Interest Rate Models: BGM

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Modeling Forward LIBOR Rates

- Brace Gatarek Musiela (1997): model for LIBOR (London Inter-Bank Offer Rate)
  - HJM framework.
  - Finite number, N of time periods.
  - LIBOR over each period lognormal.
  - