# Robust Dynamic Spanning

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# Motivation

- The classic prescription for hedging the market risk associated with derivatives positions is to restrict the possible process dynamics sufficiently so that the payoff can be spanned by (completely) dynamic trading in the underlying asset(s). This approach introduces model risk.
- We explore alternative strategies for hedging claims written on the price path of a single underlying asset.
- We are especially interested in replicating strategies that have some degree of robustness built in.

# Robust Hedging

- A robust hedge strategy is one that theoretically works for two or more models.
- For example, PCP implies that the sale of a European call can be robustly hedged by buying the underlying stock on margin and also buying the right put.
- As the example shows, robust hedging strategies typically work only for a small class of claims being hedged. Also, they succeed even under stochastic volatility and jumps. In fact, they succeed even though volatility of volatility and the jump arrival rates are unknown. Hence, model risk is largely overcome.
- There may be restrictions on the stochastic process for the underlying asset price. For example, a long forward position can be robustly replicated by buying and borrowing only by assuming that dividends are suitably restricted.
- Robust hedging strategies may or may not be (fully) dynamic.

# Overview

There are 3 parts to this talk:

1. Hedging with Static Positions in European Options and their Underlying
2. Hedging with (just) Semi-Static Trading in the Underlying
3. Static Option Hedging plus Semi-Static Trading in their Underlying

Given the time constraint, I will give just one or two examples of each.

# Assumptions

- We assume that there are no frictions, no illiquidity, no default risk on the overlying, and no arbitrage opportunities.
- The assets which trade continuously include:
  - bonds of all maturities
  - stocks
  - equity forwards of all maturities and delivery prices
  - equity futures of all maturities
  - standard European equity options of all strikes and maturities.
- In general, future stock prices, interest rates, and dividends are arbitrarily random, unless specifically indicated otherwise.

# Part 1: Static Hedging with European Options

- A put with strike $K$ has payoff $(K - S_T)^+$

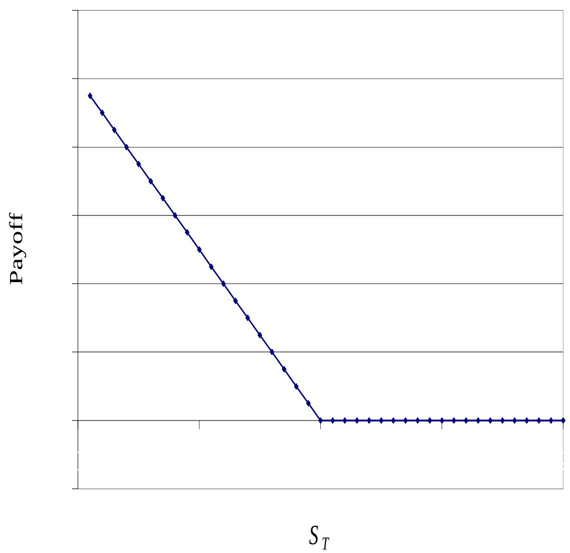


Figure 1: Put Payoff

# Static Hedging with European Options

- A call with strike $K$ has payoff $(S_T - K)^+$

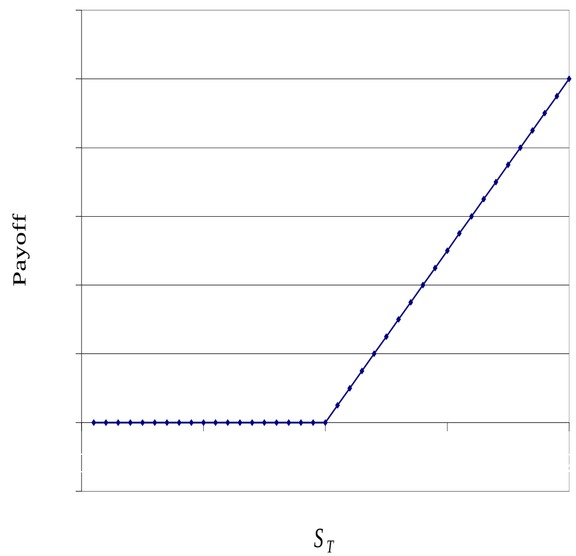


Figure 2: Call Payoff

# Static Hedging of P-I Payoffs

- Appendix 1 proves that for any generalized function $f(S)$ and any scalar $\kappa \geq 0$:

