010

that the approximation does not significantly reduce the accuracy of path-dependent option prices.

## Conclusion

We have extended the hybrid pricing framework introduced in the forex context by Piterberg (2006) by making the underlying instantaneous volatility a quadratic function of the log moneyness, thereby adding a degree of freedom. This allows for control of the slope and curvature of the smile around the forward. By approximating the forward stock dynamics by a quadratic volatility model rather than a shifted lognormal, tractability and calibration speed is retained while more long-dated convex smiles are reproduced more accurately. Still, biases are observed for very long-dated options (for example, 25 and 30 years). One of the major sources of these errors is the Gaussian approximation. Further research could be done to derive a more accurate approximation step for the Markovian projection. Finally, with the constant interest in mixed stochastic/local volatility models among practitioners, extending the calibration technique to incorporate stochastic volatility in the equity/forex rate dynamics is the next step. ■

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