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Lecture 9

Volatility

Forward Equations (1)

- BWD Equation:

  price of one option $C(K_0, T_0)$ for different (S, t)

- FWD Equation:

  price of all options $C(K, T)$ for current (S_0, t_0)

- Advantage of FWD equation:
  - If local volatilities known, fast computation of implied volatility surface,
  - If current implied volatility surface known, extraction of local volatilities,
  - Understanding of forward volatilities and how to lock them.

Forward Equations (2)

- Several ways to obtain them:
  - Fokker-Planck equation:
    - Integrate twice Kolmogorov Forward Equation
  - Tanaka formula:
    - Expectation of local time
  - Replication
    - Replication portfolio gives a much more financial insight

Fokker-Planck

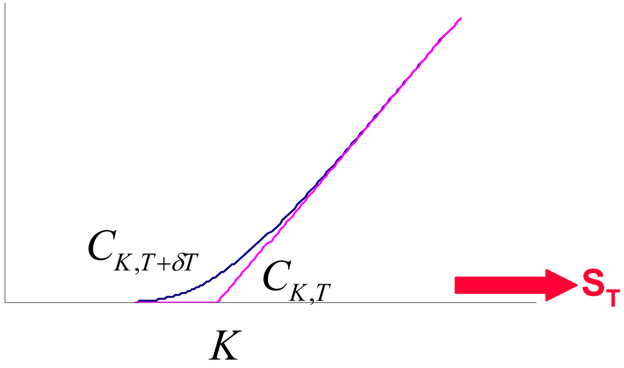
- If $dx = b(x,t)dW$
- Fokker-Planck Equation: $\frac{\partial \varphi}{\partial t} = \frac{1}{2}\frac{\partial^2 (b^2 \varphi)}{\partial x^2}$
- Where φ is the Risk Neutral density. As $\varphi = \frac{\partial^2 C}{\partial K^2}$

$$\frac{\partial^2 \left(\frac{\partial C}{\partial t}\right)}{\partial x^2} = \frac{\partial \left(\frac{\partial^2 C}{\partial K^2}\right)}{\partial t} = \frac{1}{2}\frac{\partial^2 \left(b^2 \frac{\partial^2 C}{\partial K^2}\right)}{\partial x^2}$$

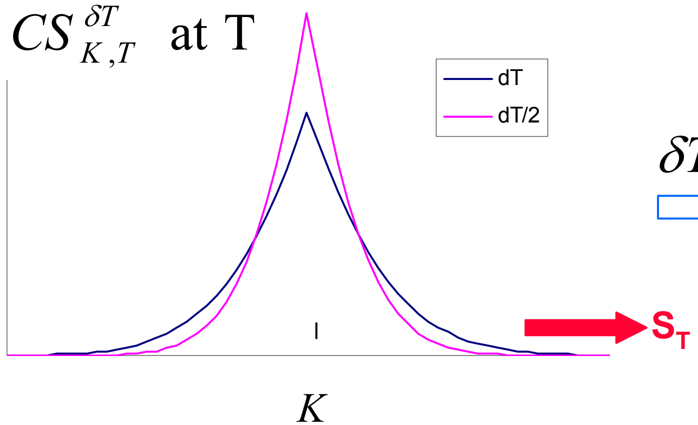
- Integrating twice w.r.t. x: $\frac{\partial C}{\partial t} = \frac{b^2}{2}\frac{\partial^2 C}{\partial K^2}$

FWD Equation: dS/S = σ(S,t) dW

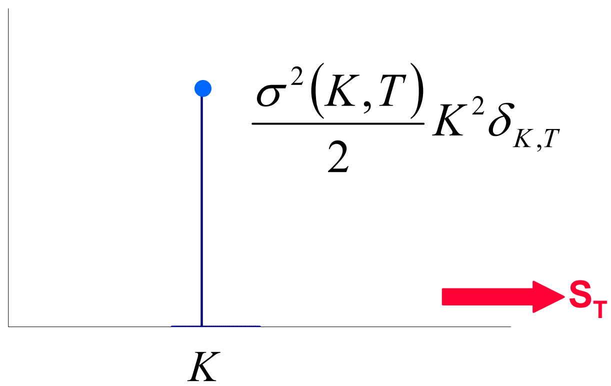
Define $CS_{K,T}^{\delta T} \equiv \dfrac{C_{K,T+\delta T} - C_{K,T}}{\delta T}$



$CS_{K,T}^{\delta T}$ at T

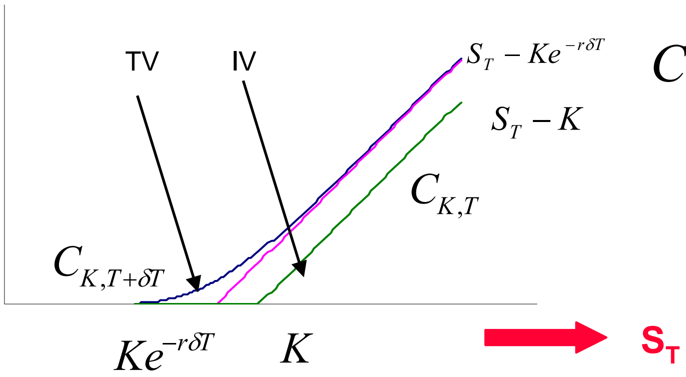


$\delta T \to 0$

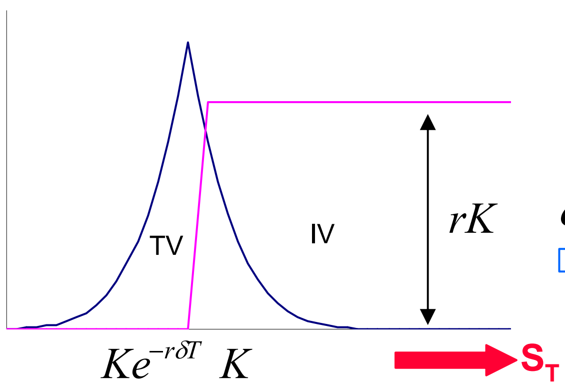


Equating prices at t_0: $$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2}$$

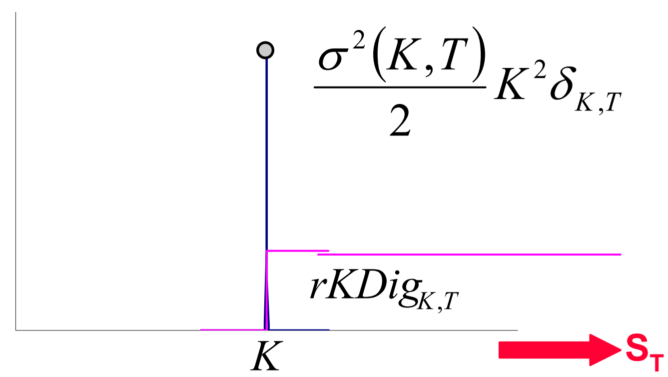
FWD Equation: dS/S = r dt + σ(S,t) dW



$CS_{K,T}^{\delta T}$ at T = Time Value (Strike Convexity) + Intrinsic Value (Interest on Strike)



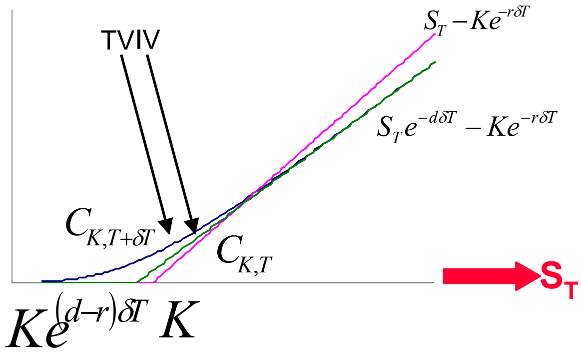
$\delta T \to 0$



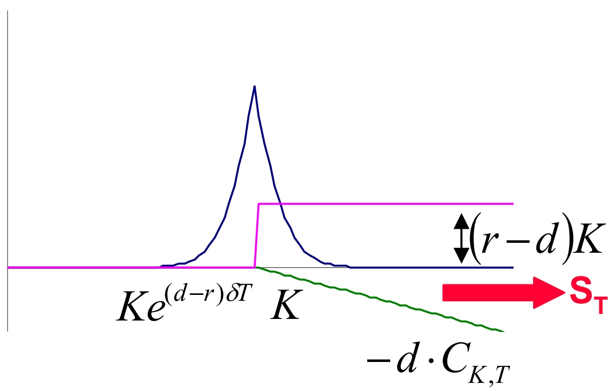
Equating prices at t_0:

$$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2}K^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}$$

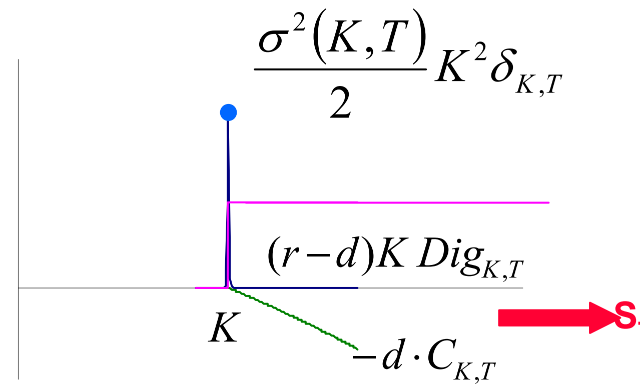
FWD Equation: $dS/S = (r-d)\,dt + \sigma(S,t)\,dW$



$CS^{\delta T}_{K,T}$ at T = TV + Interests on K – Dividends on S



$\delta T \to 0$



Equating prices at t_0: $$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - (r-d)K\frac{\partial C}{\partial K} - d \cdot C$$

Stripping Formula

$$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 C}{\partial K^2} - (r-d)K\frac{\partial C}{\partial K} - d\cdot C$$

– If $\sigma(K,T)$ known, quick computation of all $C_{K,T}(S_0,t_0)$ today,

– If all $C_{K,T}(S_0,t_0)$ known:

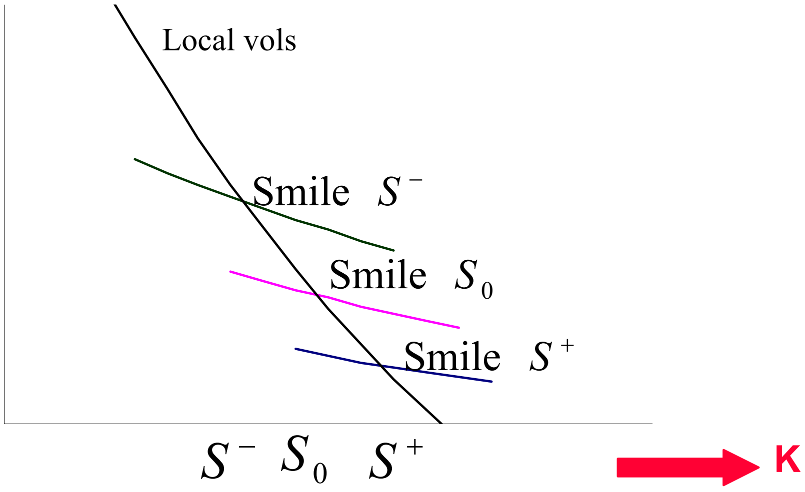
$$\sigma(K,T) = \sqrt{\frac{2\dfrac{\partial C}{\partial T} + (r-d)K\dfrac{\partial C}{\partial K} + dC}{K^2\dfrac{\partial^2 C}{\partial K^2}}}$$

Local volatilities extracted from vanilla prices and used to price exotics.

Smile dynamics: Local Vol Model (1)

- Consider, for one maturity, the smiles associated to 3 initial spot values

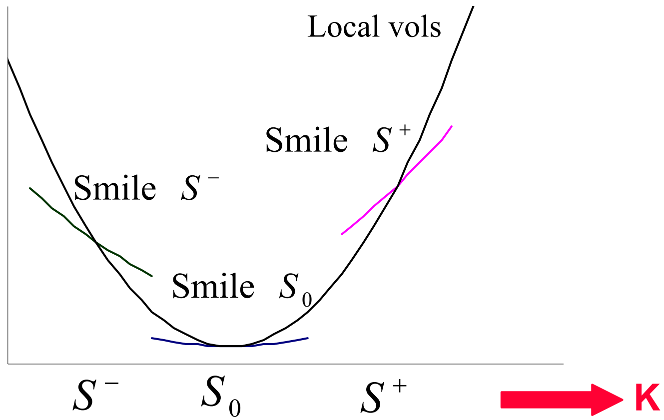
Skew case



- ATM short term implied follows the local vols
- Similar skews

Smile dynamics: Local Vol Model (2)

- Pure Smile case



  - ATM short term implied follows the local vols
  - Skew can change sign

Summary of LVM Properties

Σ_0 is the initial volatility surface

- $\sigma(S,t)$ compatible with $\Sigma_0 \Leftrightarrow \sigma =$ local vol
- $\sigma(\omega)$ compatible with $\Sigma_0 \Leftrightarrow E\left[\sigma^2 \middle| S_T = K\right]$ = (local vol)²
- $\hat{\sigma}_{k,T}$ deterministic function of (S,t)

  $\Leftrightarrow$ future smile = FWD smile from local vol

Volatility Replication

Volatility Replication

$$\frac{dS}{S} = \sigma_t dW$$ Apply Ito to f(S,t).

$$df = f_S dS + f_t dt + \tfrac{1}{2} f_{SS} \sigma_t^2 S^2 dt$$

$$\Rightarrow \int_0^T f_{SS}(S_t, t)\sigma_t^2 S^2 dt = 2\left[\underbrace{f(S_T, T) - f(S_0, 0) - \int_0^T f_t(S_t, t) dt}_{\text{European PF}} - \underbrace{\int_0^T f_S(S_t, t) dS_t}_{\Delta\text{-hedge}} \right]$$

To replicate $\int_0^T g(S,t)\sigma_t^2 dt$,find f : $g(S,t) = f_{SS}(S,t)S^2$: $f = \iint \frac{g}{S^2}$

Examples

Variance Swap	$g(S,t)=1$	$f(S,t)=-\ln(\frac{S}{S_0})$
Corridor Variance Swap	$g(S,t)=1_{[a,b]}(S_t)$	$f(S,t)=-\ln(\frac{S}{S_0})$ on [a,b] + linear extrapolation
FWD Variance Swap	$g(S,t)=1_{[T_1,T_2]}(t)$	$f(S,t)=-\ln(\frac{S}{S_0})\times 1_{[T_1,T_2]}(t)$
Absolute Variance Swap	$g(S,t)=S^2$	$f(S,t)=\frac{(S-S_0)^2}{2}$
Local Time at level K	$g(S,t)=\delta_K(S)$	$f(S,t)=\frac{(S-K)^+}{K^2}$

Conditional Instantaneous FWD Variance

From local time:

$$E\left[\int_0^T \sigma_t^2 \delta_K(S)dt\right] = 2\times\frac{C(K,T)}{K^2}$$

Differentiating wrt T:

$$E\left[\sigma_T^2\delta_K(S_T)\right] = E\left[\sigma_T^2 \middle| S_T = K\right]\cdot E\left[\delta_K(S_T)\right] = \frac{2}{K^2}\times\frac{\partial C}{\partial T}(K,T)$$

And, as:

$$E\left[\delta_K(S_T)\right] = \frac{\partial^2 C}{\partial K^2}(K,T)$$

$$E\left[\sigma_T^2 \middle| S_T = K\right] = \frac{2}{K^2}\times\frac{\frac{\partial C}{\partial T}(K,T)}{\frac{\partial^2 C}{\partial K^2}(K,T)} = \sigma_{loc}^2(K,T)$$

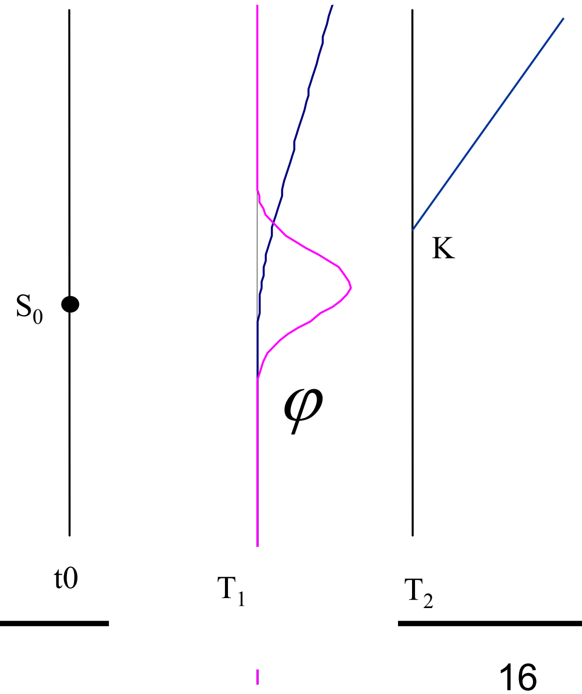
Deterministic future smiles

It is not possible to prescribe just any future smile

If deterministic, one must have

$$C_{K,T_2}(S_0,t_0)=\int \varphi(S_0,t_0,S,T_1)\,C_{K,T_2}(S\ ,T_1)\,dS$$

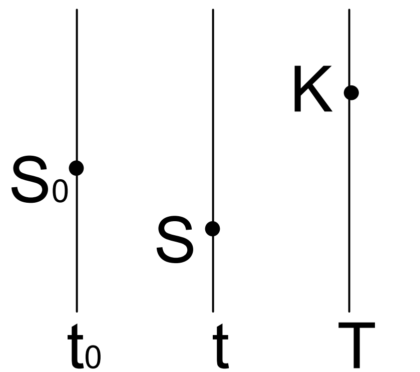
Not satisfied in general



Det. Fut. smiles & no jumps => = FWD smile

If $\exists(S,t,K,T)\,/\,V_{K,T}(S,t) \neq \overline{\sigma}^2(K,T) \equiv \lim_{\substack{\delta K \to 0 \\ \delta T \to 0}} \sigma^2_{imp}(K,T,K+\delta K,T+\delta T)$

stripped from SmileS.t



Then, there exists a 2 step arbitrage:
Define $PL_t \equiv \left(\overline{\sigma}^2(K,T) - V_{K,T}(S,t)\right)\dfrac{\partial^2 C}{\partial K^2}(S,t,K,T)$

At t0 : Sell $PL_t \cdot \left(Dig_{S-\varepsilon,t} - Dig_{S+\varepsilon,t}\right)$

At t: if $S_t \in [S-\varepsilon, S+\varepsilon]$ buy $\dfrac{2}{K^2}\mathrm{CS}_{\mathrm{K,T}}$, sell $\overline{\sigma}^2(K,T)\delta_{K,T}$

gives a premium = PLt at t, no loss at T

Conclusion: $V_{K,T}(S,t)$ independent of $(S,t) = V_{K,T}(S_0,t_0) = \sigma^2(K,T)$ from initial smile

Consequence of det. future smiles

- Sticky Strike assumption: Each (K,T) has a fixed $\sigma_{impl}(K,T)$ independent of (S,t)
- Sticky Delta assumption: $\sigma_{impl}(K,T)$ depends only on moneyness and residual maturity

- In the absence of jumps,
  - Sticky Strike is arbitrageable
  - Sticky Δ is (even more) arbitrageable

Example of arbitrage with Sticky Strike

Each CK,T lives in its Black-Scholes ($\sigma_{impl}(K,T)$)world

$$C_1 \equiv C_{K_1,T_1} \quad C_2 \equiv C_{K_2,T_2} \quad \text{assume } \sigma_1 > \sigma_2$$

