# Hull & White Convexity Adjustments for Credit – Riskless Interest Rate Swaps Under CSA

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### Abstract
Using a multicurve pricing framework has become standard market practice for investment banks. A new IBOR curve bootstrapping procedure is now in use, consisting of discounting using an OIS curve. The curve obtained allows us to recover market forwards corresponding to a "CSA" world which is the interbank world. Currently, this curve is being used to calculate forwards for non-CSA trades, which represents an important approximation since it does not take into account the convexity adjustment implied when changing from "CSA" to "non-CSA" probability measure. By reducing counterparty risk using CSA contracts, new risk factors have arisen that are left unhandled. We show in this article how to calculate non-CSA forwards convexity adjustment. This adjustment depends on collateral and funding rates volatilities as well as correlation between both. We take into account these parameters under one-factor Gaussian short-rate models, and give a detailed development for the Hull & White specific case. These closed formulae allow in turn fast bootstrapping procedures and therefore potential risk management for non-CSA swaps.

### Keywords
CSA Convexity, IRS, OIS-IBOR dynamics, Hull-White, CSA Unmanaged Risk

## Framework
The letters $C$ and $F$, when used as subscripts, will refer respectively to "COLLATERAL" and "FUNDING." Note as well that we will use the terms "funding rate" and "IBOR rate" indifferently, as well as the terms "collateral rate" and "OIS rate."

## Financial notation
$r_C$ short rate at which collateral grows
$r_F$ short rate corresponding to unsecured funding rate
$P_C(t,T)$ value at time $t$ of a zero coupon maturing at time $T$, associated with cost of collateral
$P_F(t,T)$ value at time $t$ of a zero coupon maturing at time $T$, associated with cost of funding
$f_C(t,T)$ instantaneous forward collateral rate defined by $f_C(t,T) = -\dfrac{\partial \ln P_C(t,T)}{\partial T}$
$f_F(t,T)$ instantaneous forward funding rate defined by $f_F(t,T) = -\dfrac{\partial \ln P_F(t,T)}{\partial T}$
$L(T_1,T_2)$ IBOR rate fixing at time $T_1$ and whose tenor is $T_2-T_1$
$F_C(t,T_1,T_2)$ value at time $t$ of the CSA forward rate associated with IBOR rate $L(T_1,T_2)$
$F_F(t,T_1,T_2)$ value at time $t$ of the non-CSA forward rate associated with IBOR rate $L(T_1,T_2)$

## Mathematical notation
The mathematical notation used is summarized in the following table. For the sake of simplicity, we used the same notation for Brownian motions and filtrations under risk-neutral and forward measures.

| | Monocurve | Funding | CSA |
|---|---|---|---|
| Risk-neutral measure | $\mathbb{Q}$ | $\mathbb{Q}_F$ | $\mathbb{Q}_C$ |
| Numéraires associated | $B$ | $B_F$ | $B_C$ |
| Brownian motions | $W$ | $W_F$ | $W_C$ |
| Filtrations associated | $\mathcal{F}$ | $\mathcal{F}_F$ | $\mathcal{F}_C$ |
| Expectations | $\mathbb{E}^{\mathbb{Q}}$ | $\mathbb{E}^{\mathbb{Q}_F}$ | $\mathbb{E}^{\mathbb{Q}_C}$ |
| $T$-forward measure | $\mathbb{Q}_T$ | $\mathbb{Q}_F^T$ | $\mathbb{Q}_C^T$ |
| Numéraires associated | $P(.,T)$ | $P_F(.,T)$ | $P_C(.,T)$ |
| Brownian motions | $W$ | $W_F$ | $W_C$ |
| Filtrations associated | $\mathcal{F}$ | $\mathcal{F}_F$ | $\mathcal{F}_C$ |
| Expectations | $\mathbb{E}^{\mathbb{Q}_T}$ | $\mathbb{E}^{\mathbb{Q}_F^T}$ | $\mathbb{E}^{\mathbb{Q}_C^T}$ |

We use subscript $t$ for conditional expectations. For instance, $\mathbb{E}_t^{\mathbb{Q}}$ denotes expectation conditional on $\mathcal{F}(t)$.

To distinguish Brownian motions under the same measure, we use the subscript linked to the asset. For instance, $W_S$ will denote asset $S$'s Brownian motion.

## Motivation

Before CSA swaps became a market standard, futures were already subject to margin calls implying a convexity adjustment. The latter can be seen as the difference between risk-neutral and forward-neutral expectations of an IBOR rate (see Andersen and Piterbarg, 2010, section 16.8 for a detailed demonstration):

$$CVX(t) = E_t^{\mathbb{Q}}[L(T_1, T_2)] - E_t^{\mathbb{Q}_{T_2}}[L(T_1, T_2)]$$

Calculating $E_t^{\mathbb{Q}}[L(T_1,T_2)]$ is not trivial and currently investment banks use a model-dependent approach. A commonly used adjustment is implied by a Hull & White model on short rate, leading to

$$CVX(t) = \frac{\sigma^2 B(T_1, T_2)}{4\,a\,\tau(T_1, T_2)}\left[B(T_1, T_2)(1 - e^{-2aT_1}) + 2\,a\,B(0, T_1)^2\right]$$

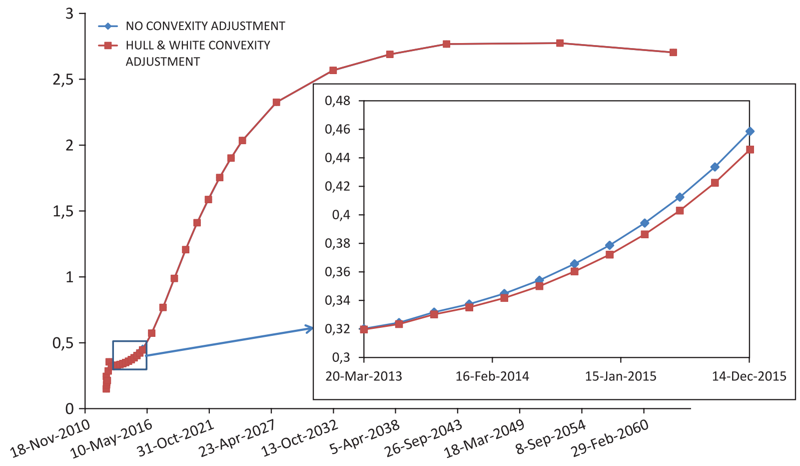
where $B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$, $a$ denotes short-rate mean reversion, and $\sigma$ its volatility as shown in Kirikos and Novak (1997). It is typically the adjustment one can find on Bloomberg using ICVS.

