The Heston–Hull–White Model Part I: Finance and Analytics

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1 Introduction

This is the first article in a series of three on financial modeling. The aim of this series is to show the full life cycle of model development. We have chosen an equity model with stochastic volatility and stochastic interest rates. This will also be the goal of a forthcoming book by one of the authors, Kienitz, Duffy et al. (2011).

The series of articles will be structured as follows:

- Financial and Mathematical Details
- Numerics and Algorithms
- Design and Implementation

The first article deals with the financial and mathematical details of the model under consideration and the reasoning for choosing a stochastic volatility model with stochastic rates. To this end we briefly review the Heston stochastic volatility model and the Hull–White short rate model, we show the impact of stochastic rates and provide code snippets for the described mathematical methods. Finally, we combine both models into a framework which is capable of stochastic volatility and stochastic rates. The algorithms for pricing and calibration are discussed in the second article. Code snippets are provided as well. The third article then gives the whole software framework and details the design of the calibration and pricing application. The source code can be ordered from the authors via joerg.kienitz(at)gmx.de.

2 The Modeling Approach

First, we consider the implied volatility surface of the German DAX index for different dates. Figure 1 shows the different shapes of this surface.

This consideration shows that it is reasonable to work with a model which is capable of modeling the observed non-flat volatilities for different strikes. This phenomenon is called *skew* or *smile*. One model which is suited for modeling such structures and applied widely is the *Heston* stochastic volatility model.

For the illustration that stochastic interest rates might become necessary let us suppose we apply the standard Black–Scholes pricing methodology. The spot price of the index is denoted by $S(0)$, T is the maturity, r the risk less rate until maturity, and σ the implied volatility. Then, we have a closed form solution for pricing calls and puts. Furthermore, we are able to calculate the derivatives of the option price with respect to implied volatility, V, and rate, ρ. We get:

$$V = S(0)\sqrt{T}n(d_1) \approx \sqrt{T} \tag{1}$$

$$\rho = TK\exp(-rT)\mathcal{N}(d_2) \approx T \tag{2}$$

with

$$d_1 = \frac{\ln(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln(S(0)/K) - (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

Equations (1) and (2) suggest that for long dated options the risk stemming from stochastic rates is bigger than that stemming from changes in volatility.

To this end we consider the following system of stochastic differential equations subject to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$.

