Black's Model With Default

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Assumptions

- Zero interest rates.
- Futures price F_t at time $t \in [0, T]$ for maturity $T' \geq T$
- F: continuous time stochastic process.
- Futures: continuous marking to market.
- $\mathbb{P}$: Statistical probability measure
- W: Standard Brownian motion under $\mathbb{P}$.
- N: Standard Poisson process under $\mathbb{P}$.

- Black model:

$$\frac{dF_t}{F_t} = \alpha dt + \sigma dW_t, \qquad t \in [0, T], \tag{1}$$

- F_0 and σ known positive constants.
- Cox Ross single jump Poisson model:

$$\frac{dF_t}{F_{t-}} = \mu dt + (e^j - 1)dN_t, \qquad t \in [0, T], \tag{2}$$

- F_0 known positive constant, μ and j real numbers of opposite sign.
- In (1) no need to know risk premium α
- In (2) no need to know arrival rate λ_p

- Both models give complete market $\Rightarrow$ Unique RN $\mathbb{Q}$
- After Measure Change $\mathbb{P} \to \mathbb{Q}$
  - Black's model: Volatility σ unchanged.
  - Cox Ross model: μ and j unchanged.
  - Black model: risk premium $\alpha = 0$ ($\mathbb{Q}$ risk-neutral measure).
  - Cox Ross model: risk-neutral arrival rate of a jump is $\lambda_q \equiv -\frac{\mu}{e^j - 1}$.
- Intuition: no need to know α in Black's model, λ_p in jump model because info contained in futures price (known).
- α in Black's model changes, μ in pure jump model does not.
- Fundamental Rules:
  1. $\mathbb{Q}$ is defined so that F is a $\mathbb{Q}$ martingale
  2. A change of measure cannot change the numerical value of a parameter that can be estimated with certainty by continuous observation of a (segment of) a single path.

Dynamic Duo: Jumping to Default

- More realistic stochastic process for F combines both processes.
- Waiting time τ to the first jump of N: exponentially distributed r.v. with constant parameter $\lambda_p > 0$.
- Let τ be the default time of the limited liability asset underlying the futures.
- $t < \tau$: F_t follows geometric Brownian motion with constant drift α, constant volatility σ.
- At τ, F drops to zero and remains there afterwards.
- Put it all together (under $\mathbb{P}$):

$$\frac{dF_t}{F_{t-}} = \alpha dt + \sigma dW_t - dN_t, \qquad t \in [0, T], \tag{3}$$

- F_0 and σ are known positive constants.
- Comparing (3) with (2), jump size j set to negative infinity, and B.M. has been introduced.
- Once F hits zero, it absorbs there: increments multiplied by $F_{t-} = 0$.
- No need to know α or λ_p - can actually assume α real-valued stochastic process, λ_p positive stochastic process.
- Default Indicator Process D: defined by $D_t = 1(N_t > 0)$ gives:

$$\frac{dF_t}{F_{t-}} = \alpha dt + \sigma dW_t - dD_t, \qquad t \in [0, T], \tag{4}$$

- (4) is a continuous time trinomial model up to τ:
  - Brownian increments generate moves up and down of order $\sqrt{dt}$
  - The Poisson process generates an $O(dt)$ probability of a large down move in the price of order one.

- Perfect replication of every derivative on futures price path, requires ability to dynamically trade three assets.
- Dynamic trading in just the money market account and the futures contract may work for some payoffs, but it will not suffice for all payoffs.
- Assume futures written on a stock, and introduce a credit default swap (CDS) written on a bond issued by the stock issuer.
- Assume zero recovery rate for the bond for simplicity $\Rightarrow$ Default event causes both the bond price and the stock price to vanish (think Enron).
- For simplicity, assume that the CDS rate is constant and observable at $\lambda_q > 0$.
- Further assume that the CDS rate is paid continuously, rather than periodically.
- As a result, prior to default, an investor can access the payoff $dD_t - \lambda_q dt$ at zero cost.