$$
\begin{aligned}
f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) \leftarrow \text{ tangent approximation} \\
&\quad + \int_{\kappa}^{\infty} f''(K)(S-K)^{+} dK + \int_{0}^{\kappa} f''(K)(K-S)^{+} dK \leftarrow \text{ tangent correction.}
\end{aligned}
$$

- This decomposition may be interpreted as a Taylor series expansion with remainder of the final payoff $f(\cdot)$ about the expansion point $\kappa$.
- The first two terms give the tangent to the payoff at $\kappa$; the last two terms continuously bend this tangent so it conforms to the nonlinear payoff.
- The payoff of an arbitrary claim has been decomposed into the payoff from $f(\kappa)$ bonds, $f'(\kappa)$ forward contracts with delivery price $\kappa$, $f''(\kappa)dK$ calls struck above $\kappa$, and $f''(\kappa)dK$ puts struck below.

# From Payoffs to Prices

- Recall the decomposition of the payoff function $f(S)$:

$$
\begin{aligned}
f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) \\
&\quad + \int_0^{\kappa} f''(K)(K-S)^+ dK + \int_{\kappa}^{\infty} f''(K)(S-K)^+ dK.
\end{aligned}
$$

- No arbitrage implies that the initial value $V_0[f]$ can be expressed in terms of the initial prices of the bond $B_0$, calls $C_0(K)$, and puts $P_0(K)$ respectively:

$$
\begin{aligned}
V_0[f] &= f(\kappa)B_0 + f'(\kappa)[C_0(\kappa) - P_0(\kappa)] \\
&\quad + \int_0^{\kappa} f''(K)P_0(K)dK + \int_{\kappa}^{\infty} f''(K)C_0(K)dK.
\end{aligned}
$$

- When $\kappa = F_0$, the second term vanishes by PCP, and the value decomposes as:

$$
V_0[f] = \underbrace{f(F_0)B_0}_{\text{intrinsic value}} + \underbrace{\int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK}_{\text{time value}}.
$$

# Example 1: In-The-Money Call

- Recall the decomposition into intrinsic and time value:

$$V_0[f] = f(F_0)B_0 + \int_0^{F_0} f''(K)P_0(K)dK + \int_{F_0}^{\infty} f''(K)C_0(K)dK.$$

- For example, suppose the final payoff is that of an in-the-money European call, i.e. $f(S) = (S - K_c)^+$, $K_c < F_0$. Formally using the above decomposition with $\kappa = F_0$ gives:

$$C_0(K_c) = (F_0 - K_c)B_0 + P_0(K_c),$$

which is Put Call Parity.

- Thus the equation at the top is a generalization of PCP to multiple options.

# Part 2: Semi-Static Trading in the Underlying

- Thus far, we have only spanned path-independent payoffs. However, for certain path-dependent payoffs, the payoff can be spanned by just semi-static trading in the underlying asset.
- By semi-static, we mean that trades can occur each time that the path must be monitored to compute the payoff of the path-dependent claim. Hence, if the path is continuously monitored as for some barrier options, trading might be continuous. We will therefore focus on (the bigger class of) path-dependent claims with discrete path monitoring.
- As usual, no assumption is made regarding the stochastic process of the underlying. This is useful because even though a model may have worked well in the past, there is no guarantee that it will continue to work well in the future.

# Example: Serial Covariance Contract

- Suppose that we partition the time set $[0, T)$ into $n$ time intervals of the form $[t_i, t_{i+1})$, where:
$$0 \equiv t_0 \leq t_1 \leq t_2 \leq ...t_{n-1} \leq t_n \equiv T.$$
- Let $F_i$ denote the futures price at time $t_i$ for maturity $T$. We assume marking-to-market occurs at each $t_i$.
- Suppose that the payoff on a serial covariance contract is defined as:
$$Cov_n \equiv \frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{F_i - F_{i-1}}{F_{i-1}} \right) \left( \frac{F_{i+1} - F_i}{F_i} \right).$$
In words, the payoff is the average of the products of adjacent returns.
- How do we hedge and price this highly path-dependent payoff?