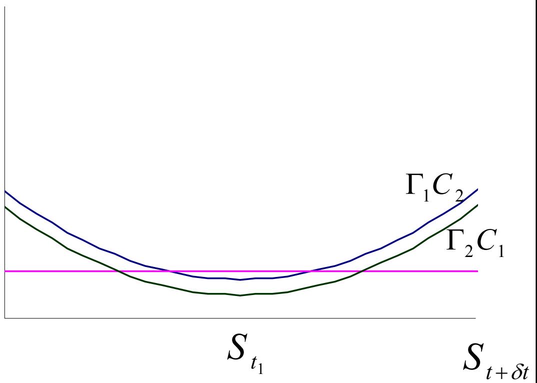
P&L of Delta hedge position over dt:

$$\delta PL(C_1) = \tfrac{1}{2}\left((\delta S)^2 - \sigma_1 S^2 \delta t\right)\Gamma_1$$

$$\delta PL(C_2) = \tfrac{1}{2}\left((\delta S)^2 - \sigma_2 S^2 \delta t\right)\Gamma_2$$

$$\delta PL(\Gamma_1 C_2 - \Gamma_2 C_1) = \frac{\Gamma_1 \Gamma_2}{2} S^2 \left(\sigma_1^2 - \sigma_2^2\right)\delta t > 0$$

$$(\text{no } \Gamma, \text{ free } \Theta)$$



⚠ If no jump

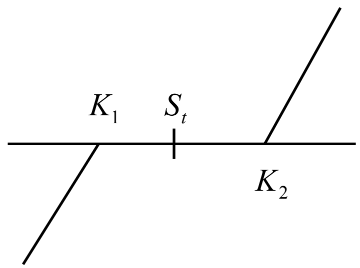
Arbitraging Skew Dynamics

- In the absence of jumps, Sticky-K is arbitrageable and Sticky-Δ even more so.
- However, it seems that quiet trending market (no jumps!) are Sticky-Δ.

⇒ In trending markets, buy Calls, sell Puts and Δ-hedge.

Example:

$$PF \equiv C_{K_2} - P_{K_1}$$



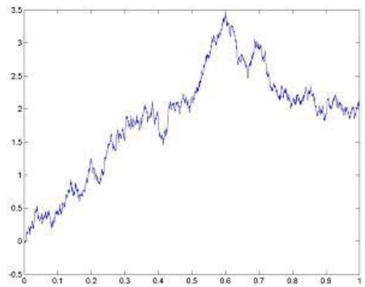
S ↗ ⇒ $\left\{ \begin{array}{c} \sigma_1, \sigma_2 \nearrow \\ \text{Vega}_{K_2} > \text{Vega}_{K_1} \end{array} \right\}$ ⇒ PF ↗

S ↘ ⇒ $\left\{ \begin{array}{c} \sigma_1, \sigma_2 \searrow \\ \text{Vega}_{K_2} < \text{Vega}_{K_1} \end{array} \right\}$ ⇒ PF ↗

Δ-hedged PF gains from S induced volatility moves.

Skew from Historical Prices

Theoretical Skew from Prices



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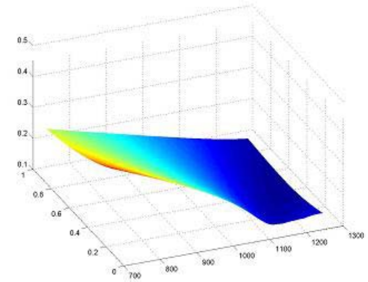


Problem : How to compute option prices on an underlying without options?

For instance : compute 3 month 5% OTM Call from price history only.

1) Discounted average of the historical Intrinsic Values.

   Bad : depends on bull/bear, no call/put parity.

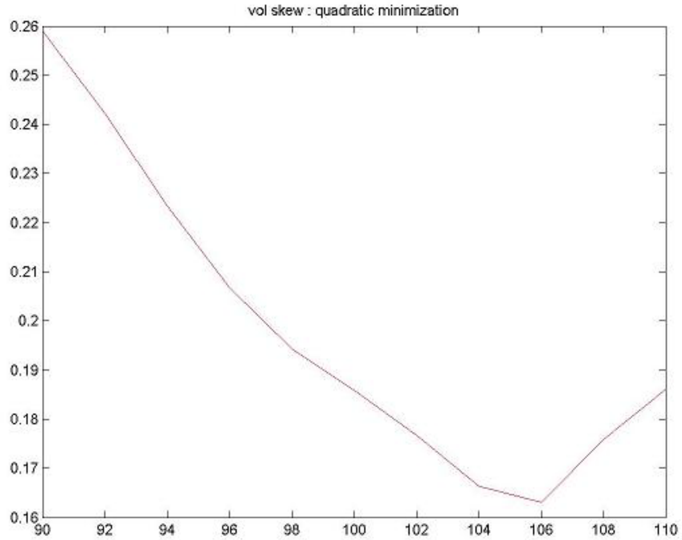
2) Generate paths by sampling 1 day return recentered histogram.

   Problem : CLT ■converges quickly to same volatility for all strike/maturity; breaks autocorrelation and vol/spot dependency.

Theoretical Skew from Prices (2)

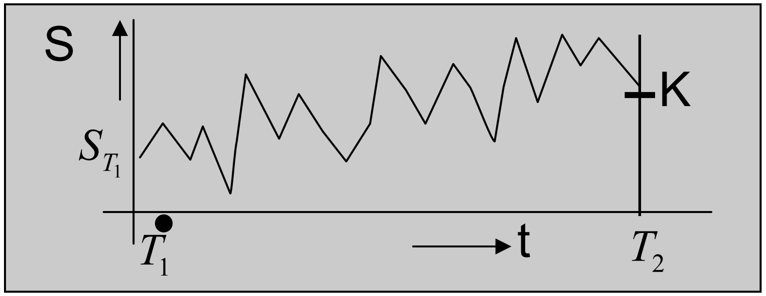
3) Discounted average of the Intrinsic Value from recentered 3 month histogram.

4) Δ-Hedging : compute the implied volatility which makes the Δ-hedging a fair game.



Theoretical Skew from historical prices

How to get a theoretical Skew just from spot price history?



Example:

3 month daily data

1 strike $K = k\, S_{T_1}$

- a) price and delta hedge for a given σ within Black-Scholes model
- b) compute the associated final Profit & Loss: $PL(\sigma)$
- c) solve for $\sigma(k) / PL(\sigma(k)) = 0$
- d) repeat a) b) c) for general time period and average
- e) repeat a) b) c) and d) to get the “theorical Skew”

IV. Volatility Expansion

Introduction

- This talk aims at providing a better understanding of:
  - How local volatilities contribute to the value of an option
  - How P&L is impacted when volatility is misspecified
  - Link between implied and local volatility
  - Smile dynamics
  - Vega/gamma hedging relationship

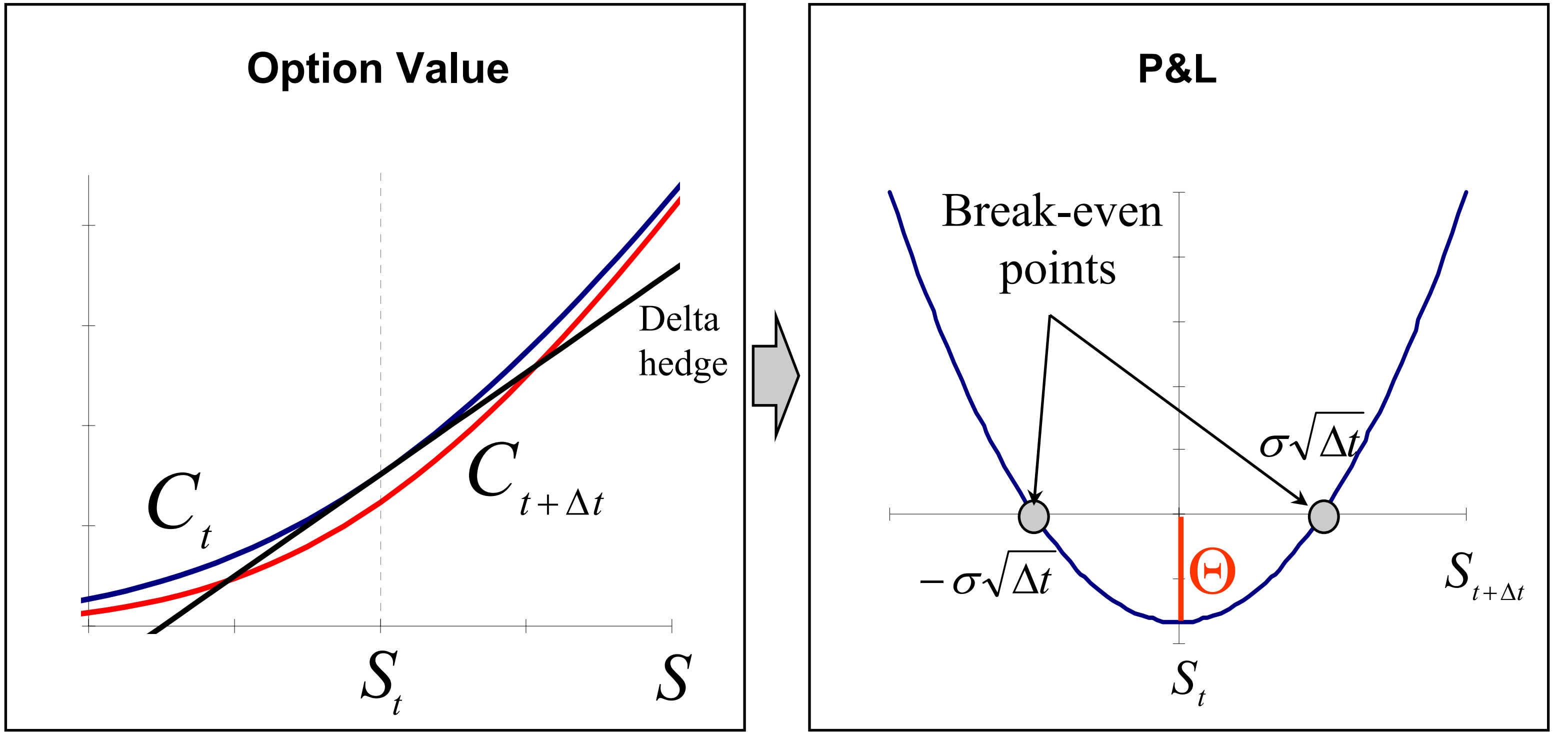
Framework & definitions

- In the following, we specify the dynamics of the spot in absolute convention (as opposed to proportional in Black-Scholes) and assume no rates:

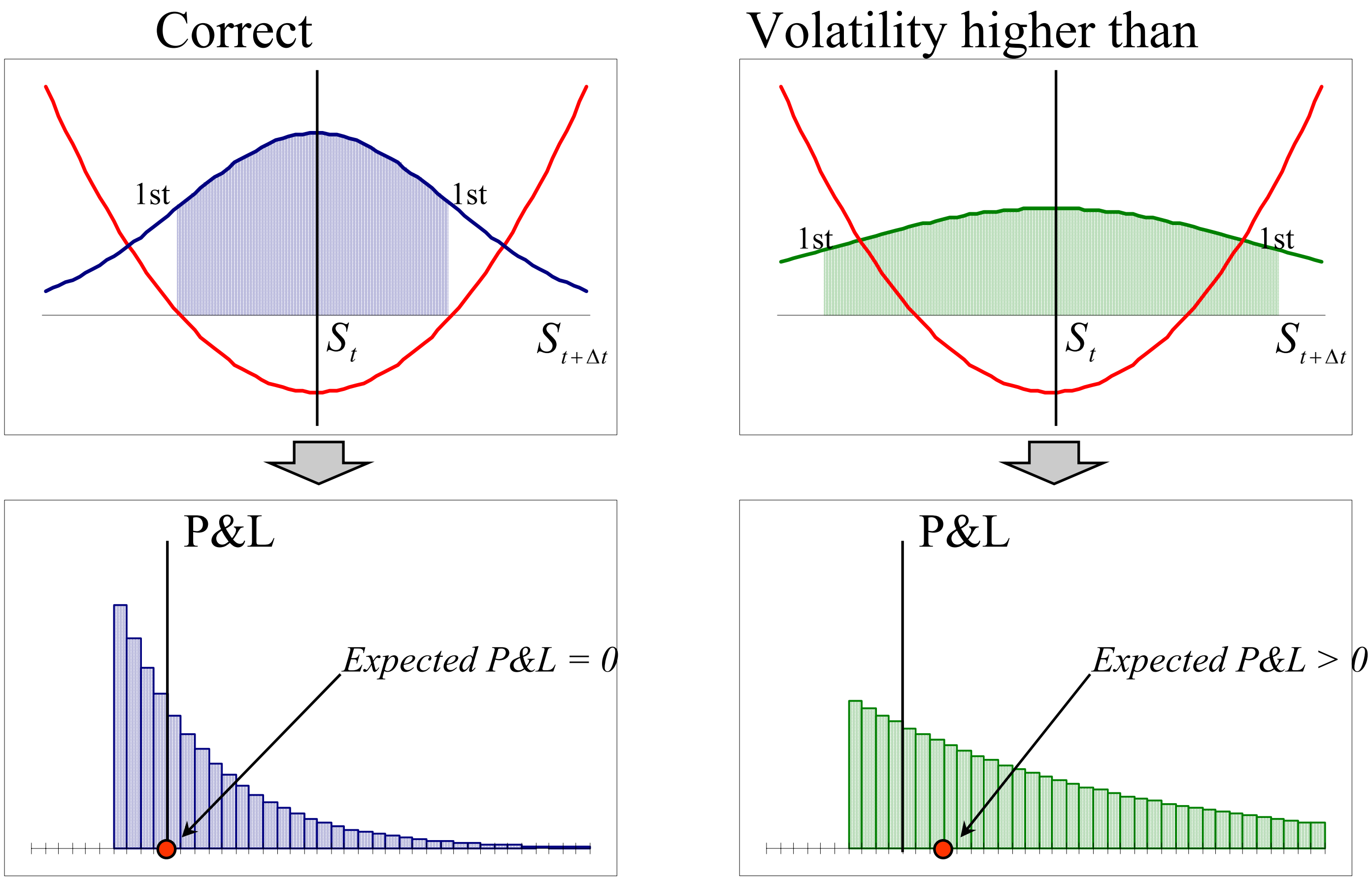
$$dS_t = \sigma_t \, dW_t$$

- σ : local (instantaneous) volatility (possibly stochastic)
- Implied volatility will be denoted by $\hat{\sigma}$

P&L of a delta hedged option



P&L of a delta hedged option (2)



Ito: When $\Delta t \to 0$, spot dependency disappears

Black-Scholes PDE

P&L is a balance between gain from Γ and

$$P\&L_{(t,t+dt)} = \left(\frac{\sigma^2}{2}\Gamma_0 + \Theta_0\right)dt$$

From Black-Scholes PDE: $\Theta_0 = -\frac{\sigma_0^2}{2}\Gamma_0$

=> discrepancy if σ different from

$$\text{gain over dt} = \frac{1}{2}\left(\sigma^2 - \sigma_0^2\right)\Gamma_0 dt$$

- $\sigma > \sigma_0$: Profit
- $\sigma < \sigma_0$: Loss

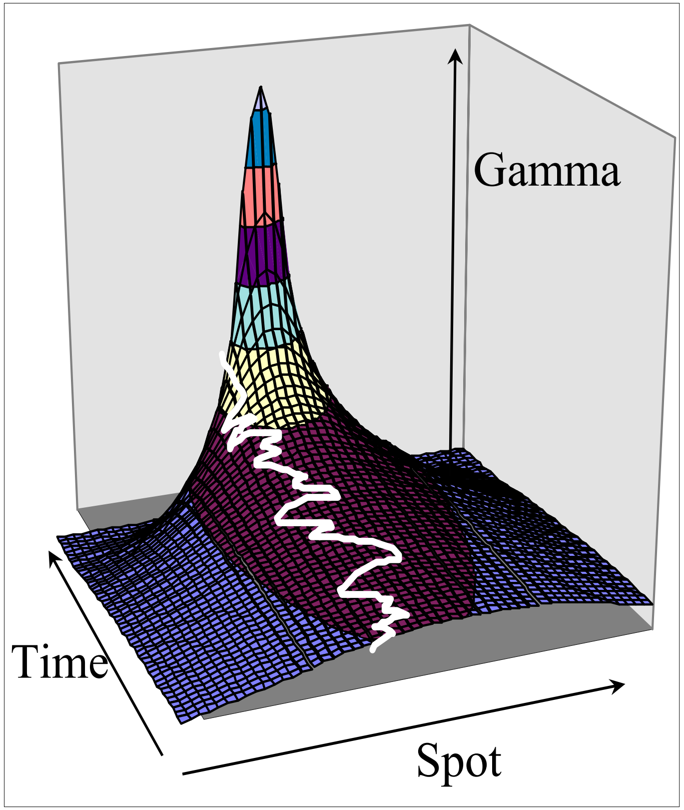
} Magnified by Γ_0

P&L over a path

Total P&L over a path

= Sum of P&L over all small time intervals

$$P\&L = \frac{1}{2}\int_0^T (\sigma^2 - \sigma_0^2)\,\Gamma_0\,dt$$



No assumption is made on volatility so far

General case

- Terminal wealth on each path is:

$$\text{wealth}_{\mathrm{T}} = X\left(\Sigma_0\right) + \frac{1}{2}\int_0^T (\sigma^2 - \sigma_0^2)\,\Gamma_0 dt$$

($X\left(\Sigma_0\right)$ is the initial price of the option)

- Taking the expectation, we get:

$$E^{\varphi}\left[\text{wealth}_{\mathrm{T}}\right] = X(\Sigma_0) + \frac{1}{2}\int_0^T \int_0^\infty E[\Gamma_0(\sigma^2 - \sigma_0^2)|S]\,\varphi\, dS\, dt$$

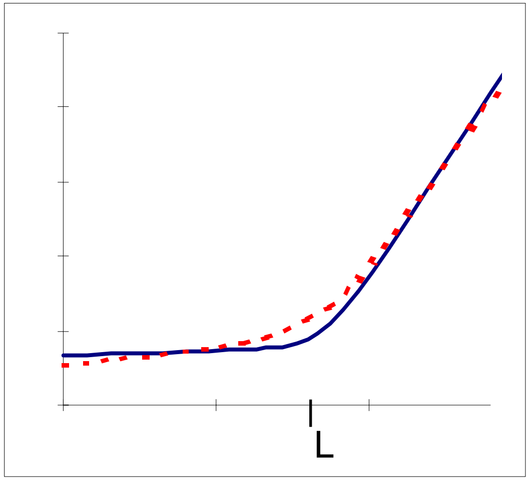
- The probability density φ may correspond to the density of a NON risk-neutral process (with some drift) with volatility σ.

Non Risk-Neutral world

• In a complete model (like Black-Scholes), the drift does not affect option prices but alternative hedging strategies lead to different expectations

Example: mean reverting process towards L with high volatility around L

We then want to choose K (close to L) T and σ_0 (small) to take advantage of it.



Profile of a call (L,T) for different vol assumptions

In summary: gamma is a volatility collector and it can be shaped by:

- a choice of strike and maturity,
- a choice of σ_0 , our hedging volatility.

Average P&L

- From now on, φ will designate the risk neutral density associated with $dS_t = \sigma dW_t$.
- In this case, E[wealth$_T$] is also $X(\Sigma)$ and we have:

$$X(\Sigma) = X(\Sigma_0) + \frac{1}{2}\int_0^T \int_0^\infty E[\Gamma_0(\sigma^2 - \sigma_0^2)|S]\,\varphi\,dS\,dt$$

- Path dependent option & deterministic vol:

$$X(\Sigma) = X(\Sigma_0) + \frac{1}{2}\int\int(\sigma^2 - \sigma_0^2)\,E[\Gamma_0|S]\,\varphi\,dS\,dt$$

- European option & stochastic vol:

$$C(\Sigma) = C(\Sigma_0) + \frac{1}{2}\int\int(E[\sigma^2|S] - \sigma_0^2)\,\Gamma_0\,\varphi\,dS\,dt$$

Quiz

- Buy a European option at 20% implied vol
- Realised historical vol is 25%
- Have you made money ?

