Effects Of The Choice Of Numeraire

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Continuous Time Finance — *Lecture 6*

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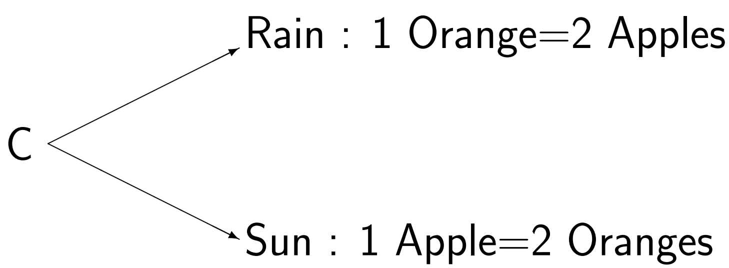
Apples and Oranges



- **Can you compare apples and oranges?**
  - Assume $\mathbb{P}(Rain) = \mathbb{P}(Sun) = 0.5$
  - Choose oranges for numeraire
  - Take expectation to get $1Apple = \frac{1}{2}0.5Orange + \frac{1}{2}2Orange = 1.25Orange$
  - Now switch numeraire to apples: $1Orange = 1.25Apple$!!!

- Conclusion: A probability measure cannot be risk neutral w.r.t. two numeraires.

- **FTAP:**
  - Choice of asset A as numeraire
  - Denote price of any other asset X in units of A as X_A
  - $\mathbb{Q}_{\mathbb{A}}$ is a risk neutral measure if X_A is a martingale $\forall X$.
  - **Theorem:** No arbitrage + Completeness $\Longleftrightarrow \exists ! \mathbb{Q}_{\mathbb{A}}$

- Writing the price of an asset in a different numeraire:

  –
  $$X_A(0) = \mathbb{E}^{\mathbb{Q}_\mathbb{A}}[X_A(T)]$$

  –
  $$\frac{X_0}{A_0} = \mathbb{E}^{\mathbb{Q}_\mathbb{A}}[\frac{X(T)}{A(T)}]$$

  –
  $$X(0) = A(0)\mathbb{E}^{\mathbb{Q}_\mathbb{A}}[\frac{X(T)}{A(T)}]$$

- Correlation between assets also changes when changing numeraire

Examples of Numeraires

- $\beta_t = e^{\int_0^t r_s ds}$: The Money Market Account
- $$X(0) = \mathbb{E}^{\mathbb{Q}_\beta}[X(T)e^{-\int_0^t r_s ds}]$$
- ZC_T : The Zero coupon Bond
- $$X(0) = B(0,T)\mathbb{E}^{\mathbb{Q}_\mathbb{T}}[X(T)]$$
- Roll Over The Bond: "Jumping Numeraire" $\prod_{i=1}^{p} \frac{1}{B_{t_{i-1},t_i}}, T = t_p$

- $$X(0) = \mathbb{E}^{\mathbb{Q}_\pi}[X(T)\prod_{i=1}^{p} B_{t_{i-1},t_i}]$$

Statistical as Risk Neutral Measure

- Use the Radon Nikodym derivative to change between numeraires:

- $$X(0) = A(0)\mathbb{E}^{\mathbb{Q}_\mathbb{A}}[\frac{X(T)}{A(T)}]$$

- $$X(0) = B(0)\mathbb{E}^{\mathbb{Q}_\mathbb{B}}[\frac{X(T)}{B(T)}]$$

- $$X(0) = B(0)\mathbb{E}^{\mathbb{Q}_\mathbb{A}}[\frac{X(T)}{B(T)}\frac{d\mathbb{Q}_\mathbb{B}}{d\mathbb{Q}_\mathbb{A}}]$$

- Comparing the first and last $\mathbb{E}^{\mathbb{Q}_{\mathbb{A}}}$-expectations, we have:

$$\frac{d\mathbb{Q}_{\mathbb{B}}}{d\mathbb{Q}_{\mathbb{A}}} = \frac{B(T)}{A(T)} \frac{A_0}{B_0}$$

- B chosen s.t. $\mathbb{P} = \mathbb{Q}_{\mathbb{B}}$ (physical/statistical measure)

- $$B(T) = A(T) \frac{d\mathbb{P}}{d\mathbb{Q}_{\mathbb{A}}}$$

- It is possible to choose a numeraire (asset) for which the price of all assets are martingales under physical measure.

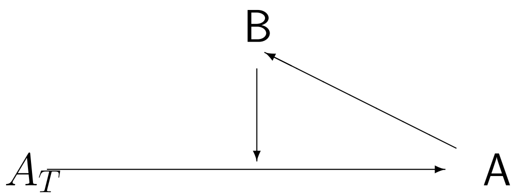
- Example:

  - $$\frac{dS}{S} = \mu dt + \sigma dW_t^{\mathbb{P}}$$
  - $B = S^{\alpha}$ is a numeraire in which assets are $\mathbb{P}$-martingales.

Market Price of Risk

- The above implies, that with appropriate choice of units of account, the market price of risk is 0

- One can represent the risk/return relationship of assets as two dimensional vectors:



  - Projection is expected return
  - Orthogonal is diversifiable risk

- The risk premium is the length of the projection from the numeraire A

- Efficient frontier is the set of assets on the line between from the asset to A_T, "Atlantis".

- The frontier changes with change of numeraire.