# Robust Hedging of Covariance Contracts

- Recall that the payoff on a covariance contract was defined as:

$$Cov_n \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right).$$

- Let $r(t)$ be the deterministic spot interest rate at time $t$.

- Suppose we do nothing from day 0 to day 1. If we hold $\frac{e^{-\int_{t_{i+1}}^{t_n} r(u)du}(F_i - F_{i-1})}{(n-1)F_iF_{i-1}}$ futures contracts from time $t_i$ to time $t_{i+1}, i = 1, \ldots, n-1$, then we receive $\left(\frac{e^{-\int_{t_{i+1}}^{t_n} r(u)du}(F_i - F_{i-1})}{(n-1)F_iF_{i-1}}\right) \times (F_{i+1} - F_i)$ in marking-to-market proceeds at time $t_{i+1}$.

- The future value of these proceeds are $\left(\frac{(F_i - F_{i-1})}{(n-1)F_iF_{i-1}}\right) \times (F_{i+1} - F_i)$ by time $t_n$.

- Summing over $i = 1, 2, \ldots, n-1$, the sum of the future values of the marking-to-market proceeds by time $t_n$ is $\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right)$.

# Pricing Covariance Contracts

- Recall that the payoff on a covariance contract was defined as:

$$Cov_n \equiv \frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{F_i - F_{i-1}}{F_{i-1}} \right) \left( \frac{F_{i+1} - F_i}{F_i} \right).$$

- The last page showed that by semi-statically trading futures, this payoff could be perfectly replicated.
- As the initial position is zero futures and as the futures trading strategy is trivially self-financing, the arbitrage-free value of the payoff on the covariance contract is zero.

# Why?

- Recall that the payoff on a covariance contract was defined as:

$$Cov_n \equiv \frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{F_i - F_{i-1}}{F_{i-1}} \right) \left( \frac{F_{i+1} - F_i}{F_i} \right).$$

- It is well known that no arbitrage implies the existence of a probability measure $\mathbb{Q}$ equivalent to the original measure $\mathbb{P}$ such that the futures price is a martingale.
- This martingale is adapted to the futures price process.
- Hence, payoffs of the form $\int_0^T N_t^f dF_t$ are priced at 0 so long as $N_t^f$ just depends on time and the futures price path up to $t$.
- The covariance contract payoff defined above is just a special case.
- All martingales have increments which are uncorrelated. All we have done is to demonstrate the trading strategy in futures that enforces this result.
- One can also trade cross auto-covariance (for zero if written on futures prices).

# Part 3: Static Option + Semi-static Underlying

- For certain path-dependent payoffs,the payoff can be spanned by combining a static position in options with semi-static trading in the underlying.
- As usual, no assumption on the stochastic process for the underlying is needed.
- We illustrate with three examples. The first has discrete path monitoring while the second and third have continuous path monitoring.

# Example 1: Local Variation

- Consider a finite set of discrete times $\{t_0, t_1, \ldots, t_n\}$ at which one can trade futures contracts.
- Let $F_i$ denote the price traded at on day $i$, for $i = 0, 1, \ldots, n$.
- For any $K > 0$, consider the payoff:

$$\sum_{i=1}^{n} [1(F_{i-1} \leq K)(F_i - K)^+ + 1(F_{i-1} > K)(K - F_i)^+],$$

  Thus for each time $i$, the payoff is zero if there is no cross of $K$. If there is a cross from below, the payoff is $F_i - K > 0$. If there is a cross from above, the payoff is $K - F_i > 0$.

- We refer to this payoff as the variation of the $F$ process at time $n$, localized to the strike $K$.

