Replicating a Defaultable Bond

by Peter Carr

Bloomberg/NYU Courant Institute

Initial version: February 9, 2005

Current version: February 9, 2005

File reference: repdefbond.tex

We show how to replicate the payoff of a defaultable bond by dynamic trading in a money market account and a credit default swap.

I Introduction to Defaultable Bond Replication

A *money market account* is a fictitious bank account whose balance grows at the random spot rate r_t over time. Hence, one dollar deposited into a money market account at time 0 results in a bank balance of $\beta_t \equiv e^{\int_0^t r_u du}$ thereafter. A *zero coupon bond* is a fictitious financial security that pays one dollar at a fixed time T. For $t \in [0, T]$ and $r_t \geq 0$, the price P_t of this pure discount bond fluctuates randomly in $[0, 1]$, but $P_T = 1$ with certainty.

A defaultable bond is a pure discount bond which *promises* to pay one dollar at its fixed expiry date. Since promises aren't always kept, we define a defaultable bond as a fictitious financial security which pays one dollar at a fixed time T if no default occurs prior to that time and it pays a random recovery rate $R \in [0, 1)$ at the default time $\tau \in [0, T]$ otherwise.

A credit default swap (commonly abbreviated as CDS) is a zero cost contract whose payoffs depend on the defaultable bond. More specifically, the long position (called the protection buyer) pays a fixed rate (called the CDS rate) periodically over time (usually every 6 months) until the earlier of default and the fixed maturity date. In return, the protection buyer receives one minus the recovery rate at the default time if this time occurs at or before expiration, and zero otherwise. If someone is long the defaultable bond, then buying a CDS is akin to buying insurance against default. The periodic fixed payment corresponds to the insurance premium and the intent of the default payoff of the CDS is to make the bondholder whole.

II Assumptions

As the focus of this note is on default, we assume zero interest rates for simplicity, i.e. $r_t = 0, t \geq 0$. Hence one dollar deposited into a money market account results in a constant bank balance of $\beta_t = 1$ dollar thereafter. To avoid arbitrage, it follows that the price of a default-free pure discount bond $P_t = 1$ for $t \in [0, T]$.

For simplicity, we will also assume zero recovery rate, $R = 0$, on the underlying defaultable bond. Hence, our defaultable bond pays one dollar at a fixed time T if no default occurs prior to that time and it pays zero otherwise. Thus, our defaultable bond pays off $1(\tau < T)$ at T. For $t \in [0, \infty)$, let D_t be the default indicator process. This stochastic process has constant value of zero prior to the default time τ and it has constant value of one after τ. Thus, the payoff on our defaultable bond can also be written as $1 - D_T$ at T.

For simplicity, we will also assume that the protection buyer of the CDS makes the fixed payment continuously over time rather than periodically. Hence, the protection buyer of our idealistic CDS pays a fixed rate continuously over time until the earlier of τ and T. In return, the protection buyer receives one dollar at τ if $\tau \leq T$, and zero otherwise (recall $R = 0$).

We also assume that the random default time τ is exponentially distributed. Mathematically, a random variable X has an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by:

$$
\mathbb{P}\{X \in (x, x + dx)\} = \begin{cases} \lambda e^{-\lambda x} dx, & \text{if } x \in \mathbb{R}^+ \\ 0 & \text{otherwise.} \end{cases}
\tag{1}
$$

It's slightly easier to remember that for $x > 0$, the probability that $X > x$ is just $e^{-\lambda x}$. If a standard Poisson process N has constant intensity λ, then the interarrival times are IID exponential with parameter λ.

Suppose that we let λ_p be the constant pararameter associated with the exponentially distributed default time τ under the real world probability measure $\mathbb{P}$. In our model, we will not need to know λ_p in order to replicate and price the defaultable bond. Hence, we are actually free to assume that λ_p is some positive random variable with an unknown distribution.

In practice, the CDS rate for a newly issued CDS fluctuates randomly over time. For simplicity, we will assume that the CDS rate is constant over time at $\lambda_q > 0$. All of our results will extend in an obvious way to the case of deterministic CDS rates $\lambda_q(t), t \in [0, T]$.

III Analysis

In this section, we will show that our setting is such that an investor can replicate the payoff to the defaultable bond by dynamically trading in the money market account and credit default swaps. It follows as a consequence of no arbitrage that the price of the defaultable bond is just the cost of the replicating portfolio.