Not necessarily!

High vol with low gamma, low vol with high gamma

Expansion in volatility

• An important case is a European option with deterministic vol:

$$C(\Sigma) = C(\Sigma_0) + \frac{1}{2}\int\int(\sigma^2 - \sigma_0^2)\Gamma_0\,\varphi\,dS\,dt$$

• The corrective term is a weighted average of the volatility differences

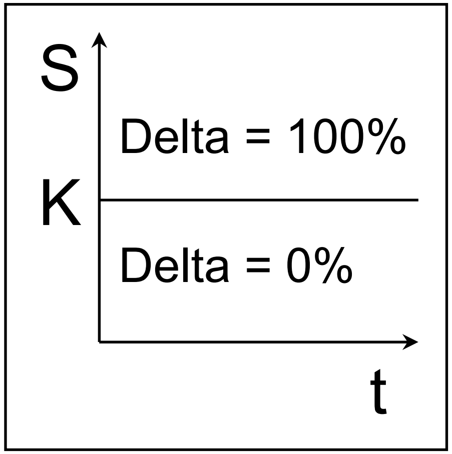
• This double integral can be approximated numerically

P&L: Stop Loss Start Gain

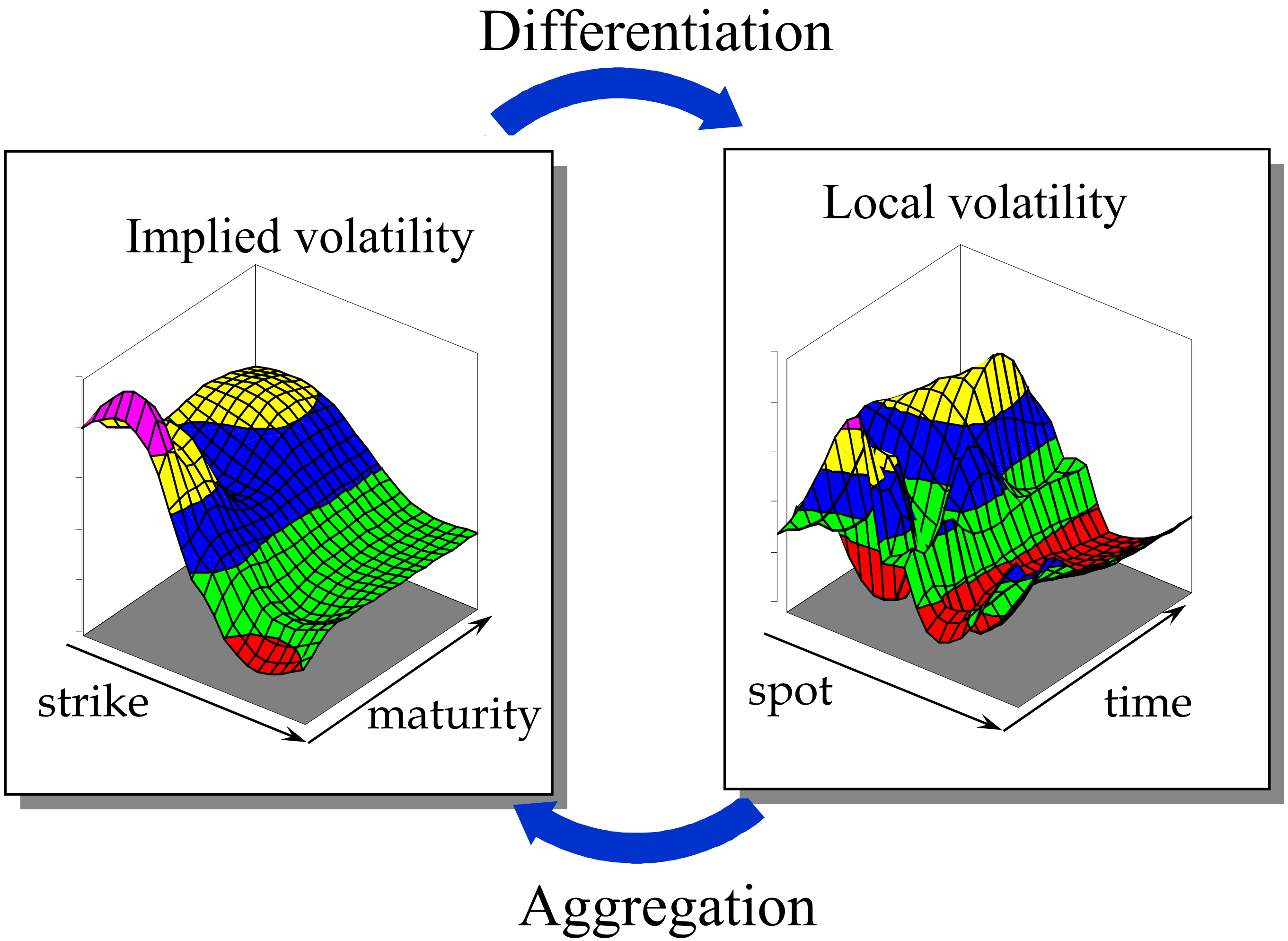
- Extreme case: $\sigma_0 = 0 \Rightarrow \Gamma_0 = \delta_K$

$$C(\Sigma) = (S_0 - K)^+ + \frac{1}{2}\int_0^T \sigma(K,t)^2 \varphi(K,t)dt$$

- This is known as Tanaka's formula



Local / Implied volatility relationship



Smile stripping: from implied to local

•Stripping local vols from implied vols is the inverse operation:

$$\sigma^2(S,T) = 2\frac{\frac{\partial C}{\partial T}}{\frac{\partial^2 C}{\partial K^2}} \qquad \text{(Dupire 93)}$$

•Involves differentiations

From local to implied: a simple case

Let us assume that local volatility is a deterministic function of time only:

$$dS_t = \sigma(t)\, dW_t$$

In this model, we know how to combine local vols to compute implied vol:

$$\hat{\sigma}(T) = \sqrt{\frac{\int_0^T \sigma^2(t)dt}{T}}$$

Question: can we get a formula with $\sigma(S,t)$?

From local to implied volatility

- When σ_0 = implied vol

$$\frac{1}{2}\iint (\sigma^2 - \sigma_0^2)\Gamma_0 \varphi \, dS dt = 0 \implies \sigma_0^2 = \frac{\iint \sigma^2 \Gamma_0 \varphi \, dS dt}{\iint \Gamma_0 \varphi \, dS dt}$$

- Γ_0 depends on σ_0 $\implies$ solve by iterations

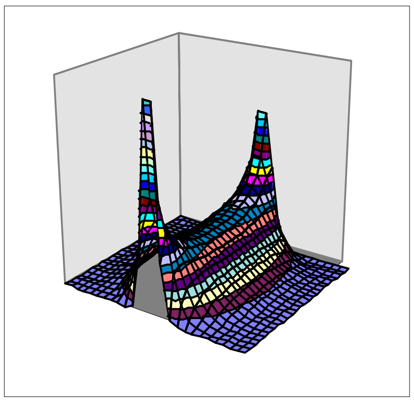
•Implied Vol is a weighted average of Local Vols

(as a swap rate is a weighted average of FRA)

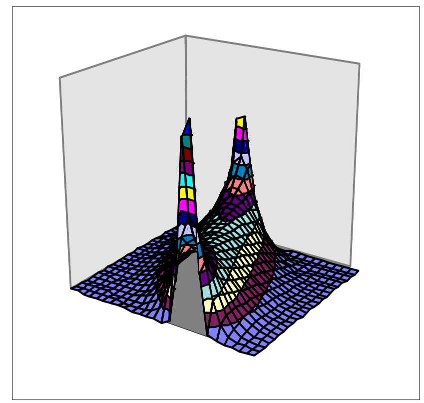
Weighting scheme

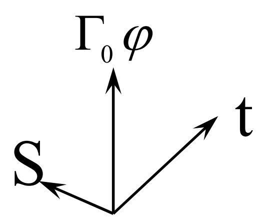
• Weighting Scheme: proportional to $\Gamma_0 \varphi$

At the money case:

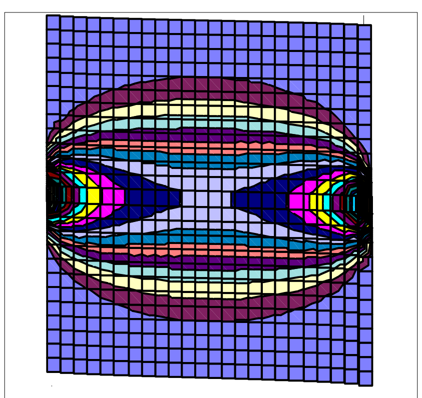


Out of the money case:

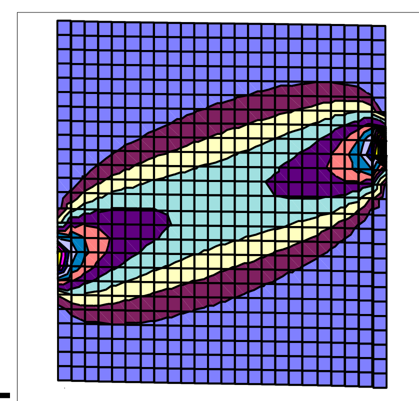




S_0=100
K=100

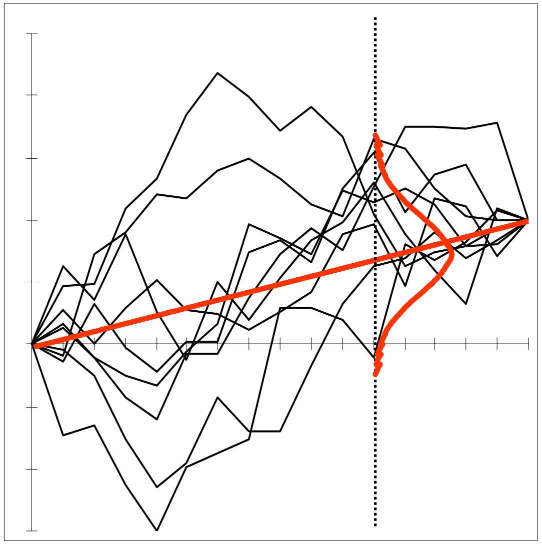


S_0=100
K=110



Weighting scheme (2)