Now the question we have to ask ourselves is, "How does this adjustment impact the yield curve?"

Figure 1 answers our question quickly: using a fixed volatility at 1%, we obtain an adjustment magnitude at about a basis point for a 3-year maturity. Although this impact is not neglectable and affects forwards, it does not affect significantly the overall yield curve shape.

Let us now focus on collateralized swaps, which are used for the long term of the yield curve. Just like futures, collateralized swaps are subject to margin calls and therefore their pricing involves a convexity adjustment. Currently, banks take into account this adjustment by separating discounting – using an OIS curve corresponding to the rate at which collateral grows – from forwards calculation. However, the forwards obtained this way correspond to the "CSA-forward" measure, that is, $F_C(t, T_1, T_2) = \mathbb{E}_t^{\mathbb{Q}_C^{T_2}}[L(T_1, T_2)]$. These are the forwards we can retrieve from the – interbank – market, since swaps are collateralized.

**Figure 1: Impact of futures convexity adjustment on zero-coupon rates**

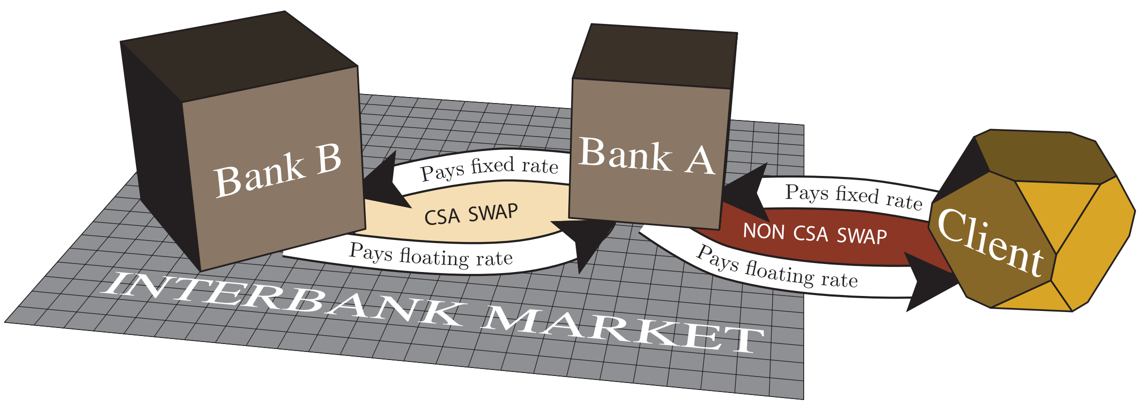



In order to price non-collateralized deals, we need forwards under the "non-CSA-forward" measure, that is, $F_F(t, T_1, T_2) = \mathbb{E}_t^{\mathbb{Q}_F^{T_2}}[L(T_1, T_2)]$. These are not directly observed in the market, and therefore have to be deduced from $F_C(t,T_1,T_2)$. Although it is clear in the literature that this convexity adjustment exists (see Mercurio, 2010; Piterbarg, 2010; Bianchetti and Carlicchi, 2011, for instance), there are currently no studies of its impact on the long end of the yield curve.

The approach we present in this article consists of modeling OIS and IBOR short rates separately using correlated Hull & White dynamics. Under this framework we obtain closed formulae linking the CSA forward $F_C(t,T_1,T_2)$ to the non-CSA forward $F_F(t,T_1,T_2)$. Our aim is not to specify a new yield curve bootstrap procedure since the adjustment depends on OIS volatility and OIS–IBOR correlation which are difficult to observe, but rather to quantify the risk implied by these parameters which is currently being underestimated. By reducing counterparty risk using CSA contracts, new risk factors have arisen that are left unhandled.

Most importantly, a typical setup involves a bank having a non-collateralized trade with a client, hedged with a collateralized trade (see Figure 2). Therefore, in order to hedge properly, the non-collateralized trade sensitivities have to be monitored as precisely as possible. Our work shows how important it is to take into account OIS–IBOR correlation, as well as their respective volatilities.

**Figure 2: Non-CSA swap hedged by a CSA swap**



## Theoretical CSA convexity adjustment and market practice

Our developments are based on the collateralized derivatives valuation framework introduced by Fujii *et al.* (2009) and generalized to the case of partial collateralization by Piterbarg (2010). Note that, for the sake of simplicity, we do not take into account different types of collateralization. When we mention a "collateralized deal" we mean a bilateral fully cash collateralized deal with daily margin calls. Moreover, we focus on the specific case of mono-currency swaps collateralized in their own currency. We therefore do not incorporate the general case of collateral currency different from the swap's currency, nor the option of collateral currency choice. While this generalization is important as it concerns many CSA contracts, it would considerably complicate things.

Consider now an IBOR rate $L(T_1,T_2)$ paid at $T_2$. If this cashflow is part of a collateralized deal its present value writes as $V_C(t) = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^{T_2} r_C(s)ds}\, L(T_1,T_2)]$, $r_C$ being the rate at which collateral grows. Expressing the previous result under the "CSA-forward" measure $\mathbb{Q}_C^{T_2}$ associated with the numéraire $P_C(.,T_2)$ leads to $V_C(t) = P_C(t,T_2)\mathbb{E}_t^{\mathbb{Q}_C^{T_2}}[L(T_1,T_2)]$.