For $t \in [0, T]$, let B_t be the value of the defaultable bond maturing at T. We assume no arbitrage and we have zero interest rates so by standard results, there exists at least one probability measure $\mathbb{Q}$ under which the stochastic process $\{B_t, t \in [0, T]\}$ is a $\mathbb{Q}$ martingale.

Suppose that we guess that in our setting, $B_t = B(t)1_(D_t = 0)$ for $t \in [0, T]$, where $B(t) : [0, T] \mapsto [0, 1]$ is a function of time to be determined. We will refer to the function $B(t)$ as the value of the defaultable bond conditional on no default up to time t. Note that the deterministic process $\{B(t), t \in [0, T]\}$ is not a $\mathbb{Q}$ martingale, while the stochastic process $\{B_t, t \in [0, T]\}$ is.

Suppose that at some time $t \in [0, T]$ prior to default, an investor goes long one defaultable bond and completely finances the cost by borrowing at the riskfree rate of zero. Then the infinitessimal gain on the leveraged bond position is:

$$B'(t)dt(1 - dD_t) - B(t)dD_t,$$

where recall that D is the default indicator.

Suppose instead that an investor goes long a CDS at some time $t \in [0, T]$. For λ_q, constant, the P&L arising from going long one CDS at time t and then closing the position an instant later at time $t + dt$ is:

$$dD_t - \lambda_q dt.$$

Since this position is costless, it follows that for any risk-neutral probability measure $\mathbb{Q}$, λ_q is the risk-neutral arrival rate under $\mathbb{Q}$ of a default arriving at any time $t \in [0, T]$. Since the probability of default is

$O(dt)$, the infinitessimal gain on the CDS can also be written as

$$dD_t - \lambda_q dt (1 - dD_t).$$

Now suppose that an investor sets up the following zero cost portfolio:

- long one defaultable bond completely financed by borrowing at the riskfree rate of zero, and

- long N credit default swaps.

Putting our results together, the infinitessimal gain on the zero cost portfolio is:

$$B'(t)dt(1 - dD_t) - B(t)dD_t + N[dD_t - \lambda_q dt(1 - dD_t)].$$

Suppose that the investor chooses $N = B(t)$. Then the gain simplifies to:

$$[B'(t)dt - \lambda_q B(t)dt](1 - dD_t).$$

To prevent arbitrage, we require that this gain vanish, or equivalently that the function $B(t)$ solves the ODE:

$$B'(t) = \lambda_q B(t), \qquad t \in [0, T].$$

We also have as a terminal condition that $B(T) = 1$. The solution to this simple terminal value problem is clearly:

$$B(t) = e^{-\lambda_q(T-t)}, \qquad t \in [0, T].$$

The foregoing motivates the following martingale representation for the defaultable bond:

$$1(\tau < T) = e^{-\lambda_q T} - \int_0^{\tau \wedge T} e^{-\lambda_q(T-t)}[dD_t - \lambda_q dt]. \tag{2}$$

It is a straightforward (and recommended) exercise to check that (2) holds when $1(\tau < T) = 1$ and when $1(\tau < T) = 0$. In words, (2) says that to replicate the time T payoff $1(\tau < T)$ of a (long position in one) defaultable bond in our simple economy, one should initially invest $e^{-\lambda_q T}$ dollars in a money market

4

account and short $e^{-\lambda_q(T-t)}$ credit default swaps at each time t prior to $\tau \wedge T$. If the reason that one wants to replicate the payoff is that a defaultable bond has been shorted, then the issuer should initially charge $e^{-\lambda_q T}$ (at least) in order to finance the initial money market position. If the gains and losses on the CDS position are deposited into and financed out of the money market account, then the money market account balance will coincide at every time with the arbitrage-free value of the defaultable bond. In other words, the balance in the money market account at time $t \in [0, T]$ will be $e^{-\lambda_q(T-t)}1(D_t = 0)$.

Taking risk-neutral expectations on both sides of (2) leads to the pricing result:

$$B_0 = e^{-\lambda_q T}. \tag{3}$$

Clearly, the risk-neutral arrival rate of default λ_q enters the defaultable bond valuation formula in the same way as does the risk-free interest rate.

Some obvious extensions which you should do to check your understanding is:

1. Add positive deterministic interest rates $r(t)$.

2. Make the CDS rate λ_q a deterministic function of time.

3. Price a defaultable bond at an arbitrary time t rather than at time 0.

4. Give the defaultable bond a positive deterministic recovery rate R.

5. Show how to replicate a (long position in one) CDS by dynamic trading in defaultable bonds and the money market account.

6. Do all of the above together.

7. Think about what trades you would do if the actual market price of the defaultable bond differed from its arbtrage free value.