- Weighting scheme is roughly proportional to the brownian bridge density



Brownian bridge density:

$$BB\varphi_{K,T}(x,t) = P\left[S_t = x \middle| S_T = K\right]$$

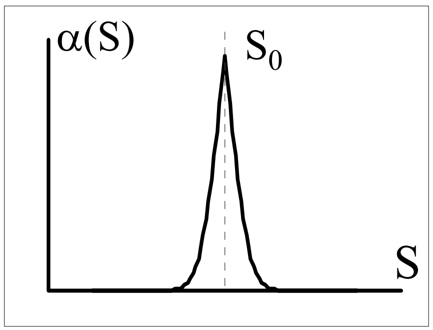
Time homogeneous case

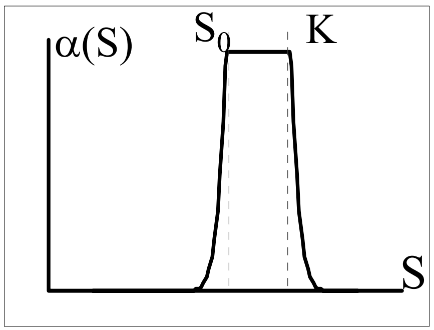
$$\hat{\sigma}^2 = \int \alpha(S)\sigma^2(S)dS$$

$$\alpha(S) = \frac{\int \Gamma_0 \varphi dt}{\iint \Gamma_0 \varphi \, dSdt}$$

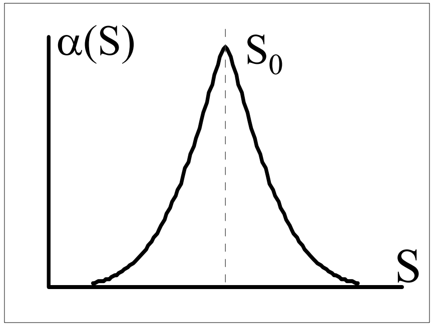
ATM ($K=S_0$) | OTM ($K>S_0$)

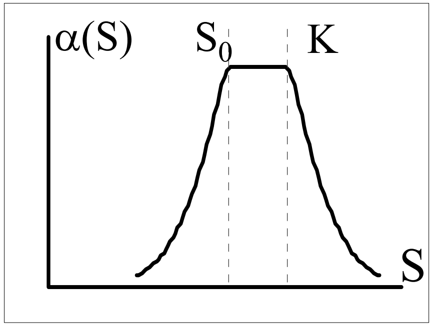
$\sigma\sqrt{T}$ small





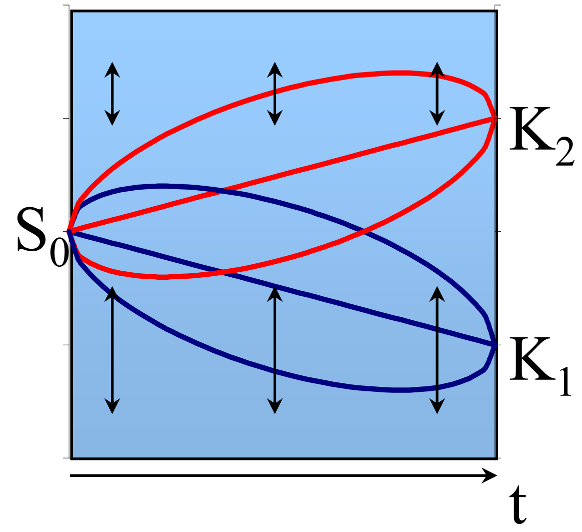
$\sigma\sqrt{T}$ large





Link with smile

$\hat{\sigma}_{K_1}$ and $\hat{\sigma}_{K_2}$ are averages of the same local vols with different weighting schemes



=> New approach gives us a direct expression for the smile from the knowledge of local volatilities

But can we say something about its dynamics?

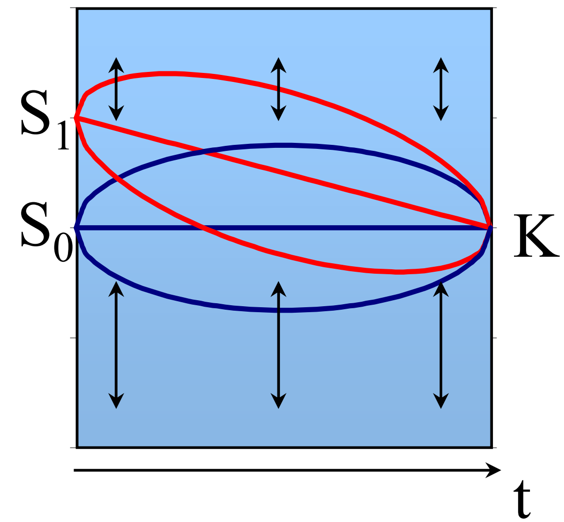
Smile dynamics

Weighting scheme imposes some dynamics of the smile for a move of the spot:

For a given strike K,

$$S\uparrow \Rightarrow \hat{\sigma}_K \downarrow$$

(we average lower volatilities)



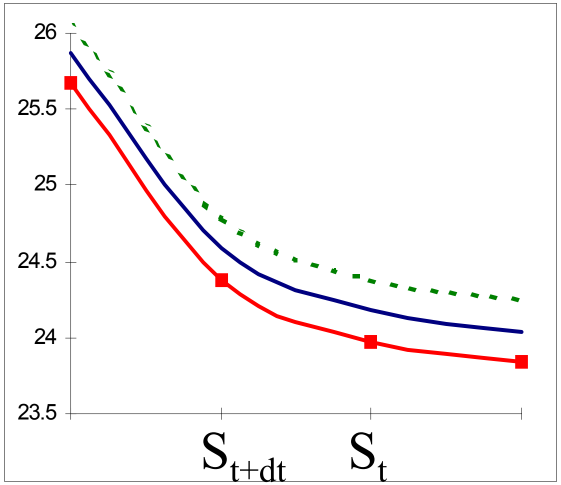
Smile today (Spot S_t)

&

Smile tomorrow (Spot S_{t+dt}) in sticky strike model

Smile tomorrow (Spot S_{t+dt}) if σ_{ATM}=constant

Smile tomorrow (Spot S_{t+dt}) in the smile model

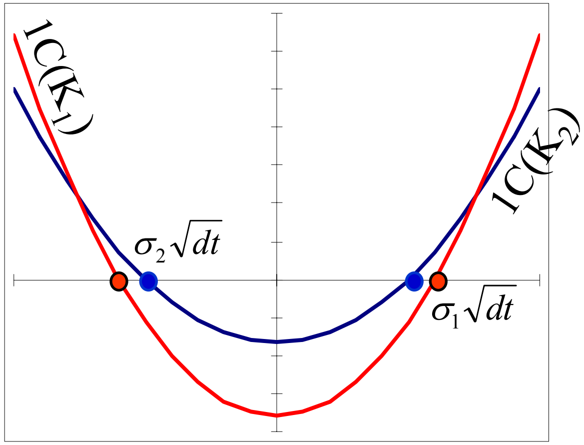


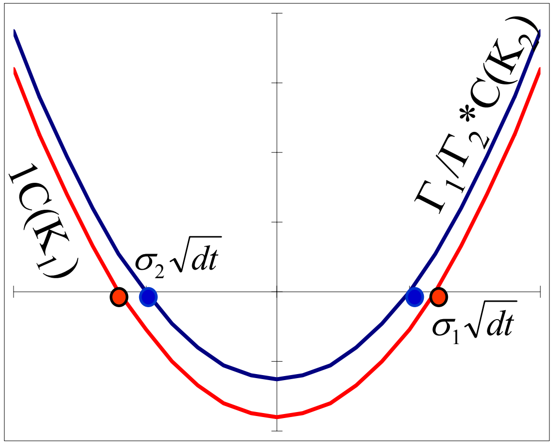
Sticky strike model

A sticky strike model ($\hat{\sigma}_K(t) = \hat{\sigma}_K$) is arbitrageable.