$$dS_t = r_t S_t dt + \sqrt{v_t} S_t dW_t^1 \tag{3}$$

$$dv_t = \kappa(\overline{v} - v_t)dt + \omega\sqrt{v_t}dW_t^2 \tag{4}$$

$$dr_t = \lambda(\theta(t) - r_t)dt + \eta dW_t^3 \tag{5}$$

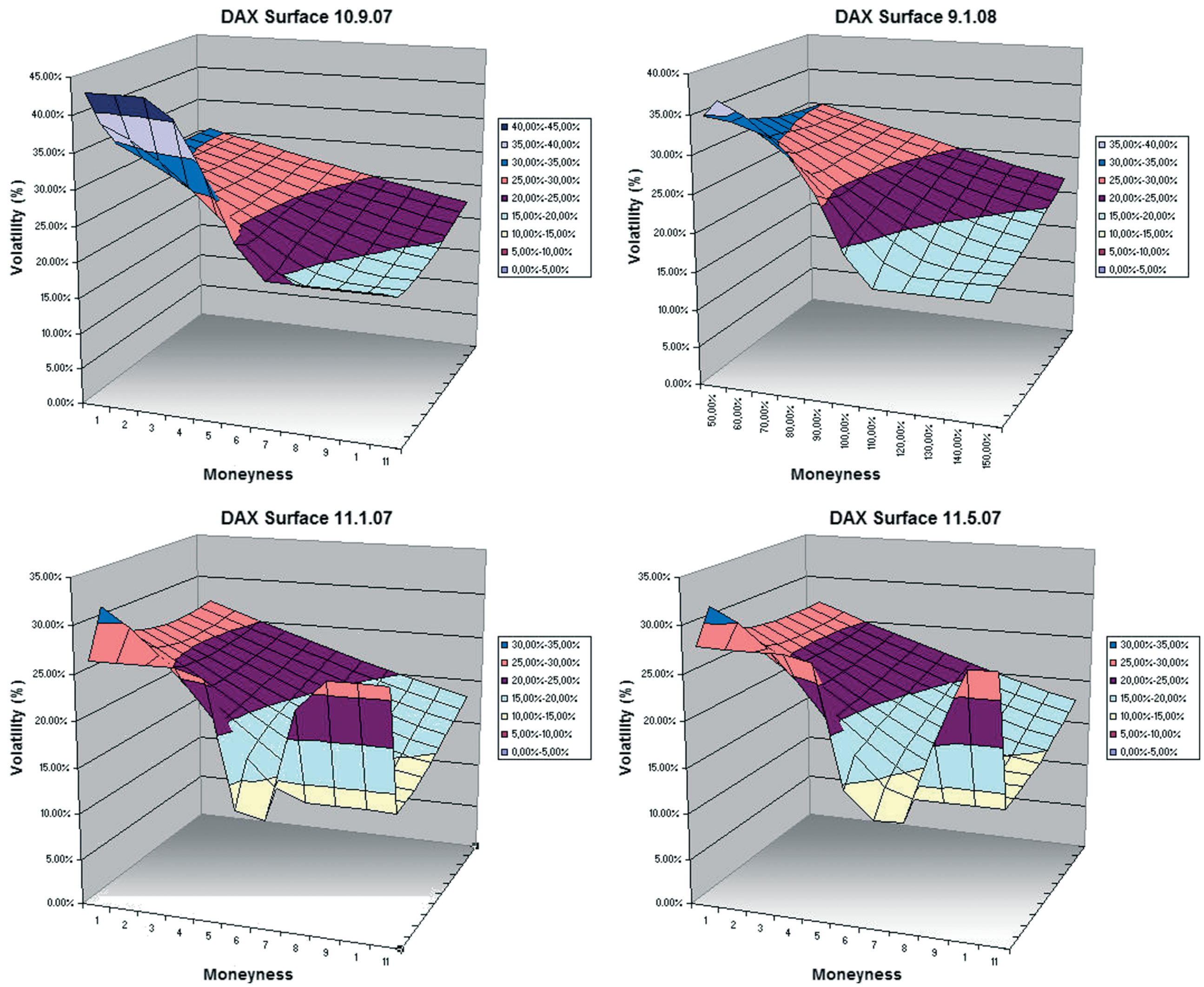
Equation (3) describes the evolution with respect to time t of the price of an equity $S(t)$. Its volatility is given $\sqrt{v_t}$ by where v_t evolves as a CIR mean reverting process determined by (4). This is exactly the dynamic from the well-known *Heston* stochastic volatility model. Instead of choosing the short rate $r(t)$ being constant it is given by a mean reverting Ornstein Uhlenbeck process (5) with time dependent but deterministic mean reversion level $\theta(t)$. The latter dynamic is known as the *Hull–White* model. That is why the model (3) – (5) is called the hybrid *Heston–Hull–White* model.

Any cashflow at a future date T has to be discounted by the stochastic factor $D(t,T) = \exp\left(-\int_t^T r(s)ds\right)$ to obtain the value at time t. The bank account B_t modeled by the stochastic dynamic $dB_t = r_t B_t dt$ is used as a numeraire. The probability measure Q is associated with this numeraire. This makes it possible to compute all present payoffs as

$$V(0) = \mathbb{E}_Q[D(t,T)V(T)] \tag{6}$$

Let us shortly comment on the used parameters. The process v_t reverts to the long-term variance $\bar{v}$ in the mean. The mean reversion speed k adjusts the velocity of this convergence. The volatility of variance (second-order volatility) is given by the constant ω. The long-run mean of interest rate is given by the function θ. Its time dependence effects that the system (3), (4), (5) is not

Figure 1: The implied volatility surface of the DAX on 10.9.07, 9.1.08, 11.1.07, and 11.5.07.



time-homogenous and thus it is not an Itô diffusion in the sense of Oksendal (2007). The constant λ determines the speed of mean reversion for the interest rate. Finally the constant η represents the volatility of the interest rate.

The improvement that is gained by introducing stochastic interest rates compared to the pure Heston model will be significant as soon as one prices derivatives with rather long maturities since in this case the assumption of a prevailing constant short rate evidently is no more appropriate.

We are interested in deriving a closed formula for the characteristic function of the process S_t. If the three driving Wiener processes are correlated arbitrarily this is not possible. For an extension of the model to cover this case see Grzelak et al. (2009a), Grzelak et al. (2009b), and Grzelak et al. (2009c). We therefore assume that the noise of the interest rate is independent of those determining asset price and variance, i.e., $dW_t^i dW_t^j = \rho_{ij} dt$ where

$$(\rho_{ij}) = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In many cases this turns out to be a reasonable assumption and we can argue that we model the risk inherent in stochastic movement of the rates but neglect the effect of correlated moves in the index and in the rates. If we set $\lambda = \eta = 0$ we obtain the Heston model. If we set in addition $\kappa = \omega = 0$ we have the Black–Scholes model.

Now, to apply the Heston–Hull–White model we have to consider the pricing of liquid options to be used to deduce the model parameters from market prices, the pricing of exotic path dependent structures and options which involve early exercise possibilities. In this series of articles we do not discuss the pros and cons of calibration and the issue of solving backward problems. We just remark that this is a common practice and we describe the methodology and the implementation necessary to fulfill this task. The results must of course be questioned and reasonable assumptions on the range of parameters and the influence of the parameters have to be considered. For instance the movement of the smile in the Heston model suggests that there are parameters which have the same effect on the volatility surface. Furthermore, we do not discuss numerical methods for solving early exercise problems. This is beyond the scope of this article series.