Similarly, the value of the same cashflow not being collateralized writes as, under the "non-CSA-forward" measure $\mathbb{Q}_F^{T_2}$ associated with the numéraire $P_F(.,T_2)$, $V_F(t) = P_F(t,T_2)\mathbb{E}_t^{\mathbb{Q}_F^{T_2}}[L(T_1,T_2)]$.

As interbank swaps are collateralized, non-CSA forwards are not directly observed in the market and have to be deduced from CSA forwards using a convexity adjustment, which writes as $CVX_{CSA}(t) = E_t^{\mathbb{Q}_C^{T_2}}[L(T_1,T_2)] - E_t^{\mathbb{Q}_F^{T_2}}[L(T_1,T_2)]$. Currently, market practitioners neglect this impact and use CSA forwards directly for non-CSA swaps pricing. This is due to the difficulty of calculating the adjustment, and also to the fact that the main concern for non-CSA swaps is counterparty risk.

While counterparty risk for non-CSA swaps is fundamental, we demonstrate hereafter that non-CSA forwards convexity adjustment should not be neglected. To measure purely the impact of this adjustment, we do not account for counterparty risk. In the next section we set up a framework using one-factor Gaussian short-rate models for collateral and funding rates. Our framework is similar to those introduced in Piterbarg (2010) and Kenyon (2010), but we focus specifically on calculating non-CSA forwards for which we obtain closed formulae. These formulae allow, in turn, fast boostrapping procedures and potential risk management via sensitivities computations.

## Evaluating CSA adjustment using Gaussian short-rate models

Note that we are assuming banks raise funds at IBOR rate and therefore we use indifferently "funding rate" and "IBOR rate." In practice, banks raise funds at IBOR plus a funding spread which must be taken into account, as explained in Whittall (2010). Moreover, the funding spread is stochastic and therefore a proper framework would imply modeling this spread separately and correlating it to IBOR and OIS rates.

However, our first aim is to measure the impact of OIS–IBOR joint distribution on non-CSA forwards. The next step would be to extend the model by incorporating a stochastic funding spread.

### Modeling framework

In order to model the funding short rate $r_F$ and collateral short rate $r_C$, we use one-factor Gaussian short-rate models. The $r_F$ and $r_C$ dynamics under $\mathbb{Q}$ write as

$$dr_C(t) = \mu_C(t, r_C(t))\,dt + \sigma_C(t)\,dW_C^{\mathbb{Q}}(t)$$

$$dr_F(t) = \mu_F(t, r_F(t))\,dt + \sigma_F(t)\,dW_F^{\mathbb{Q}}(t)$$

where $\sigma_C$, $\sigma_F$, $\mu_C$, and $\mu_F$ are deterministic functions. The $\mu$ functions determine the type of Gaussian model considered. For instance, $\mu_C(t, r_C(t)) = \frac{\partial f_C}{\partial t}(0,t) + \sigma_C^2 t$ defines a Ho & Lee model for $r_C$.

Let $\rho_{C,F}(t) = \langle dW_C^{\mathbb{Q}}(t), dW_F^{\mathbb{Q}}(t)\rangle/dt$ denote the instantaneous correlation between $r_C$ and $r_F$ Brownian motions. Under this framework, we can express the zero-coupon dynamics as

$$\frac{dP_C(t,T)}{P_C(t,T)} = r_C(t)\,dt + \Gamma_C(t,T)\,dW_C^{\mathbb{Q}}(t)$$

$$\frac{dP_F(t,T)}{P_F(t,T)} = r_F(t)\,dt + \Gamma_F(t,T)\,dW_F^{\mathbb{Q}}(t)$$

$\Gamma_C$ and $\Gamma_F$ being deterministic. Moreover, zero coupons can be expressed as follows:

$$P_C(t,T) = A_C(t,T)\,e^{-B_C(t,T)\,r_C(t)}$$

$$P_F(t,T) = A_F(t,T)\,e^{-B_F(t,T)\,r_F(t)}$$

with $A_C$, $A_F$, $B_C$, and $B_F$ being deterministic as well.

### Adjustment calculation

Using the previous results we obtain the following dynamics for $r_F$ under $\mathbb{Q}_C^{T_2}$ and $\mathbb{Q}_F^{T_2}$:

$$dr_F(t) = [\mu_F(t, r_F(t)) - \rho_{C,F}(t)\,\sigma_F(t)\,\Gamma_C(t,T_2)]\,dt + \sigma_F(t)\,dW_F^{\mathbb{Q}_C^{T_2}}(t)$$

$$dr_F(t) = [\mu_F(t, r_F(t)) - \sigma_F(t)\,\Gamma_F(t,T_2)]\,dt + \sigma_F(t)\,dW_F^{\mathbb{Q}_F^{T_2}}(t)$$

In order to link the $r_F$ distribution under $\mathbb{Q}_C^{T_2}$ with its distribution under $\mathbb{Q}_F^{T_2}$, we introduce $\widehat{r}_F(T_1) = r_F(T_1) - cvx(t,T_1,T_2)$ with $cvx$ such that the $\widehat{r}_F(t)$ distribution under $\mathbb{Q}_C^{T_2}$ matches the $r_F$ distribution under $\mathbb{Q}_F^{T_2}$. The idea of introducing $\widehat{r}_F$ is based on the fact that $r_F$ has deterministic drifts under both measures $\mathbb{Q}_C^{T_2}$ and $\mathbb{Q}_F^{T_2}$, and therefore we can define a deterministic function $cvx$ linking these dynamics. This allows, in turn, a straightforward calculation of the adjustment:

$$\begin{aligned}
L(T_1,T_2) &= \frac{1}{T_2-T_1}\left(\frac{1}{P_F(T_1,T_2)} - 1\right)\\
&= \frac{1}{T_2-T_1}\left(\frac{1}{A_F(T_1,T_2)}\,e^{B_F(T_1,T_2)\,r_F(T_1)} - 1\right)\\
&= e^{B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\,\frac{1}{T_2-T_1}\\
&\quad\times\left(\frac{1}{A_F(T_1,T_2)}\,e^{B_F(T_1,T_2)\,\widehat{r}_F(T_1)} - e^{-B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\right)\\
&= e^{B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\left[\frac{1}{T_2-T_1}\left(\frac{1}{A_F(T_1,T_2)}\,e^{B_F(T_1,T_2)\,\widehat{r}_F(T_1)} - 1\right)\right.\\
&\quad\left.+\frac{1}{T_2-T_1}\left(1 - e^{-B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\right)\right]\\
&= e^{B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\left[\widehat{L}(T_1,T_2) + \frac{1}{T_2-T_1}\left(1 - e^{-B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\right)\right]
\end{aligned}$$

where we introduced the notation

$$\widehat{L}(T_1,T_2) = \frac{1}{T_2-T_1}\left(\frac{1}{A_F(T_1,T_2)}\,e^{B_F(T_1,T_2)\,\widehat{r}_F(T_1)} - 1\right).$$

As the $\widehat{r}_F(t)$ distribution under $\mathbb{Q}_C^{T_2}$ matches the $r_F$ distribution under $\mathbb{Q}_F^{T_2}$, we have that $L(T_1,T_2)$ under $\mathbb{Q}_F^{T_2}$ is equally distributed with $\hat{L}(T_1,T_2)$ under $\mathbb{Q}_C^{T_2}$. Therefore:

$$\mathbb{E}_t^{\mathbb{Q}_C^{T_2}}[L(T_1,T_2)] = e^{B_F(T_1,T_2)\,cvx(t,T_1,T_2)} \times \left[\mathbb{E}_t^{\mathbb{Q}_F^{T_2}}[L(T_1,T_2)] + \frac{1}{T_2-T_1}\left(1-e^{-B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\right)\right]$$

This finally allows us to link the CSA forward $F_C$ to the non-CSA forward $F_F$:

$$F_C(t,T_1,T_2) = e^{B_F(T_1,T_2)\,cvx(t,T_1,T_2)}\,F_F(t,T_1,T_2) + \frac{1}{T_2-T_1}\left(e^{B_F(T_1,T_2)\,cvx(t,T_1,T_2)}-1\right)$$

Now, the task that is left is to specify a one-factor Gaussian model in order to calculate the *cvx* term. We show hereafter how to calculate this term using Hull & White models.

## Using Hull & White models

Considering the Hull & White dynamics for $r_F$, we have

$$\mu_F(t,r_F(t)) = \theta_F(t) - a_F r_F(t)$$

with the $\theta_F$ function allowing us to recover the initial interest rate term structure:

$$\theta_F(t) = \frac{\partial f_F}{\partial t}(0,t) + a_F f_F(0,t) + \frac{\sigma_F^2}{2a_F}\left(1-e^{-2a_F t}\right)$$

$\Gamma_F$, $B_F$, and $A_F$ write as

$$\Gamma_F(t,T_2) = \sigma_F(t)\left(\frac{1-e^{-a_F(T_2-t)}}{a_F}\right)$$

$$B_F(T_1,T_2) = \frac{1}{a_F}\left[1-e^{-a_F(T_2-T_1)}\right]$$

$$A_F(T_1,T_2) = \frac{P_F(0,T_2)}{P_F(0,T_1)}\exp\left(B_F(T_1,T_2)f_F(t,T_1) - B_F(T_1,T_2)^2\,\frac{1}{2}\int_0^{T_1} e^{-2a_F(T_1-t)}\sigma_F^2(t)\,dt\right)$$

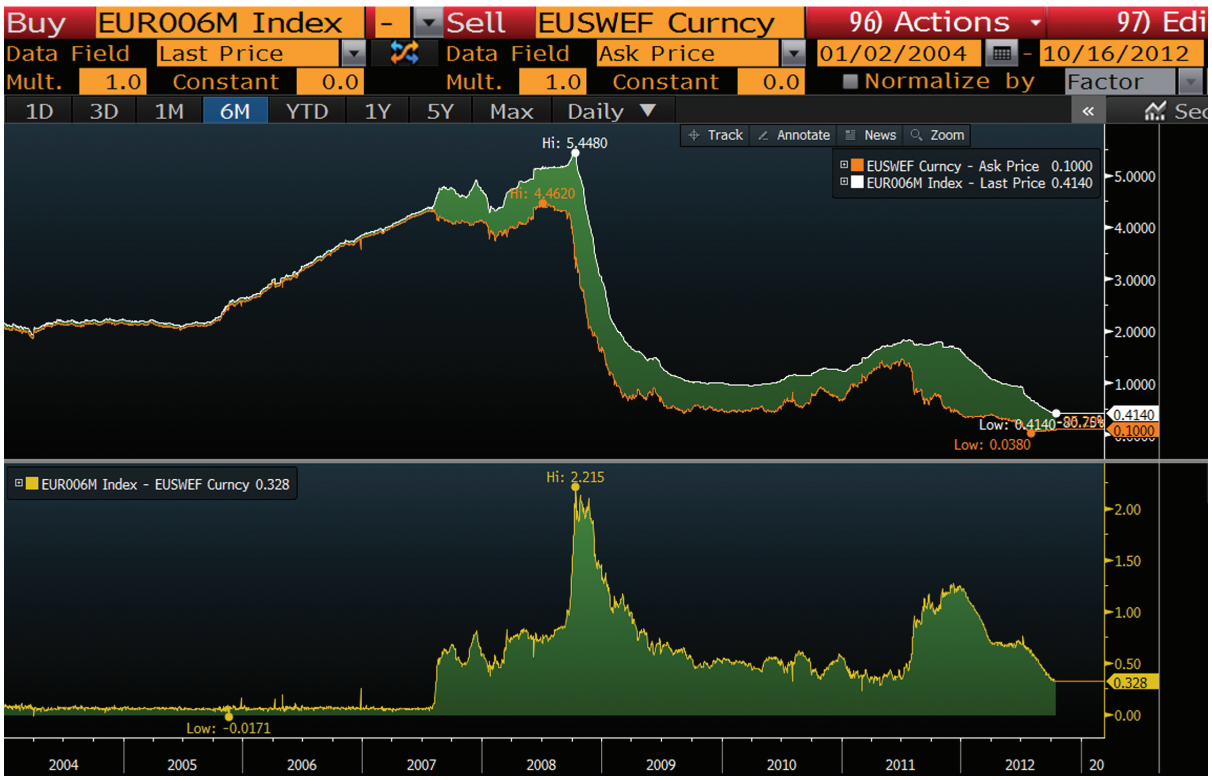
Replacing the subscript $F$ with $C$ allows us to write the corresponding functions for the collateral short rate $r_C$. After calculation (see Appendix B.1 for more details), we obtain