# Robust Hedging of Local Variation

- It is a tautology that the target payoff

$$\sum_{i=1}^{n} \left[ 1(F_{i-1} \leq K)(F_i - K)^+ + 1(F_{i-1} > K)(K - F_i)^+ \right]$$

$$= \ (F_n - K)^+ - (F_0 - K)^+ - \sum_{i=1}^{n} 1(F_{i-1} > K)(F_i - F_{i-1}).$$

- Hence, the claim paying the local variation can be hedged by buying a call and eliminating its intrinsic value whenever it is positive by shorting the forward contract with delivery price $K$.
- The fair price of the local variation is the initial premium of the OTM option with the same underlying, strike, and maturity.

# Observations and Generalizations

- By integating $K$ from 0 to infinity, one can also create the payoff $\sum_{i=1}^{n}(F_i - F_{i-1})^2$.
- Dividing by $n$, this payoff is the floating part of a price variance swap.
- More generally, one can create the payoff $\sum_{i=1}^{n} g(F_i, F_{i-1})$ by combining semi-static trading in the underlying with a static position in options of all strikes maturing at $t_n$ if and only if $g_{11}(F_i, F_{i-1})$ is independent of $F_{i-1}$.
- This condition is violated for standard variance swaps and hence exact replication requires further assumptions.

# Example 2: Hyper Options

- To the pantheon of exotic options, we introduce HYPER options (High Yielding Performance Enhancing Reversible options).
- As usual, a hyper option is issued as either a call or a put.
- A hyper option is similar to an American option in that it can be exercised early, but it also differs from an American option in that it can be exercised an unlimited number of times.
- Exercising a hyper option not only locks in the exercise value, but it also turns a hyper call into a hyper put and vice versa.
- Thus after a hyper call is first exercised, it can be exercised next as a put, then as a call, etc. The strike, maturity, and underlying are never changed.
- Since a hyper option can be exercised an unlimited number of times, all of the exercise proceeds are deferred without interest to maturity.
- As usual, a hyper option need never be exercised, so it has nonnegative value.

# Hyper Options on Forward Prices

- In this presentation, we will only consider hyper options written on the forward price $F$ of some underlying asset. We assume that both the hyper option and the forward contract mature at some fixed date $T$.
- If a hyper call is exercised at any time $t \in [0, T]$, then the owner will receive $F_t - K$ at $T$, where $K$ denotes the strike price of the hyper option.
- Exercising the hyper call converts it into the corresponding hyper put.
- We do not require that the hyper call be ITM for it to be exercised. If the owner exercises his hyper call while $F < K$ to obtain the ITM hyper put, then $F - K$ is negative so the owner owes $K - F$ to the writer at maturity.
- If a hyper put is exercised at any time $t \in [0, T]$, then the owner will receive $K - F_t$ at $T$ and the hyper put reverses into the corresponding hyper call.
- At maturity, the hyper option can be exercised for the final time or it can expire worthless.

# Get Plenty of Exercise

- We restrict ourselves to exercise strategies which include exercising at maturity if and only if it is ITM.
- We refer to such a strategy as *sensible*. Sensible strategies permit exercise prior to maturity as well.
- We say that a sensible exercise strategy is *optimal* if it is value maximizing.
- Depending on the price path which is realized, we will show that it can be optimal for the owner of a hyper option to exercise early one or more times.
- In fact, at any time prior to maturity, there is always positive probability of multiple optimal early exercises.
- Thus, the writer of a hyper option must find a hedging strategy which defends against these multiple optimal early exercises.
- Ideally, this hedging strategy would also be immune to model risk.

# The Hyper American

- Recall that a hyper option is a multiply exerciseable American option whose polarity switches on each exercise.
- Since hyper options can potentially be exercised infinitely often, all exercise proceeds are deferred without interest to maturity.
- When the hyper option is written on a forward price as we assume, then at any time there is positive probability of multiple optimal early exercises.
- All of this suggests that a hyper option has greater value than a standard American option on the forward price (which has a positive early exercise premium).