Analysis

- Let $V(F,t) : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ be a $C^{2,1}$ function. Itô's lemma for semi-martingales implies:

$$V(F_T,T) = V(F_0,0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-},t)dF_t + \int_0^T \left[\frac{\partial V}{\partial t}(F_{t-},t) + \frac{\sigma^2 F_{t-}^2}{2}\frac{\partial^2 V}{\partial F^2}(F_{t-},t)\right] dt$$
$$+ \int_0^T \left[V(0,t) - V(F_{t-},t) - \frac{\partial V}{\partial F}(F_{t-},t)(0 - F_{t-})\right] dD_t. \quad (5)$$

- Add and subtract so that last term in (5) is gain from dynamically trading a CDS:

$$V(F_T,T)$$
$$= V(F_0,0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-},t)dF_t$$
$$+ \int_0^T \left\{\frac{\partial V}{\partial t}(F_{t-},t) + \frac{\sigma^2 F_{t-}^2}{2}\frac{\partial^2 V}{\partial F^2}(F_{t-},t) + \lambda_q \left[V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-}\right]\right\} dt$$
$$+ \int_0^T \left[V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-}\right](dD_t - \lambda_q dt). \quad (6)$$

- Suppose $V(F,t)$ solves the following partial differential difference equation (PDDE):

$$\frac{\partial V}{\partial t}(F,t) + \frac{\sigma^2 F^2}{2}\frac{\partial^2 V}{\partial F^2}(F,t) + \lambda_q \left[V(0,t) - V(F,t) + \frac{\partial V}{\partial F}(F,t)F \right] = 0, \qquad (7)$$

on the domain: $F > 0, t \in [0,T]$ and with terminal condition:

$$V(F,T) = f(F), \qquad F > 0. \qquad (8)$$

- The solution to this Cauchy problem exists and is unique.
- (6) reduces to:

$$\begin{aligned} f(F_T) &= V(F_0,0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-},t)dF_t \\ &\quad + \int_0^T \left[V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-} \right] (dD_t - \lambda_q dt). \qquad (9) \end{aligned}$$

- Charge $V(F_0,0)$ dollars initially hold $\frac{\partial V}{\partial F}(F_{t-},t)$ futures and $V(0,t) - V(F_{t-},t) + \frac{\partial V}{\partial F}(F_{t-},t)F_{t-}$ CDS at each $t \in [0,T]$
- Achieve final payoff $f(F_T)$.

- Recall:

$$
\begin{aligned}
f(F_T) &= V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t)dF_t \\
&\quad + \int_0^T \left[V(0,t) - V(F_{t-}, t) + \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-} \right] (dD_t - \lambda_q dt).
\end{aligned}
$$

- Positions in the two risky hedge instruments will vanish after the default time.
- Replication is achieved without knowledge of the drift or the arrival rate of jumps (under $\mathbb{P}$).
- If the futures price behaved as in (4), it would be trivial to estimate σ.
- This model has all of the econometric advantages of the simpler Black model: parameters needed to price are easily determined from sample path, quantities which are difficult to estimate from the path are not needed for pricing.

Pricing a Call

- $C(F,t) = V(F,t)$ value function when terminal payoff is $f(F) = (F-K)^+$.
- $f(0) = 0 \Longrightarrow C(0,t) = 0$.
- PDDE (7) simplifies to the following PDE:

$$\frac{\partial C}{\partial t}(F,t) + \frac{\sigma^2 F^2}{2}\frac{\partial^2 C}{\partial F^2}(F,t) - \lambda_q C(F,t) + \lambda_q F \frac{\partial C}{\partial F}(F,t) = 0 \qquad (10)$$

  on domain $F > 0, t \in [0,T]$, w. terminal condition $C(F,T) = (F-K)^+, F > 0$.
- This is Black Scholes boundary value problem with F replacing S & λ_q replacing r.
- Defaultable call value is thus:

$$C(F,t) = FN(d_1) - Ke^{-\lambda_q (T-t)} N(d_2), \qquad (11)$$

$$d_1 \equiv \frac{\ln(F/K) + (\lambda_q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 \equiv d_1 - \sigma\sqrt{T-t}. \qquad (12)$$

Incomplete Market

- Suppose that there is no credit default swap.
- Assume (unrealistically) that investors can trade only the futures and the money market account.
- In such a setting, the market is incomplete and the parameter λ_q is not known.
- Call payoff cannot be perfectly replicated $\Leftrightarrow$ There exists an infinite number of martingale measures $\mathbb{Q}$, all consistent with the initial observed futures price F_0.
- For pricing calls, there is a one to one correspondence between martingale measures $\mathbb{Q}$ and the parameter λ_q appearing in (11).
- Each martingale measure produces a call value $C(F,t;\lambda_q)$ obtained by evaluating (11) at the associated λ_q.

Call Value Range in Incomplete Market

- Recall that the Black Scholes call value increases in r, so C increases in λ_q.
- As λ_q approaches zero, the call value approaches the Black model value with volatility σ.
- As λ_q approaches infinity, the call value approaches F.
- The range of arbitrage-free call values is between the Black model value and F.
- This range reduces to a single point, once the market price of the CDS or the market price of another option becomes known.