$$cvx(t,T_1,T_2) = \int_t^{T_1} e^{-a_F(T_1-s)}\left[\Gamma_F(s,T_2)\,\sigma_F(s) - \rho_{C,F}(s)\Gamma_C(s,T_2)\sigma_F(s)\right]ds$$

which can be analytically integrated. Assuming constant volatilities[1] leads to (see Appendix B.2 for calculation details)

$$\begin{aligned} cvx(t,T_1,T_2) = {}& \frac{\sigma_F}{a_F}\left(\frac{\sigma_F}{a_F} - \rho_{C,F}\frac{\sigma_C}{a_C}\right)\left(1-e^{-a_F(T_1-t)}\right) \\ & - \frac{\sigma_F^2}{2a_F^2}e^{-a_F(T_1+T_2)}\left(e^{2a_F T_1} - e^{2a_F t}\right) \\ & + \rho_{C,F}\frac{\sigma_C\sigma_F}{a_C(a_F+a_C)}e^{-a_F T_1 - a_C T_2}\left(e^{(a_F+a_C)T_1} - e^{(a_F+a_C)t}\right) \end{aligned}$$

**Figure 3: EONIA 6M swap vs. EURIBOR 6M from 2004 to 2012**




**Source: Bloomberg**

Now we have obtained a closed formula for the *cvx* term, we have explicitly linked the CSA forward $F_C(t,T_1,T_2)$ to the non-CSA forward $F_F(t,T_1,T_2)$ and we are able to study the impact of CSA adjustment on the yield curve.

## Impact measurement under Hull & White framework

### Decorrelation impact

Decorrelation between OIS and IBOR rates is the key to understanding the impact of the adjustment. The decorrelation that appeared in 2007 was an important factor that led banks to separate OIS discounting from forwards calculation. Nothing can make it more obvious than the historical chart of the OIS–IBOR spread presented as Figure 3. More precisely, the EURIBOR 6-month rate is plotted against a 6-month EONIA swap from 2004 to 2012.

The graphs in Figure 4 show the impact on the yield curve obtained when taking into account decorrelation between OIS and IBOR rates. The blue curve corresponds to traditional monocurve bootstrapping of a 6-month EURIBOR swap curve. The red curve is obtained using OIS discounting. We see that they have similar levels; the difference can be seen if we look at the zero-coupon rates shown in Table A.1, in which we note a basis point difference for the 30-year maturity.

All the other curves are obtained taking into account convexity with constant 1% volatility for OIS and IBOR rates and for different correlation levels varying from 100% to 70%. We can note the importance of the impact as decorrelation becomes more important.

The violet curve is obtained taking into account convexity with constant 1% volatility for OIS and IBOR rates and for a correlation level of 80%. 100% correlation would correspond to a pre-crisis scenario (before 2007) and 80% would correspond to a decorrelation that could be observed under stressed market conditions.

We notice an important downward shift on both the yield curve and the forward curve as decorrelation becomes more important. Even with 90% correlation we obtained for the 50Y maturity a rate 16 basis points lower than what we obtained with a typical OIS bootstrap and a 34 basis points impact on the forward rate (absolute difference). Numerical results are presented in Tables A.1 and A.2.

This alerts us to the importance of taking into account the adjustment for risk management purposes: all non-collateralized swaps are highly sensitive to this adjustment. Therefore, in order to manage risks on non-CSA swaps, one should monitor as precisely as possible OIS and IBOR correlation, as well as their respective volatilities.