To see the impact of stochastic rates on the implied volatility surface we illustrate fix the parameters of the stochastic volatility component, the Heston model. Furthermore, we fix a yield curve and calibrate the Hull–White model to match the initial curve. Then, we study the effect by varying the parameters for mean reversion and volatility which are the free parameters for the chosen dynamic. First, Figure 2 shows the effect of stochastic interest rates. We have plotted the implied volatility calculated from option prices generated by a Heston stochastic volatility model and by a Heston–Hull–White model having the same stochastic volatility parameters. We have chosen $\overline{v} = 0.02$, $v_0 = 0.02$, $\kappa = 0.2$, $\rho = -0.6$ and $\omega = 0.5$ for the Heston model parameters and $\lambda = 0.1$ and $\eta = 0.05$ for the Hull–White model. This is the base scenario.

Figure 2 shows the impact on the long end of the smile and furthermore shows that the short-term smile is more pronounced.

Fixing all parameters but changing the volatility of the rates which is a measure for the randomness for the curve over time, Figure 3 shows the effect on the implied volatility surface. Decreasing randomness decreases the volatility level of the long end whereas increasing the level also increases the volatility level of the long end.

Fixing all parameters but changing the mean reversion speed which is a measure for tending to the initial curve over time, Figure 4 shows the effect on the implied volatility surface. Decreasing the mean reversion increases the level of the long end of the volatility since decreasing this parameter

Figure 2: Implied Volatility Surface for a Heston-Hull-White Model and the associated Heston model. The parameters are: $v = 0.02$, $v_0 = 0.02$, $\kappa = 0.2$, $\rho = -0.6$, and $\omega = 0.5$ for the Heston model parameters, and $\lambda = 0.1$ and $\eta = 0.05$.

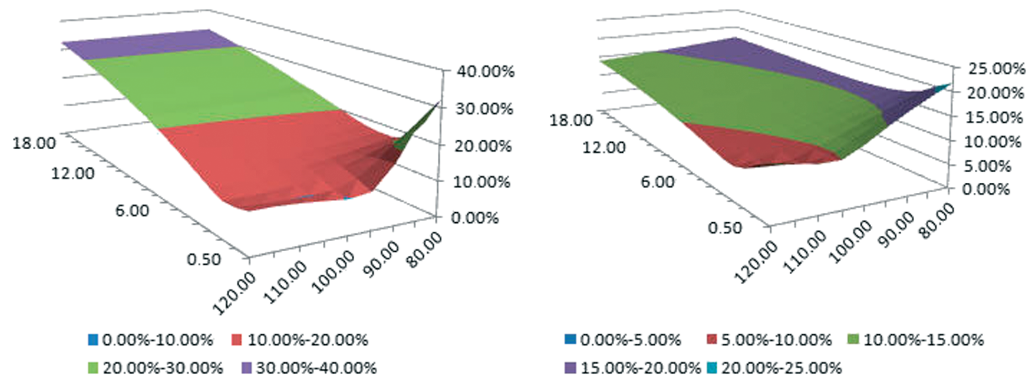


Figure 3: Illustration for changing volatility. Left is the base scenario, mid is increasing the volatility and right is decreasing it.

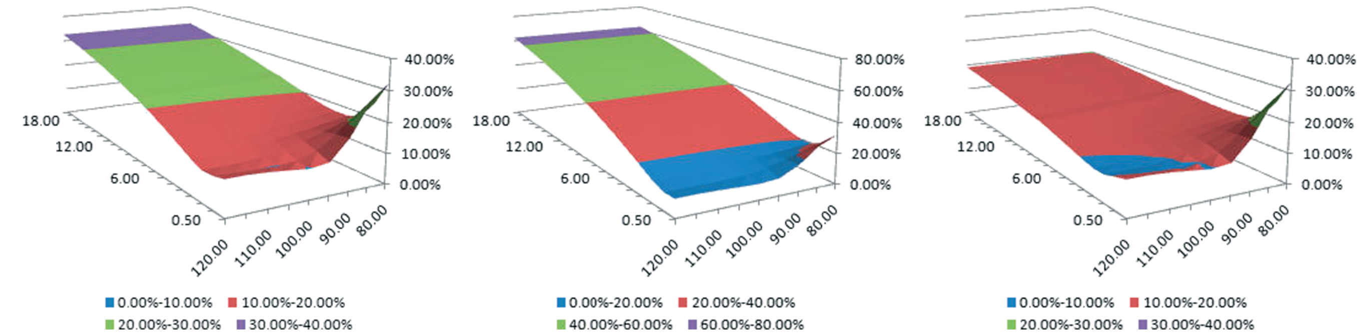
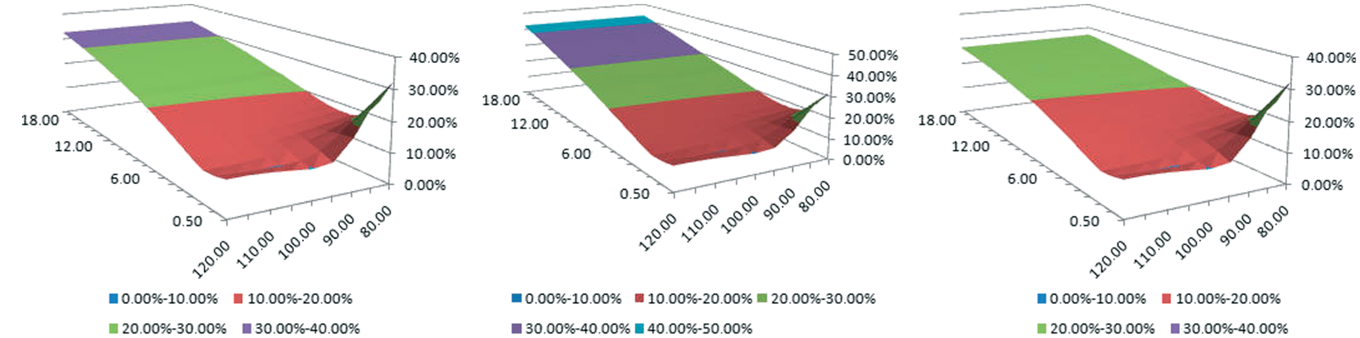