# Objects May Appear Larger...

- Assuming only frictionless markets and no arbitrage, we show that a hyper option has exactly the same value as the corresponding European option, regardless of the model.
- Thus, no arbitrage forces the hyper call to have the same value as the European call with the same underlying, strike, and maturity. The analogous statement holds for puts.
- The reason for these surprising results is that all sensible exercise strategies are also optimal.
- Note that this result differs from Merton's classical result for American calls on non-dividend paying stocks. For these options, the optimal exercise strategy is to wait to maturity and exercise if and only if the call is ITM then.

# Robust Hedging of Hyper Options

- Let $C_t^h$ and $P_t^h$ denote the respective prices at time $t \in [0,T]$ of hyper calls & puts with fixed strike $K$ & fixed maturity $T$.
- Let $C_t^e$ and $P_t^e$ denote the corresponding European option prices, which satisfy:

$$C_t^e - P_t^e = B_t(F_t - K), \qquad t \in [0,T].$$

- Consider the following *polarity matching strategy* for hedging the sale of a hyper option: Buy a call, and
  1. If the owner is holding the hyper option as a call, hold nothing else, otherwise:
  2. If the owner is holding the hyper option as a put, also be short a forward contract struck at $K$. Thus, the net position is long 1 synthetic put.
- From put call parity written above, this strategy perfectly replicates the payoffs to a hyper option and hence we conclude from no arbitrage that:

$$C_t^h = C_t^e \qquad P_t^h = P_t^e, \qquad t \in [0,T].$$

# Concluding Remarks on Hyper Options

- The two examples in this part are linked. The local variation of the price path arises if the owner of the hyper option adopts an exercise strategy which monitors the path discretely and exercises as soon as the option is ITM.
- Other exercise strategies can be used to generate upper bounds on the number of upcrosses or downcrosses of a given spatial interval.
- Roger Lee and I have also looked at hyper options on the spot price and other variations on the hyper option payoff.

# Example 3: Variance Swaps

- *Variance swaps* now trade actively over the counter on both stock indices and stocks.
- The long position in a variance swap pays a fixed amount at the contract's maturity in return for a standard estimate of the realized variance of a specified underlying over the contract's life.
- The fixed amount is initially determined so that the variance swap is costless to enter. When converted to an annualized volatility, this fixed amount is called the variance swap rate.

# Variance Swap Term Sheet

Bank of America Securities LLC | Indicative Terms
(For Discussion Only) | October 8, 1999

**S&P 500 Index Realized Variance Swap**

| | |
|---|---|
| **Equity Payer:** | Bank of America, N.A. ("BofA") |
| **Equity Receiver:** | Merrill Lynch International |
| **Trade Date:** | October 8, 1999 |
| **Maturity Date:** | May 7, 2003 |
| **Underlying Index:** | The Standard & Poor's 500 Composite Stock Price Index |

**Equity Calculations:**

(a) "Initial Price" means 0.305

(b) "Final Price" means the actual realized index Variance defined in accordance with the following formula and definition:

$$\sqrt{\frac{\sum_{i=1}^{n-1}\left[\ln\left(\frac{P_{i+1}}{P_i}\right)\right]^2}{n-2}}\times\sqrt{52}$$

(c) "Natural Logarithm" means for any Daily Quotient, as determined by the Calculation Agent, the exponential number which equates 2.718281828 to such Daily Quotient;

(d) "n" means the total number of Valuation Dates;

(e) "$P_i$" means the closing level of the index on the ith valuation date (i.e.: $P_1$ is the closing level of the index on October 6, 1999, $P_2$ is the closing level of the index on the first Wednesday that is an Exchange Business Day following the Trade Date and $P_n$ is the closing level of the index on the Final Valuation Date.

(f) "Valuation Dates" means, commencing on October 6, 1999, and each Wednesday thereafter up to and including the Final Valuation Date and if any such date is not an Exchange Business Day, the next following day that is an Exchange Business Day, subject to the Market Disruption Events as set forth in the 1996 ISDA Equity Derivatives Definitions.

