Interest Rates: More on Numeraires, and the Ho Lee Model

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Notation

- $B_{t,T} \equiv P_t(T)$
- $D_T \equiv B_{0,T}$
- Money Market Account: $\beta_t = e^{\int_0^t r_s ds}$
- If interest rates are deterministic: $\beta_t = 1/D_t$
- Forward Rates: $B_{t,T} = e^{-\int_t^T f_{t,s} ds}$
- Solving for f: $f_{t,T} = -\frac{\partial ln B_{t,T}}{\partial T}$

The Forward Martingale Measure

- Numeraire: $Z_T \equiv B_{t,T}$
- Corresponding risk neutral measure: $\mathbb{Q}_T$.
- Forward rate is a $\mathbb{Q}_T$-martingale:

$$f_{t,T} = \lim_{\delta T \to 0} \frac{B_{t,T} - B_{t,T+\delta T}}{B_{t,T}\delta T}$$

- For the short rate, we then have:

$$\mathbb{E}_t^{Q_T}[r_T] = \mathbb{E}_t^{Q_T}[f_{T,T}] = f_{t,T}$$

- Conclusion: Under the forward measure, the expected value of the short rates is the initial forward curve.

The Money Market Measure

- Numeraire: $\beta_T \equiv e^{\int_0^T r_s ds}$
- Corresponding risk neutral measure: $\mathbb{Q}_\beta$.
- For any tradeable X:

$$X_0 = \frac{X_0}{\beta_0} = \mathbb{E}^{Q_\beta}[\frac{X_T}{\beta_T}]$$

- Under this measure, the discount factor doesn't "come out":

$$\mathbb{E}^{Q_\beta}[X_T e^{-\int_0^T r_s ds}] \neq Z_T(0)\mathbb{E}^{Q_\beta}[X_T]$$

- Conclusion: bond prices aren't martingales under this measure.
- We have a **convexity bias**: higher interest rates are discounted at higher rates.

The Ho Lee Model

- The SDE for the short rate: $dr = \sigma dW_t$.

- From above considerations, we need: $\mathbb{E}^{Q_T}[r_t] = f_{0,t}$.

- Get $r_t = f_{0,t} + \sigma W_t$.

- Under $\mathbb{Q}_\beta$,

$$dr_t = \frac{\partial f_{0,t}}{\partial t} dt + \sigma^2 t dt + \sigma dW^\beta$$

- Can be obtained from the HJM result for the forward rates:

$$df_{t,T} = \sigma_{t,T}(\int_t^T \sigma_{t,T} ds)dt + \sigma_{t,T} dW^\beta$$

- Combine $r_T = f_{T,T}$, with the constant volatility in the model, which implies:

$$df_{t,T} = \sigma^2 (T-t)dt + \sigma dW^\beta$$

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- Get:

$$r_T = f_{0,T} + \sigma^2 \int_0^T (T-s)ds + \sigma W_T^\beta$$

- Which gives the SDE we had:

$$dr_t = \frac{\partial f_{0,t}}{\partial t} dt + \sigma^2 t dt + \sigma dW^\beta$$