Figure 4: Illustration for changing the mean reversion. Left is the base scenario, mid is decreasing mean reversion and right is increasing mean reversion.



implicitly increases the influence of the randomness. Thus, increasing the mean reversion leads to a decrease in the level of the long end of the volatility.

3 Mathematical Background

In this section we will solve and discuss the Heston and the Hull–White SDE, recall the Hull–White decomposition and review relevant formulae for pricing. Then we apply these results to deduce the characteristic function of the asset price process. The characteristic function is the main ingredient for pricing options in such modeling frameworks. The application and the necessary algorithms are detailed in the next article.

3.1 The Heston Model

The Heston stochastic volatility model is given by equations (3) and (4) where the function $r(t) = r$ is a positive constant. The model has been introduced in Heston (1993) and has been the base for further research since, see Andersen (2006), Bin (2007), Lord and Kahl (2000), Lord and Kahl (2006), or Muskulus (2007), only to mention a few.

The number of research papers on this subject suggest that the model is widely applied and there is a strong need for efficient numerical methods.

If we denote $d := \frac{4\kappa\bar{v}}{\omega^2}$ and $n := \frac{4\kappa\exp(-\kappa(T-t))}{\omega^2(1-\exp(-\kappa(T-t)))}$ for the variance we have

$$Q(v_T < x|v_t) = F_{\chi^2}\left(\frac{x\cdot n}{\exp(-\kappa(T-t))}, d, n\right) \tag{7}$$

with $F_{\chi^2}(y, d, n)$ denoting the non-central χ^2 distribution with d degrees of freedom and n as the non-centrality parameter. Let T and t be given and $T > t$ then from the properties of the non-central χ^2 distribution it is known that:

$$\mathbb{E}[v_T|v_t] = \bar{v} + (v_t - \bar{v})\exp(-\kappa(T-t)) \tag{8}$$

$$\begin{aligned}\mathbb{V}[v_T|v_t] = {} & \frac{v_t\omega^2\exp(-\kappa(T-t))}{\kappa}(1-\exp(-\kappa(T-t))) \\ & + \frac{\bar{v}\omega^2}{2\kappa}(1-\exp(-\kappa(T-t)))^2\end{aligned} \tag{9}$$

Thus, the variance grows with growing ω and decreases with increasing κ. A well-known fact is that if $V_0 > 0$ and if $2\kappa\bar{v} \geq \omega^2$ - known as the *Feller condition* - the process v_t can never reach 0. If the Feller condition is not fulfilled, zero is accessible but strongly reflecting. As for a general class of financial models the prices for European calls and puts can be computed semi-analytically. A formula for the expectation of the call option value at time t denoted by $C(t)$ is

$$C(t) = \pm\left[e^{-d(T-t)}S(0)P_1(\pm) - e^{-r(T-t)}KP_2(\pm)\right] \tag{10}$$

with

$$\mathcal{D}_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}\left(\frac{\varphi_j(z)e^{-izy}}{iz}\right)dz; \quad P_j(\pm) = \frac{1-\pm 1}{2} \pm \mathcal{D}_j$$

We assumed that φ is the characteristic function of the underlying model and that it is analytic and bounded in a strip $0 \leq I(z) \leq 1$. We consider

$$\varphi_1(z) = e^{-\ln(S(t))-(r-d)(T-t)}\varphi(z-i) \text{ and } \varphi_2(z) = \varphi(z)$$

There is another version only involving one integration. We assume that the logarithmic asset price $x(T)$ has an analytic characteristic function φ in the strip $S_z = \alpha \leq I(z) \leq \beta$ and we assume that the payoff function $e^{-\alpha}f(x)$ is integrable with $c \in S_f$ where S_f is the strip on which the payoffs fourier transform, $\hat{f}$, exists and is analytic. If $S_F = S_f \cap S_z$ is not empty, the time t option value $C(t)$ is given by:

$$C(t) = \frac{e^{-r(T-t)}}{2\pi}\int_{c-\infty}^{c+\infty}\varphi(-z)\hat{f}(z)dz \tag{11}$$

with $c \in I(z)$, $z \in S_F$.