(g) "$\sum_{t=1}^{n}$" means the summation from $t=1$ to $t=n$.

| | |
|---|---|
| **Notional:** | 111,230,666 |
| **Equity Payment:** | Notional * [ Final Price$^2$ – Initial Price$^2$ ]<br>If the Equity Payment is a positive value, then the Equity Payer pays the Equity Receiver this value.<br>If the Equity Payment is a negative value, then the Equity Receiver pays the Equity Payer the absolute value of this number. |
| **Credit Terms:** | n/a |

# Variance Swap Quotes

GRAB Govt **BXVS**

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13:46 **S&P 500** PAGE 1 / 1

| Index Ticker | Currency | Bid | Ask | Time |
|---|---|---|---|---|
| 1) Jun05 | EUR | 12.035 | 13.535 | 3/04 |
| 2) Dec05 | EUR | 13.427 | 14.927 | 3/04 |
| 3) Dec06 | EUR | 14.267 | 15.767 | 3/04 |
| 4) Dec07 | EUR | 14.881 | 16.381 | 3/04 |
| 5) Dec08 | EUR | 15.314 | 16.814 | 3/04 |

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# Robust Replication of Variance Swaps

- Suppose no frictions, arbitrage, or rates and that the underlying spot price $S$ is positive and continuous. Then under $\mathbb{Q}$:

$$dS_t = \sigma_t S_t dW_t, \quad t \in [0, T].$$

- The payoff on a (continuously monitored) variance swap on \$1 of notional is:

$$\frac{1}{T}\int_0^T \left(\frac{dS_t}{S_t}\right)^2 - VS_0^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt - VS_0^2.$$

# Robust Replication of Variance Swaps

- Recall that the payoff on a (continuously monitored) variance swap is:

$$\frac{1}{T}\int_0^T \left(\frac{dS_t}{S_t}\right)^2 - VS_0^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt - VS_0^2,$$

  where $VS_0$ is the initial variance swap rate.

- This payoff can be spanned without specifying a stochastic process for $\sigma$.

- As we will see, the robust replicating strategy combines a static position in $T-$maturity options with dynamic trading in the underlying over $[0, T]$.

# Reminder: Static Hedging of the Log Payoff

- Recall that for general $f$, for any expansion point $\kappa \geq 0$, and for any $S \geq 0$,

$$f(S) = f(\kappa) + f'(\kappa)(S-\kappa) + \int_\kappa^\infty f''(K)(S-K)^+ dK + \int_0^\kappa f''(K)(K-S)^+ dK$$

- In terms of the bond price $B_0$, and call and put prices $C_0(K)$ and $P_0(K)$, a claim on any "European-style" payoff $f(S_T)$ has time-0 value

$$f(\kappa)B_0 + f'(\kappa)[C_0(\kappa) - P_0(\kappa)] + \int_\kappa^\infty f''(K)C_0(K)dK + \int_0^\kappa f''(K)P_0(K)dK.$$

- There are no restrictions on the underlying price process, but we do assume the existence of $T$-maturity European options of all strikes. See Breeden-Litzenberger (78) and Carr-Madan (98) for further details.

# **Example:** $\log(S_T/S_0)$

- Suppose the payoff to be replicated is $X_T = \log(S_T/S_0)$.
- Then expand $f(S) := \log(S/S_0)$ about $\kappa = S_0$.
- The initial value of a claim on $X_T$ is therefore

$$-\int_0^{S_0} \frac{1}{K^2} P_0(K)dK - \int_{S_0}^{\infty} \frac{1}{K^2} C_0(K)dK.$$

  So hold $-(1/K^2)dK$ units of each out-of-the-money option.
- This decomposition will be useful when we robustly replicate a variance swap.

# Further Assumptions

- Assume no frictions, no arbitrage, and no rates for simplicity.
- Assume a +ve continuous price process $S$ for $t \in [0, T]$, so that under $\mathbb{Q}$

$$dS_t = \sigma_t S_t dW_t, \quad t \in [0, T],$$

  where $W$ is a $\mathbb{Q}$ standard Brownian motion.
- While $S$ is continuous over time, $\sigma$ need not be.
- Let $X_t := \log(S_t/S_0)$. We want to create the payoff

$$\langle X \rangle_T \equiv \int_0^T (dX_t)^2 = \int_0^T \left(\frac{dS_t}{S_t}\right)^2 = \int_0^T \sigma_t^2 dt.$$