Let us consider two strikes $K_1 < K_2$

The model assumes constant vols $\sigma_1 > \sigma_2$ for example





By combining K_1 and K_2 options, we build a position with no gamma and positive theta (sell 1 K_1 call, buy Γ_1/Γ_2 K_2 calls)

Vega analysis

- If σ & σ_0 are constant

$$C(\Sigma) = C(\Sigma_0) + \frac{1}{2}(\sigma^2 - \sigma_0^2)\iint \Gamma_0 \varphi \, dSdt$$

- $\sigma^2 = \sigma_0^2 + \varepsilon$

$$C(\sigma_0^2 + \varepsilon) = C(\sigma_0^2) + \varepsilon \underbrace{\frac{1}{2}\iint \Gamma_0 \varphi \, dSdt}_{\frac{\partial^2 C}{\partial \sigma^2}}$$

$$\text{Vega} = \frac{\partial C}{\partial \sigma} = \frac{\partial C}{\partial \sigma^2} \cdot \frac{\partial \sigma^2}{\partial \sigma} = \frac{\partial C}{\partial \sigma^2} \cdot 2\sigma$$

Gamma hedging vs Vega hedging

- Hedge in $\Gamma \longrightarrow$ insensitive to realised historical vol
- If $\Gamma=0$ everywhere, no sensitivity to historical vol => no need to Vega hedge
- Problem: impossible to cancel Γ now for the future
- Need to roll option hedge
- How to lock this future cost?
- Answer: by vega hedging

Superbuckets: local change in local vol

For any option, in the deterministic vol case:

$$X(\Sigma) = X(\Sigma_0) + \frac{1}{2}\int\int (\sigma^2 - \sigma_0^2)\, E[\Gamma_0 | S]\, \varphi\, dS\, dt$$

For a small shift ε in local variance around (S,t), we have:

$$X(\Sigma) = X(\Sigma_0) + \frac{1}{2}\varepsilon\, E[\Gamma_0 | S]\, \varphi$$

$$\Rightarrow \frac{dX}{d\left(\sigma^2_{(S,t)}\right)} = \frac{1}{2} E[\Gamma_0(S,t) | S]\, \varphi(S,t)$$

For a european option: $\dfrac{dC}{d\left(\sigma^2_{(S,t)}\right)} = \dfrac{1}{2}\Gamma_0(S,t)\varphi(S,t)$

Superbuckets: local change in implied vol

Local change of implied volatility is obtained by combining local changes in local volatility according a certain weighting

$$\frac{dC}{d\left(\hat{\sigma}^2\right)} = \int \frac{dC}{d\left(\sigma^2\right)} \frac{d\left(\sigma^2\right)}{d\left(\hat{\sigma}^2\right)}$$

sensitivity in local vol

weighting obtain using stripping formula

Thus:

cancel sensitivity to any move of implied vol

<=> cancel sensitivity to any move of local vol

<=> cancel all future gamma in expectation

Conclusion

- This analysis shows that option prices are based on how they capture local volatility
- It reveals the link between local vol and implied vol
- It sheds some light on the equivalence between full Vega hedge (superbuckets) and average future gamma hedge

Delta Hedging

- We assume no interest rates, no dividends, and absolute (as opposed to proportional) definition of volatility
- Extend f(x) to f(x,v) as the Bachelier (normal BS) price of f for start price x and variance v:

$$f(x,v) \equiv E^{x,v}[f(X)] \equiv \frac{1}{\sqrt{2\pi v}} \int f(y) e^{-\frac{(y-x)^2}{2v}} dy$$

with f(x,0) = f(x)

- Then,

$$f_v(x,v) = \frac{1}{2} f_{xx}(x,v)$$

- We explore various delta hedging strategies

Calendar Time Delta Hedging

- Delta hedging with constant vol: P&L depends on the path of the volatility and on the path of the spot price.
- Calendar time delta hedge: replication cost of

$$f(X_t, \sigma^2.(T-t))$$

$$f(X_0, \sigma^2 T) + \frac{1}{2}\int_0^t f_{xx}(dQV_{0,u} - \sigma^2 du)$$

- In particular, for sigma = 0, replication cost of

$$f(X_t)$$

$$f(X_0) + \frac{1}{2}\int_0^t f_{xx} dQV_{0,u}$$

Business Time Delta Hedging

- Delta hedging according to the quadratic variation: P&L that depends <u>only</u> on quadratic variation and spot price

$$df(X_t, L - QV_{0,t}) = f_x dX_t - f_v dQV_{0,t} + \frac{1}{2} f_{xx} dQV_{0,t} = f_x dX_t$$

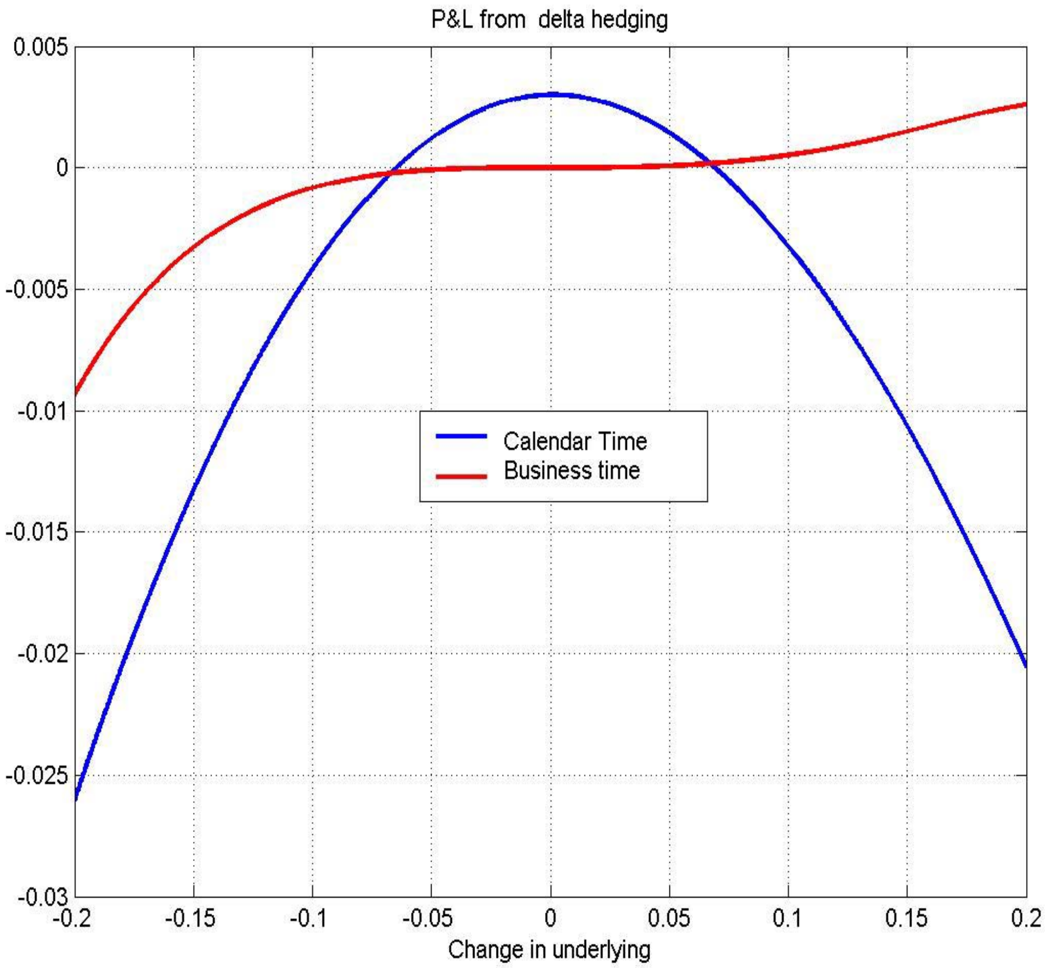
- Hence, for $QV_{0,T} \leq L,$

$$f(X_t, L - QV_{0,t}) = f(X_0, L) + \int_0^t f_x(X_u, L - QV_{0,u})\, dX_t$$

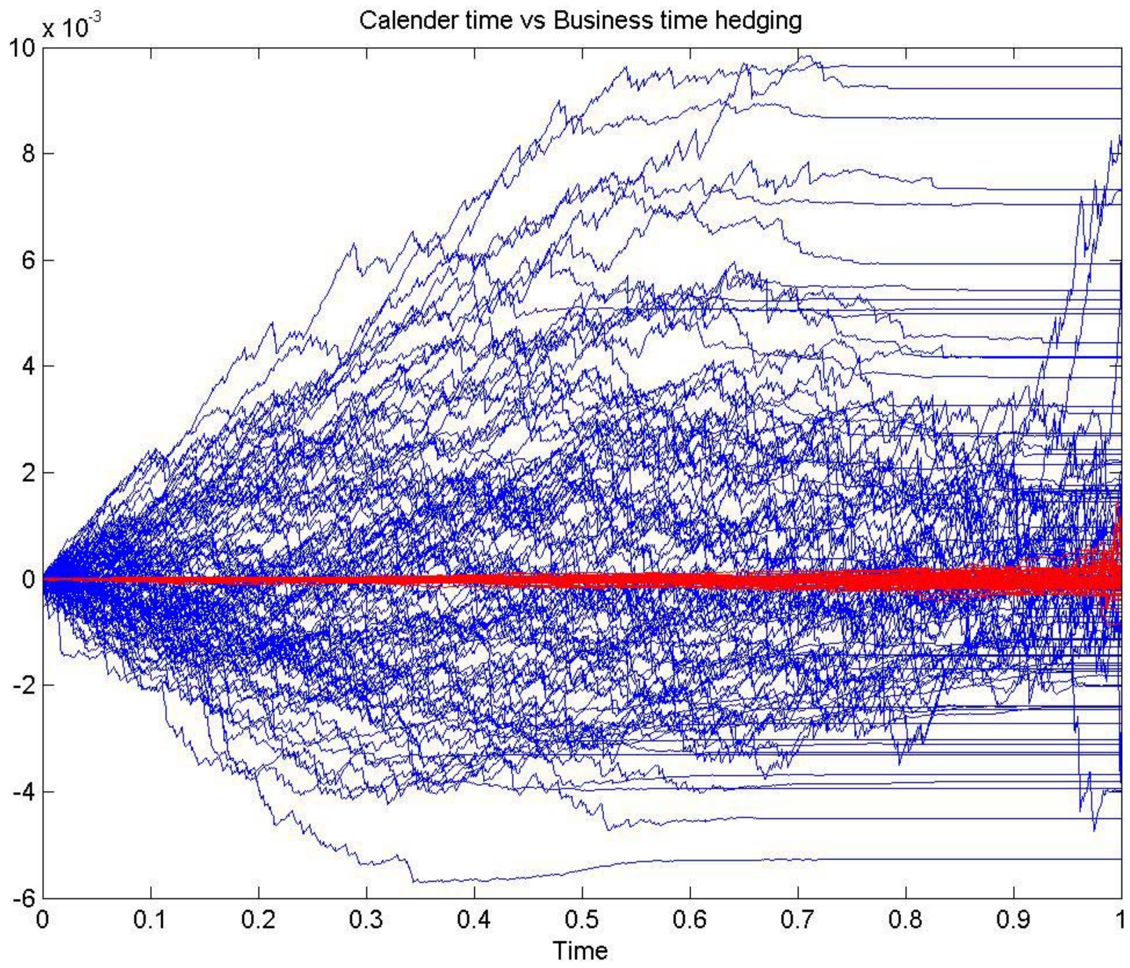
And the replicating cost of $f(X_t, L - QV_{0,t})$ is $f(X_0, L)$