Therefore, the important notion is the function φ which is the *characteristic function* of the model. This is known in closed form for the Heston model and has already been given in Heston (1993). However, the function in the original text suffers from instabilities coming from the representation in the complex plane. There is a more stable version of the characteristic function given in Albrecher et al. (2006). The characteristic function is given by:

$$\phi_H(u,t,T) = \exp(A_H(u,t,T) + B_\sigma(u,t,T)v_t + iuX_t) \text{ with the functions} \tag{12}$$

$$\begin{aligned}
A_H(u,t,T) &= \frac{\kappa\bar{v}}{\omega^2}\left((\beta-D)(T-t) - 2\log\left(\frac{1-G\exp(-D(T-t))}{1-G}\right)\right),\\
B_\sigma(u,t,T) &= \frac{\beta-D}{\omega^2}\left(\frac{1-\exp(-D(T-t))}{1-G\exp(-D(T-t))}\right),\\
G &= \frac{\beta-D}{\beta+D},\\
\beta &= \kappa - \rho\omega ui,\\
D &= \sqrt{\beta^2 - 2\alpha\omega^2},\\
\alpha &= -\frac{1}{2}u(i+u)
\end{aligned}$$

and v_t denoting the initial variance while X_t is the logarithm of the spot.

3.2 The Hull–White Model

Recall the Hull–White short rate model is given by Equation (5). We follow the description in Brigo and Mercurio (2006). Applying Itô's formula to the exponential function we have

$$\begin{aligned}
d(\exp(\lambda t)r_t) &= \lambda\exp(\lambda t)r_t dt + \exp(\lambda t)dr_t\\
&= \lambda\exp(\lambda t)r_t dt + \exp(\lambda t)\lambda(\theta_t - r_t)dt + \exp(\lambda t)\eta dW_t\\
&= \exp(\lambda t)(\lambda\theta_t dt + \eta dW_t).
\end{aligned}$$

Integrating, we have

$$r_t = \exp(-\lambda(t-s))r_s + \lambda\int_s^t\exp(-\lambda(t-u))\theta_u du + \eta\int_s^t\exp(-\lambda(t-u))dW_u. \tag{13}$$

Therefore, the distribution of r_t conditioned on $\mathcal{F}_s$ is given by applying the following result:

Lemma 1 *Let $f: \Omega \times [s,t] \to \mathbb{R}$ be Itô integrable and deterministic, i.e., independent of $\omega \in \Omega$. Then*

$$\int_s^t f(u)dW_u \sim \mathcal{N}\left(0, \int_s^t f^2(u)du\right).$$

Thus, $r_t \mid \mathcal{F}_s$ is normally distributed with parameters

$$\mathbb{E}[r_t \mid \mathcal{F}_s] = \exp(-\lambda(t-s))r_s + \lambda \int_s^t \exp(-\lambda(t-u))\theta_u du \text{ and} \tag{14}$$

$$\mathbb{V}[r_t \mid \mathcal{F}_s] = \frac{\eta^2}{2\lambda}(1 - \exp(-2\lambda(t-s))). \tag{15}$$

This observation leads to the following consideration which is known as the *Hull–White decomposition*.

Lemma 2 *Defining* $\psi_t = \mathbb{E}[r_t \mid \mathcal{F}_0]$ *as in (14) we have*

$$r_t = \tilde{r}_t + \psi_t \tag{16}$$

where $\tilde{r}_t$ *is basic Ornstein Uhlenbeck mean reverting*

$$d\tilde{r}_t = -\lambda\tilde{r}_t dt + \eta dW_t, \quad \tilde{r}_0 = 0. \tag{17}$$

From (13) we have $\tilde{r}_t = \eta\int_0^t \exp(-\lambda(t-u))dW_u$ so $\tilde{r}_t$ has zero mean and variance as in (15). With some more work we arrive at

$$\mathbb{E}\left[\exp\left(-\int_s^t \tilde{r}_u du\right)\right] = \exp(C(s,t)) \text{ with}$$

$$C(s,t) = \frac{\eta^2}{2\lambda^3}\left(\lambda(s-t) - 2(1-\exp(-\lambda(t-s)) + \frac{1}{2}(1-\exp(-2\lambda(t-s))\right).$$