**Figure 4: OIS–IBOR decorrelation impact on spot rates and forward rates**

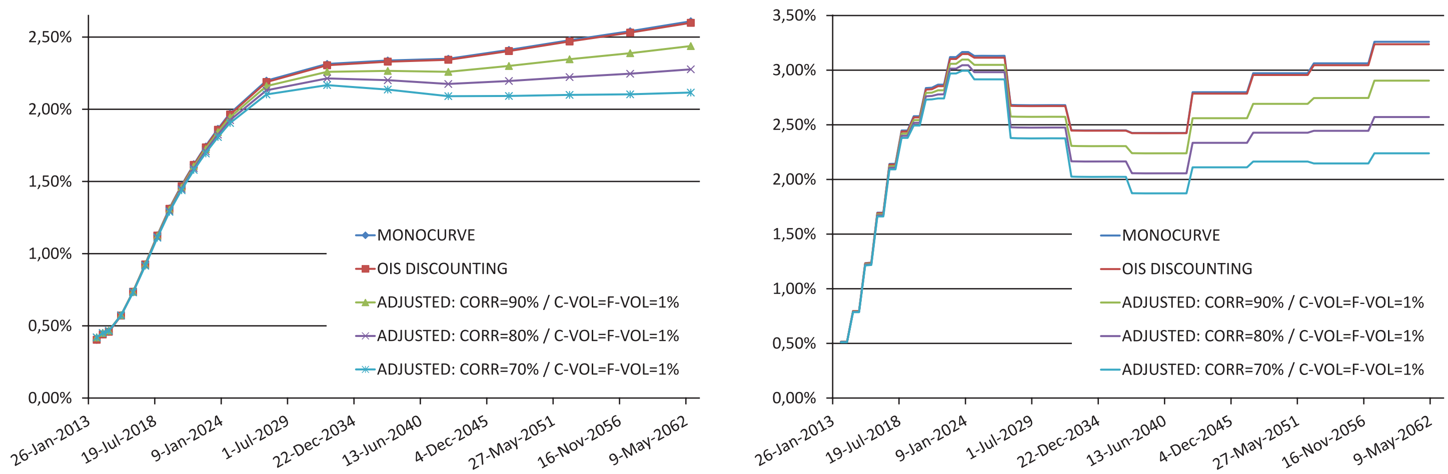
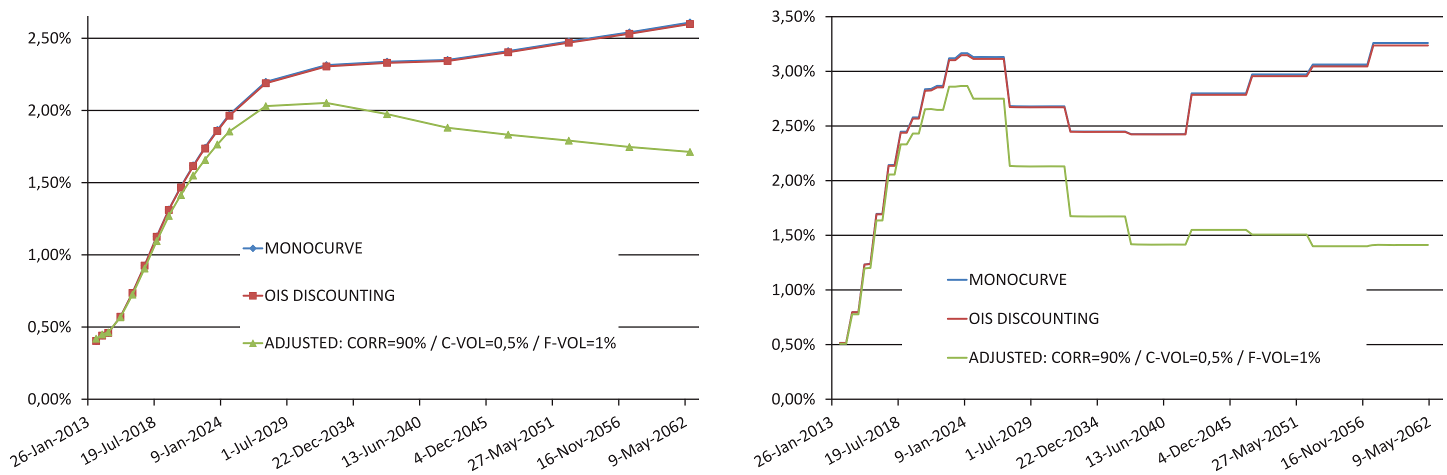



**Figure 5: Adjustment impact with $\sigma_C < \sigma_F$**



## Assuming OIS volatility lower than IBOR volatility

In the previous section we analyzed the correlation's impact using collateral rate volatility – which is unobservable – equal to funding rate volatility at a 1% level. In this section we take a look at the behavior of the adjustment under the assumption of collateral rate volatility lower than funding rate volatility.

Although it is a difficult task to define a level of collateral rate volatility $\sigma_C$, it seems logical to keep it lower than the funding rate volatility $\sigma_F$. The reason is that the collateral rate is driven by central bank rates that are less volatile than interbank rates. In this case the downward shift in spot rates and forward rates observed in the previous section is increased.

In the current example we used as previously $\sigma_F = 1\%$ and $\rho = 90\%$, but we took $\sigma_C = 50\%$ (see Figure 5). From the *cvx* formula obtained in Section 3.3, we would expect the downward shift to be more significant since $\rho$ and $\sigma$ are interchangeable. This is indeed what we observe: the 50Y spot rate is 89 basis points lower than a typical OIS bootstrap and the forward rate is 1.83% lower (absolute difference), as shown in Tables A.3 and A.4.

## Conclusion

By modeling OIS and IBOR rates separately and under the assumption of no funding spread, we managed to calculate the forwards convexity adjustment impacting non-collateralized swaps. We have shown how crucial it is to take into account OIS–IBOR decorrelation in terms of yield curve risk management. The next important steps consist of:

- trying to quantify the OIS–IBOR correlation term structure as well as the OIS volatility term structure, both being difficult to observe;
- modeling a stochastic funding rate correlated to IBOR and OIS rates.

## Appendix A: Numerical results

**Table A.1: Spot rates – decorrelation impact.**

| | | | Adjusted | | |
|---|---|---|---|---|---|
| Maturity | Monocurve | OIS discounting | $\rho_{C,F} = 90\%$ | $\rho_{C,F} = 80\%$ | $\rho_{C,F} = 70\%$ |
| 1Y | 0.40% | 0.40% | 0.42% | 0.42% | 0.42% |
| 5Y | 0.93% | 0.93% | 0.92% | 0.92% | 0.92% |
| 10Y | 1.74% | 1.74% | 1.72% | 1.71% | 1.69% |
| 20Y | 2.31% | 2.31% | 2.26% | 2.21% | 2.17% |
| 30Y | 2.35% | 2.34% | 2.26% | 2.18% | 2.09% |
| 40Y | 2.48% | 2.47% | 2.35% | 2.22% | 2.10% |
| 50Y | 2.61% | 2.60% | 2.44% | 2.28% | 2.12% |