$f(X_0, L)$ finances exactly the replication of f until $\tau : QV_{0,\tau} = L$

Daily P&L Variation



Tracking Error Comparison



V. Stochastic Volatility Models

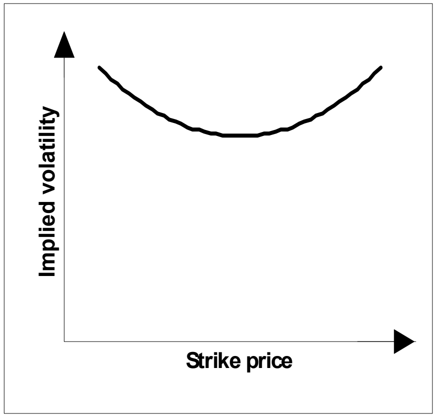
Hull & White

- Stochastic volatility model **Hull&White (87)**

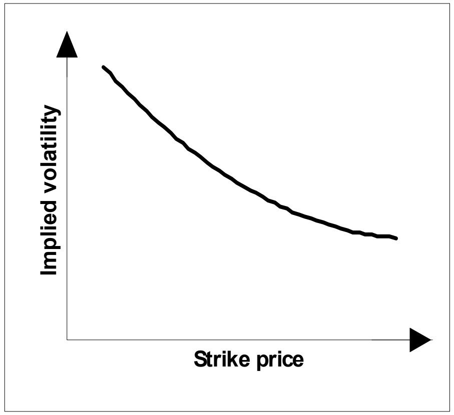
$$\frac{dS_t}{S_t} = r\,dt + \sigma_t\,dW_t^P$$

$$d\sigma_t = \alpha\,dt + \beta\,dZ_t^P$$

- Incomplete model, depends on risk premium
- Does not fit market smile



$\rho_{Z,W} = 0$



$\rho_{Z,W} < 0$

Role of parameters

- Correlation gives the short term skew
- Mean reversion level determines the long term value of volatility
- Mean reversion strength
  - Determine the term structure of volatility
  - Dampens the skew for longer maturities
- Volvol gives convexity to implied vol
- Functional dependency on S has a similar effect to correlation

Heston Model

$$\begin{cases} \frac{dS}{S} = \mu \, dt + \sqrt{v} \, dW \\ dv = \lambda(\bar{v} - v)dt + \eta\sqrt{v}dZ \quad \langle dW, dZ \rangle = \rho \, dt \end{cases}$$

Solved by Fourier transform:

$$x \equiv \ln \frac{FWD}{K} \quad \tau = T - t$$

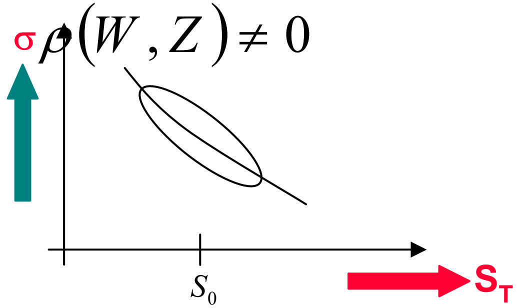
$$C_{K,T}(x, v, \tau) = e^x P_1(x, v, \tau) - P_0(x, v, \tau)$$

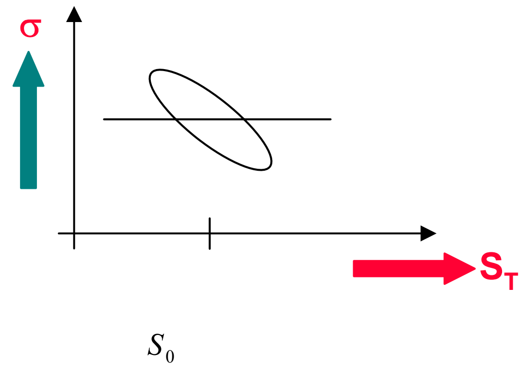
Spot dependency

2 ways to generate skew in a stochastic vol model

1) $\sigma_t = x_t f(S,t), \rho(W,Z) = 0$

2) $\rho(W,Z) \neq 0$





-Mostly equivalent: similar (S_t,σ_t) patterns, similar future evolutions

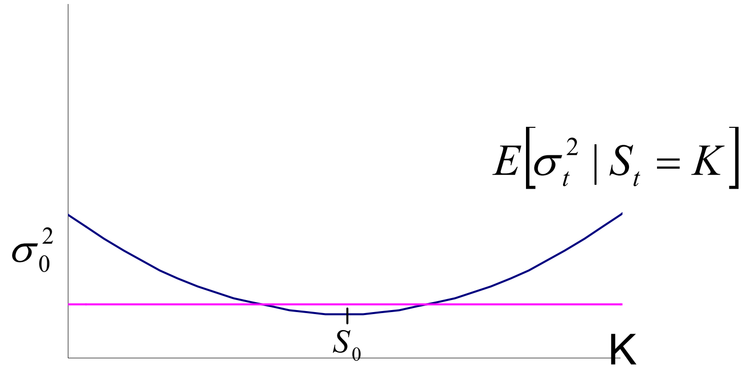
-1) more flexible (and arbitrary!) than 2)

-For short horizons: stoch vol model ⬄ local vol model + independent noise on vol.

Convexity Bias

$$\begin{cases} dS = \sigma_t dW \\ d\sigma_t^2 = \alpha dZ \\ \rho(W, Z) = 0 \end{cases} \Rightarrow \quad E\left[\sigma_t^2 \mid S_t = K\right] = \sigma_0^2 \ ?$$

NO! only $E\left[\sigma_t^2\right] = \sigma_0^2$



σ_t likely to be high if $S_t >> S_0$ or $S_t << S_0$

Impact on Models

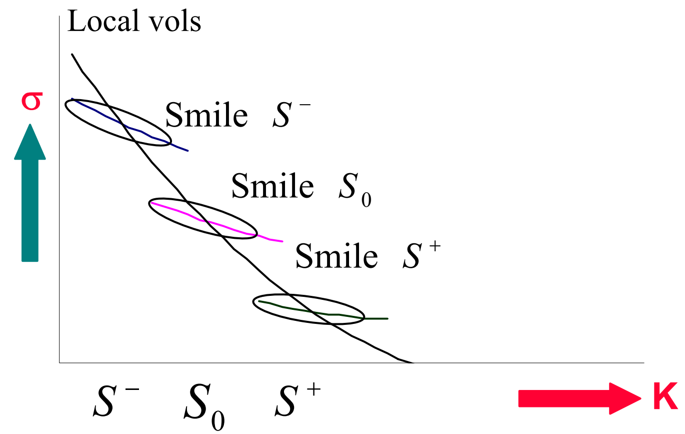
- Risk Neutral drift for instantaneous forward variance
- Markov Model:

$$\frac{dS}{S} = f(S,t)\sigma_t dW$$ fits initial smile with local vols $\sigma(S,t)$

$$\Leftrightarrow f(S,t) = \frac{\sigma^2(S,t)}{E[\sigma_t^2 \mid S_t = S]}$$

Smile dynamics: Stoch Vol Model (1)

Skew case (r<0)



- ATM short term implied still follows the local vols

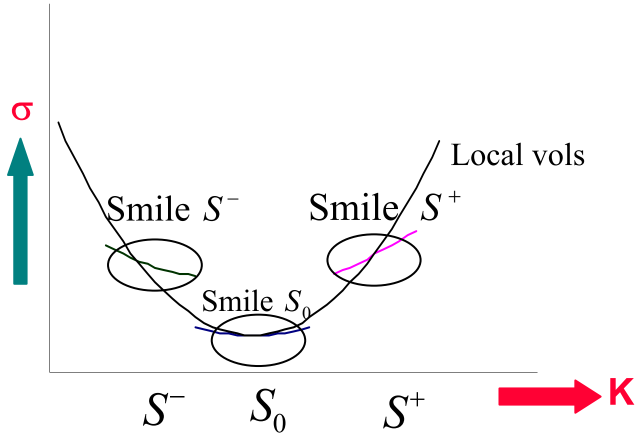
$$\left(E\left[\sigma^2{}_T \middle| S_T = K\right] = \sigma^2(K,T)\right)$$

- Similar skews as local vol model for short horizons
- Common mistake when computing the smile for another spot: just change S_0 forgetting the conditioning on σ :

if S : $S_0 \rightarrow S^+$ where is the new σ ?

Smile dynamics: Stoch Vol Model (2)

- Pure smile case (r=0)



- ATM short term implied follows the local vols
- Future skews quite flat, different from local vol model
- Again, do not forget conditioning of vol by S

Forward Skew

Forward Skews

In the absence of jump :

model fits market $\Leftrightarrow \forall K,T \quad E[\sigma_T^2 | S_T = K] = \sigma_{loc}^2(K,T)$

This constrains

a) the sensitivity of the ATM short term volatility wrt S;

b) the average level of the volatility conditioned to S_T=K.

a) tells that the sensitivity and the hedge ratio of vanillas depend on the calibration to the vanilla, not on local volatility/ stochastic volatility.

To change them, jumps are needed.

<u>But</u> b) does not say anything on the conditional forward skews.

Sensitivity of ATM volatility / S

At t, short term ATM implied volatility ~ σ_t.

As σ_t is random, the sensitivity $\dfrac{\partial \sigma^2}{\partial S}$ is defined only in average:

$$E_t[\sigma^2_{t+\delta t} - \sigma^2_t \,|\, S_{\delta t} = S_t + \delta S] = \sigma^2_{loc}(S_t + \delta S, t + \delta t) - \sigma^2_{loc}(S_t - t) \approx \frac{\partial \sigma^2_{loc}(S,t)}{\partial S} \cdot dS$$

In average, σ^2_{ATM} follows σ^2_{loc}.

Optimal hedge of vanilla under calibrated stochastic volatility corresponds to perfect hedge ratio under LVM.