# Realized Variance and Log Payoffs

- Recalling that $X_t := \log\left(\frac{S_t}{S_0}\right)$ and $dS_t = \sigma_t S_t dW_t$, Itô's rule implies:

$$X_T = \int_0^T \frac{1}{S_t} dS_t + \frac{1}{2}\int_0^T \left(\frac{-1}{S_t^2}\right)\sigma_t^2 S_t^2 dt = \int_0^T \frac{1}{S_t} dS_t - \frac{1}{2}\int_0^T \sigma_t^2 dt.$$

- So the realized and annualized variance $\frac{\langle X \rangle_T}{T} \equiv \frac{1}{T}\int_0^T \sigma_t^2 dt$ is just twice the difference between the arithmetic mean of the instantaneous returns and their geometric mean:

$$\frac{\langle X \rangle_T}{T} = \frac{2}{T}\left[\int_0^T \frac{1}{S_t} dS_t - X_T\right] = \underbrace{\int_0^T \frac{2}{T S_t} dS_t}_{\text{dynamic underlying}} + \underbrace{\frac{2}{T}\log\left(\frac{S_0}{S_T}\right)}_{\text{static options}}.$$

# Robustly Replicating a Variance Swap

- Recall that:

$$\frac{\langle X\rangle_T}{T} \equiv \frac{1}{T}\int_0^T \sigma_t^2 dt = \int_0^T \frac{2}{TS_t} dS_t + \frac{2}{T}\log\left(\frac{S_0}{S_T}\right).$$

- So, a dynamic position in $\frac{2}{TS_t}$ shares held at each time $t$ combined with a static options position with initial value:

$$\int_0^{S_0} \frac{2}{TK^2} P_0(K) dK + \int_{S_0}^{\infty} \frac{2}{TK^2} C_0(K) dK,$$

replicates the payoff to the floating part of a variance swap.

# Summary and General Conclusions

- We have shown how to robustly replicate the payoffs from path-independent payoffs, serial covariation contracts, local variation contracts, hyper options, and variance swaps.

- Many of the claims had the same price as a single European option, despite the fact that their payoff was path-dependent.

- In general, the greater the usage of options in the hedge, the less one is relying on a model.

- For other examples of robustly replicable payoffs, see my presentation "Hedging with Options" at:
http://www.math.nyu.edu/research/carrp/papers/pdf/HWOpres4.pdf

# App: Replic'g Payoffs with Bonds & Options

- For any fixed $\kappa$, the fundamental theorem of calculus implies:

$$\begin{aligned} f(S) &= f(\kappa) + 1_{S>\kappa}\int_{\kappa}^{S} f'(u)du - 1_{S<\kappa}\int_{S}^{\kappa} f'(u)du \\ &= f(\kappa) + 1_{S>\kappa}\int_{\kappa}^{S}\left[f'(\kappa) + \int_{\kappa}^{u} f''(v)dv\right]du \\ &\quad -1_{S<\kappa}\int_{S}^{\kappa}\left[f'(\kappa) - \int_{u}^{\kappa} f''(v)dv\right]du. \end{aligned}$$

- Noting that $f'(\kappa)$ is independent of $u$, Fubini's theorem implies:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + 1_{S>\kappa}\int_{\kappa}^{S}\int_{v}^{S} f''(v)dudv \\ &\quad +1_{S<\kappa}\int_{S}^{\kappa}\int_{S}^{v} f''(v)dudv. \end{aligned}$$

- Integrating over $u$ yields:

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + 1_{S>\kappa} \int_{\kappa}^{S} f''(v)(S-v)dv \\ &\quad + 1_{S<\kappa} \int_{S}^{\kappa} f''(v)(v-S)dv \\ &= f(\kappa) + f'(\kappa)(S-\kappa) + \int_{\kappa}^{\infty} f''(v)(S-v)^{+}dv \\ &\quad + \int_{0}^{\kappa} f''(v)(v-S)^{+}dv. \end{aligned}$$

- Q.E.D. (quite easily done).