**Table A.2: Forward rates – decorrelation impact.**

| | | | Adjusted | | |
|---|---|---|---|---|---|
| Maturity | Monocurve | OIS discounting | $\rho_{C,F} = 90\%$ | $\rho_{C,F} = 80\%$ | $\rho_{C,F} = 70\%$ |
| 5Y | 2.14% | 2.14% | 2.12% | 2.11% | 2.09% |
| 10Y | 3.12% | 3.10% | 3.06% | 3.01% | 2.97% |
| 20Y | 2.45% | 2.45% | 2.31% | 2.17% | 2.03% |
| 30Y | 2.80% | 2.79% | 2.56% | 2.34% | 2.11% |
| 40Y | 3.06% | 3.04% | 2.74% | 2.45% | 2.15% |
| 50Y | 3.26% | 3.24% | 2.90% | 2.57% | 2.24% |

**Table A.3: Spot rates impact with $\sigma_C < \sigma_F$.**

| Maturity | Monocurve | OIS discounting | Adjusted $\rho_{C,F} = 90\%$ $\sigma_C = 0.5\%$, $\sigma_F = 1\%$ |
|---|---|---|---|
| 1Y | 0.40% | 0.40% | 0.42% |
| 5Y | 0.93% | 0.93% | 0.91% |
| 10Y | 1.74% | 1.74% | 1.66% |
| 20Y | 2.31% | 2.31% | 2.05% |
| 30Y | 2.35% | 2.34% | 1.88% |
| 40Y | 2.48% | 2.47% | 1.79% |
| 50Y | 2.61% | 2.60% | 1.71% |

**Table A.4: Forward rates impact with $\sigma_C < \sigma_F$.**

| Maturity | Monocurve | OIS discounting | Adjusted $\rho_{C,F} = 90\%$ $\sigma_C = 0.5\%$, $\sigma_F = 1\%$ |
|---|---|---|---|
| 1Y | 0.52% | 0.52% | 0.51% |
| 5Y | 2.14% | 2.14% | 2.06% |
| 10Y | 3.12% | 3.10% | 2.86% |
| 20Y | 2.45% | 2.45% | 1.67% |
| 30Y | 2.80% | 2.79% | 1.55% |
| 40Y | 3.06% | 3.04% | 1.40% |
| 50Y | 3.26% | 3.24% | 1.41% |

## Appendix B: Adjustment calculation under Hull & White framework

### B.1 Calculating the $cvx(t,T_1,T_2)$ term

**Step 1: $r_F$ dynamics under $\mathbb{Q}_C^{T_2}$**

Let us apply the change of measure from $\mathbb{Q}$ to $\mathbb{Q}_C^{T_2}$ using the Radon–Nikodym derivative:

$$\frac{d\mathbb{Q}_C^{T_2}}{d\mathbb{Q}}|_t = Z(t) = K\frac{P_C(t,T_2)}{B_C(t)}$$

where $K$ is a constant term ($K = B_C(0)/P_C(0,T)$).

The Radon–Nikodym derivative $Z(t)$ has the same volatility term as $P_C(t,T_2)$:

$$dZ(t)/Z(t) = -\Gamma_C(t,T_2)\,dW^{\mathbb{Q}}(t)$$

Applying the Girsanov theorem we can now express the funding short-rate $r_F$ dynamics under $\mathbb{Q}_C^{T_2}$ as

$$dr_F(t) = \left[-\rho_{C,F}(t)\Gamma_C(t,T_2)\sigma_F(t) + \theta_F(t) - a_F r_F(t)\right]dt + \sigma_F(t)dW_F^{\mathbb{Q}_C^{T_2}}(t)$$

Solving this equation[2] yields

$$r_F(T_1) = r_F(t)e^{-a_F(T_1-t)} + \int_t^{T_1} e^{-a_F(T_1-s)}\left[-\rho_{C,F}(s)\Gamma_C(s,T_1)\sigma_F(s) + \theta_F(s)\right]ds$$
$$+ \int_t^{T_1}\sigma_F(t)e^{-a_F(T_1-s)}dW_F^{\mathbb{Q}_C^{T_2}}(t)ds$$

>

### Step 2: $r_F$ dynamics under $\mathbb{Q}_F^{T_2}$

Changing measure from $\mathbb{Q}$ to $\mathbb{Q}_F^{T_2}$ leads to the following dynamics for $r_F$ under $\mathbb{Q}_F^{T_2}$:

$$dr_F(t) = \left[\, -\Gamma_F(t,T_2)\,\sigma_F(t) + \theta_F(t) - a_F r_F(t)\,\right] dt + \sigma_F dW_F^{\mathbb{Q}_F^{T_2}}(t)$$

or equivalently:

$$r_F(T_1) = r_F(t)e^{-a_F(T_1-t)} + \int_t^{T_1} e^{-a_F(T_1-s)}\left[-\Gamma_F(t,T_2)\,\sigma_F(t) + \theta_F(s)\right] ds$$
$$+ \int_t^{T_1} \sigma_F(t) e^{-a_F(T_1-s)} dW_F^{\mathbb{Q}_F^{T_2}}(t) ds$$

This leads us to define

$$cvx(t,T_1,T_2) = \int_t^{T_1} e^{-a_F(T_1-s)}\left[\Gamma_F(s,T_2)\,\sigma_F(s) - \rho_{C,F}(s)\Gamma_C(s,T_2)\sigma_F(s)\right] ds$$

$$\widehat{r_F}(T_1) = r_F(T_1) - cvx_{HW}(t,T_1,T_2)$$

which makes $\widehat{r}_F(T_1)$ under $\mathbb{Q}_C^{T_2}$ equally distributed with $r_F$ under $\mathbb{Q}_F^{T_2}$.