Thus the value $P(0,T)$ of a zero coupon bond paying one unit at maturity T is

$$P(0,T) = \mathbb{E}[D(0,T)] = \exp\left(-\int_0^T \psi_u du + C(0,T)\right)$$

at present. Equivalently

$$\psi_T = -\frac{\partial}{\partial T}\log P(0,T) + \frac{\partial}{\partial T}C(0,T) = f(0,T) + \frac{\eta^2}{2\lambda^2}(1-\exp(-\lambda T))^2 \tag{18}$$

with $f(t,T)$ denoting instantaneous forward rates. From (14) we observe that θ_t determines ψ_t via the ODE $\theta_t = \frac{1}{\lambda}\frac{\partial}{\partial t}\psi_t + \psi_t$. Substituting (18) in this equation yields

$$\theta_t = f(0,t) + \frac{1}{\lambda}\frac{\partial}{\partial t}f(0,t) + \frac{\eta^2}{2\lambda^2}(1-\exp(-2\lambda t)). \tag{19}$$

Setting $f(0,T)$ in (19) equal to the market instantaneous forward rates $f^M(0,T)$ we have adjusted θ such that the Hull–White model reproduces exactly the market discount curve $P^M(0,T)$. The superscript M indicates that the values are calculated from a known yield curve. It does only involve interpolation but no further modeling. We have modeled the yield curve as a class as follows:

```
class InterestCurve
{
    public:
      //input matrix with two columns. Maturity and discount
           factor.
      InterestCurve(NumericMatrix<double>& DiscountCurve);
      ~InterestCurve();
      double YieldCurve(double t);   //Y(0,t) as
           BrigoMercurio2006, def 1.3.1
      double ZeroBondCurve(double t);//P(0,t) as
           BrigoMercurio2006, def 1.3.2
      double InstForward(double t);  //f(0,t) as
           BrigoMercurio2006, def 1.4.2
    private:
      NumericMatrix<double> itsDiscountCurve;
     //Converts zero bond values to simply-compounded rates
      inline double ZeroBondToYield(double t, double
           ZeroBond);
};
```

The simplicity of the normally distributed Hull–White short rate which results in high performance for calibration purposes which is discussed in a later article also has a serious drawback. Short rates can become negative with positive probability! This is less important in practice since negative short rates will typically occur far out of a two-sigma neighborhood of their mean. The problem lies rather in the justification of the model in principal. A model that predicts that people could demand a reward for possessing other's money might hardly be acceptable but is used in practice so. For completeness we remark that the probability of negative short rates is given by

$$Q(r_t < 0) = \Phi\left(-\frac{\psi_t}{\sqrt{\mathbb{V}[r_t \mid \mathcal{F}_0]}}\right)$$

with Φ the standard Gaussian distribution function.

In order to compute zero bond values we have to know the distribution of the integrated short rate process given by $R_{t,T} = \int_t^T r(u)du \mid \mathcal{F}_t$. Here we use the fact that integrated Gaussian processes are Gaussian again. The parameters are

$$\mathbb{E}[R_{t,T} \mid \mathcal{F}_t] = B(t,T)(r_t - \psi_t) + \log\left(\frac{P^M(0,t)}{P^M(0,T)}\right) + \frac{1}{2}(V(0,T) - V(0,t)) \tag{20}$$

and

$$\mathbb{V}[R_{t,T} \mid \mathcal{F}_t] = \frac{\eta^2}{\lambda^2}\left(T - t + \frac{2}{\lambda}\exp(-\lambda(T-t)) - \frac{1}{2\lambda}\exp(-2\lambda(T-t)) - \frac{3}{2\lambda}\right) \tag{21}$$

with

$$B(t,T) = \frac{1}{\lambda}(1-\exp(-\lambda(T-t))) \tag{22}$$

This leads to the zero bond formula

$$P(t,T) = A(t,T)\exp(-B(t,T)r_t) \text{ with} \tag{23}$$

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\left(B(t,T)f^M(0,t) - \frac{\eta^2}{4\lambda}(1-\exp(-2\lambda t))B(t,T)^2\right).$$

This formula can be implemented using the model parameters as well as the necessary computations using a yield curve. The following code snippet illustrates the computation:

```
void HullWhite::ZeroBondOption::run()
{
      //computation of price, see BrigoMercurio2006,
          section 3.3.2
```

```
        double P_T = getZeroBondValue(itsT);
        double P_S = getZeroBondValue(itsBondMaturity);
        double sigma_p = itsEta * sqrt((1.0 - exp(-2.0 *
            itsLambda * itsT)) / (2.0 * itsLambda)) *
               factorB(itsT, itsBondMaturity);
        double h = (1.0 / sigma_p) * log (P_S / (P_T *
            itsK)) + sigma_p / 2.0;

        if(itsCorP == 1)
        //equation 3.40
            itsPrice = itsN * (P_S * CumulativeNormal(h)
               - itsK * P_T * CumulativeNormal
                      (h - sigma_p));
        else
        //equation 3.41
            itsPrice = itsN * (itsK * P_T *
                 CumulativeNormal(-h + sigma_p) - P_S *
                      CumulativeNormal(-h));
}
```

Basic interest rate derivatives such as caplets and floorlets can be expressed in terms of zero bond options. Therefore, it is possible to apply Equation (23) to price such options.