## B.2 Calculating the integral

Let us now calculate the $cvx(t,T_1,T_2)$ integral

$$cvx(t,T_1,T_2) = \int_t^{T_1} e^{-a_F(T_1-s)}\left[\Gamma_F(s,T_2)\,\sigma_F(s) - \rho_{C,F}(s)\Gamma_C(s,T_2)\sigma_F(s)\right] ds$$

where the $\Gamma$ functions are given by

$$\Gamma_F(t,T_2) = \sigma_F(t)\left(\frac{1-e^{-a_F(T_2-s)}}{a_F}\right)$$
$$\Gamma_C(t,T_2) = \sigma_C(t)\left(\frac{1-e^{-a_C(T_2-s)}}{a_C}\right)$$

$$cvx(t,T_1,T_2) = \int_t^{T_1} e^{-a_F(T_1-s)}\left[\frac{\sigma_F^2}{a_F}\left(1-e^{-a_F(T_2-s)}\right)\right.$$
$$\left. -\rho_{C,F}\frac{\sigma_C\sigma_F}{a_C}\left(1-e^{-a_C(T_2-s)}\right)\right] ds$$
$$= \underbrace{\int_t^{T_1}\left(\frac{\sigma_F^2}{a_F} - \rho_{C,F}\frac{\sigma_C\sigma_F}{a_C}\right)e^{-a_F(T_1-s)}\,ds}_{I_1}$$
$$-\underbrace{\frac{\sigma_F^2}{a_F}\int_t^{T_1} e^{-a_F(T_1+T_2-2s)}\,ds}_{I_2}$$
$$+\underbrace{\rho_{C,F}\frac{\sigma_C\sigma_F}{a_C}\int_t^{T_1} e^{-a_F T_1 - a_C T_2} e^{(a_F+a_C)s}\,ds}_{I_3}$$

Let us now calculate the three integrals above:

$$I_1 = \int_t^{T_1}\left(\frac{\sigma_F^2}{a_F} - \rho_{C,F}\frac{\sigma_C\sigma_F}{a_C}\right)e^{-a_F(T_1-s)}\,ds$$
$$= \frac{1}{a_F}\left(\frac{\sigma_F^2}{a_F} - \rho_{C,F}\frac{\sigma_C\sigma_F}{a_C}\right)\left(1-e^{-a_F(T_1-t)}\right)$$

$$I_2 = \frac{\sigma_F^2}{a_F}\int_t^{T_1} e^{-a_F(T_1+T_2-2s)}\,ds = \frac{\sigma_F^2}{2a_F^2}e^{-a_F(T_1+T_2)}\left(e^{2a_F T_1} - e^{2a_F t}\right)$$

$$I_3 = \rho_{C,F}\frac{\sigma_C\sigma_F}{a_C}\int_t^{T_1} e^{-a_F T_1 - a_C T_2} e^{(a_F+a_C)s}\,ds$$
$$= \rho_{C,F}\frac{\sigma_C\sigma_F}{a_C(a_F+a_C)}e^{-a_F T_1 - a_C T_2}\left(e^{(a_F+a_C)T_1} - e^{(a_F+a_C)t}\right)$$

We finally obtain

$$cvx(t,T_1,T_2) = \frac{\sigma_F}{a_F}\left(\frac{\sigma_F}{a_F} - \rho_{C,F}\frac{\sigma_C}{a_C}\right)\left(1-e^{-a_F(T_1-t)}\right)$$
$$-\frac{\sigma_F^2}{2a_F^2}e^{-a_F(T_1+T_2)}\left(e^{2a_F T_1} - e^{2a_F t}\right)$$
$$+\rho_{C,F}\frac{\sigma_C\sigma_F}{a_C(a_F+a_C)}e^{-a_F T_1 - a_C T_2}\left(e^{(a_F+a_C)T_1} - e^{(a_F+a_C)t}\right)$$

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## ENDNOTES

1. The calculation can easily be extended to time-dependent volatilities but this leads to heavier formulae we chose not to expose for the sake of clarity.
2. Solution of an Orstein–Uhlenbeck process, simply obtained by calculating $d(r_F(t)e^{at})$ and integrating.

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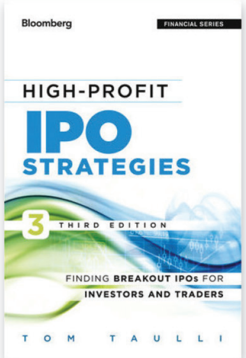


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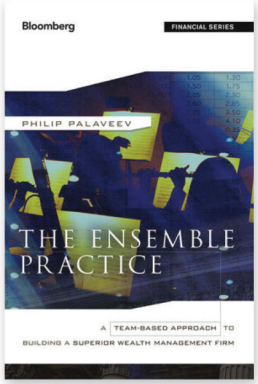


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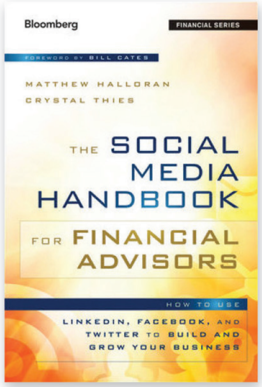


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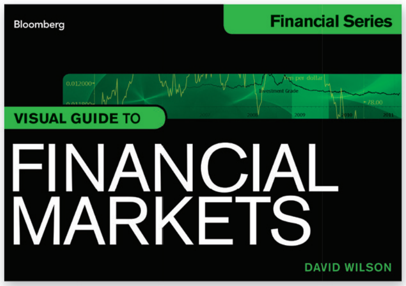


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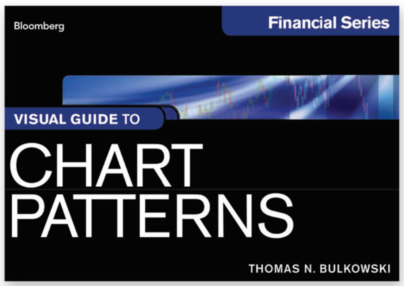


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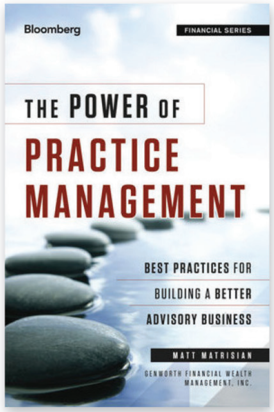


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