The following code snippet illustrates the implementation of the caplet/floorlet pricing.

```
void HullWhite::CapletFloorlet::run()
{
        itsCorP = -itsCorF + 1;
        itsT = itsStartT;
        itsBondMaturity = itsEndT;
        itsN = itsCapletFloorletN
           * (1.0 + itsCapletFloorletK * (itsEndT -
               itsStartT));
        itsK = 1.0 / (1.0 + itsCapletFloorletK * (itsEndT -
               itsStartT));
        //equivalence of caplets/floorlets and ZBPs/ZBCs,
               see [BrMe]
    HullWhite::ZeroBondOption::run();
}
```

4 Heston–Hull–White characteristic function

After reviewing the basic features of the model we proceed by reviewing a method for pricing European call and put options. To this end we have to compute an expected value

$$C_{\mathcal{F}_t}(T, K) = \mathbb{E}(D(t, T)(S_T - K)^+ \mid \mathcal{F}_t). \tag{24}$$

The possible application of the Fast Fourier Transform to derive option values relies on the availability of a closed formula for the characteristic function of the log price process. We discuss an algorithm for applying the technique to our particular problem in the second article of this series. The assumption that the driving Wiener process in (5) is independent of those in (3) and (4) will effect that the characteristic function of the Heston–Hull–White process is essentially the characteristic function of the pure Heston model up to a factor coming from the Hull–White short rate model. For this purpose note that for $X_t = \log S_t$ we have by Itô's formula

$$\begin{aligned} dX_t &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\ &= r_t dt - \frac{v_t}{2} dt + \sqrt{v_t} dW_t \\ &= r_t dt + dX_{Ht} \end{aligned} \tag{25}$$

Here X_H is the log price process of the pure Heston model (3) – (4) with $r_t = 0 = const$. Integrating (25) from s to T gives

$$X_T = R_{s,T} + X_{HT} \tag{26}$$

with initial conditions set equal, $X_s = X_{Hs}$. The additional summand $R_{s,T} = \int_s^T r_t dt$ can be interpreted as a correction of the term $r(T-s)$ that the pure Heston model would produce assuming a constant short rate r.

Note that since both the short rate process (5) as well as its integrated version are Gaussian processes their characteristic functions are given by

$$\phi_{HW}(u, t, T) = \mathbb{E}[\exp(iur_T) \mid \mathcal{F}_t] = \exp\left(iu\mu_{HW} - \frac{\sigma_{HW}^2 u^2}{2}\right) \tag{27}$$

with μ_{HW} and σ^2_{HW} as in (14) and (15) and

$$\phi_R(u, t, T) = \mathbb{E}[\exp(iuR_{t,T}) \mid \mathcal{F}_t] = \exp\left(iu\mu_R - \frac{\sigma_R^2 u^2}{2}\right) \tag{28}$$

This can be implemented in C++ using the following code: First, we need some functionality for computing the mean and the variance.

```
double HullWhite::Model::getMean(double r_s, double s,
double t) const
{
      return r_s * exp(-itsLambda * (t - s)) + alpha(t) -
        alpha(s) * exp(-itsLambda * (t - s));
}

double HullWhite::Model::getVariance(double s, double t)
const
{
      return (itsEta * itsEta) / (2.0 * itsLambda) * (1.0 -
        exp(-2.0 * itsLambda * (t - s)));
}
```

Then, the characteristic function can be computed using the following piece of code:

```
complex<double> HullWhite::Model::CharFuncOfIntegral
(complex<double> u, double T) const
{
      return exp(getMeanOfIntegral(itsRInst, 0.0, T) * u
             * IU
      - 0.5 * getVarianceOfIntegral(0.0, T) * u * u);
}
```

```
complex<double> Heston::Model::CharFunc(complex<double>
u, double S, double T) const
{
      complex<double> U1;
      U1 = u * (u + IU);

      double gamma = 0.5 * itsOmega * itsOmega;

      complex<double> alpha = -0.5 * U1;
      complex<double> beta = itsKappa - itsRho * itsOmega
         * u * IU;

      complex<double> D = sqrt(beta * beta - 4.0 * alpha
         * gamma);
      complex<double> G = (beta - D) / (beta + D);
      complex<double> B = ((beta - D) / (itsOmega *
         itsOmega)) * ((1.0 - exp(- D * T))
         / (1.0 - G * exp(- D * T)));
      complex<double> psi = (G * exp(- D * T) - 1.0) /
          (G - 1.0);
      complex<double> A = ((itsKappa * itsVLong) /
         (itsOmega * itsOmega))
         * ((beta - D) * T - 2.0 * log(psi));

     //For the HHW model itsR will be set zero
      return exp(A + B * itsVInst + IU * u * (itsR * T +
          log(S)));
}
```

with μ_R and ϕ_R^2 as in (20) and (21). Since the derivation of the Heston characteristic function is detailed in Heston (1993) and observing that $R_{t,T}$ and X_{HT} we have:

Theorem 3 *The Heston–Hull–White characteristic function* $\phi_{HHW}(u,t,T)=\mathbb{E}[\exp(iu X_T) \mid \mathcal{F}_t]$ *is given by*

$$\phi_{HHW}(u,t,T) = \phi_R(u,t,T)\phi_H(u,t,T)$$

where ϕ_H *is the Heston characteristic function with zero short rate, Equation (12).*

For some indices it is possible to get quotes of forward starting call and put options. Such options can be priced prevailing the forward characteristic function is known. A general mechanism for using the forward characteristic function is given in Beyer and Kienitz (2009). For the Heston–Hull–White model we have:

Theorem 4 *The Heston–Hull–White forward characteristic function*

$$\phi^F_{HHW}(u,t,T,S) = \mathbb{E}[\exp(iu(X_S - X_T)) \mid \mathcal{F}_t] \text{ for } t<T<S$$

is given by

$$\phi^F_{HHW}(u,t,T,S) = \phi^F_R(u,t,T,S)\,\phi^F_H(u,t,T,S)$$

with

$$\begin{aligned}\phi^F_R(u,t,T,S) &= \mathbb{E}[\exp(iu(R_{t,S} - R_{t,T})) \mid \mathcal{F}_t]\\ &= \exp\left(iu\left(-B(T,S)\psi_T + \log\left(\frac{P^M(0,T)}{P^M(0,S)}\right) + \frac{1}{2}(V(0,S) - V(0,T))\right)\right.\\ &\quad\left. - \frac{V(T,S)u^2}{2}\right)\cdot \phi_{HW}(B(T,S)u,t,T)\end{aligned}$$

(see (18), (22), (21), (27)) and with the forward characteristic function of the pure Heston model with zero short rate

$$\begin{aligned}\phi^F_H(u,t,T,S) &= \mathbb{E}[\exp(iu(X_{H_S} - X_{H_T})) \mid \mathcal{F}_t]\\ &= \exp\left[A_H(u,T,S) + \left(\frac{\exp(-\kappa(T-t))B_\sigma(u,T,S)}{1-2\gamma B_\sigma(u,T,S)}\right)v_t + iuX_t\right]\\ &\quad\times\left(\frac{1}{1-2\gamma B_\sigma(u,T,S)}\right)^{\frac{2\kappa\bar{v}}{\omega^2}}\end{aligned}$$

with A_H, B_σ, v_t *and* X_t *as in theorem 3 and* $\gamma = \frac{\omega^2}{4\kappa}(1-\exp(-\kappa(T-t)))$.

This can be shown as follows: As in (26) we decompose $X_T = R_{t,T} + H_{H_T}$ and $X_S = R_{t,S} + X_{H_S}$ to see that

$$\begin{aligned}\exp(iu(X_S - X_T)) &= \exp(iu((X_{H_S} - X_{H_T}) + (R_{t,S} - R_{t,T})))\\ &= \exp(iuR_{T,S})\exp(iu(X_{H_S} - X_{H_T}))\end{aligned}$$

which by independence gives the factorization into the forward characteristic functions of the integrated short rate process and of the Heston process with drift 0.

Now the function ϕ^F_R can be computed as follows.

$$\begin{aligned}\phi^F_R(u,t,T,S) &= \mathbb{E}[\exp(iuR_{T,S}) \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[\exp(iuR_{T,S}) \mid \mathcal{F}_T] \mid \mathcal{F}_t]\\ &\overset{(28)}{=} \exp\left(iu\left(-B(T,S)\psi_T + \log\left(\frac{P^M(0,T)}{P^M(0,S)}\right)\right.\right.\\ &\quad\left.\left. + \frac{1}{2}(V(0,S) - V(0,T))\right) - \frac{V(T,S)u^2}{2}\right)\mathbb{E}[\exp(iuB(T,S)r_T) \mid \mathcal{F}_t]\end{aligned}$$

and (27) gives the result.

For the derivation of the Heston forward characteristic function ϕ^F_H we refer to appendix C, lemma C.2.1 of Bin (2007). Note that the formulas for A_H and B_σ given there are slightly mistaken. On the right hand side the time variable should read τ instead of T.

```
complex<double> HestonHullWhite::EuropeanOp::
CharFunc(complex<double> u) const
{
      return HullWhite::Model::CharFuncOfIntegral(u + IU,
             itsT) * Heston::Model::CharFunc(u, itsS /
                   itsScale, itsT);
}
```

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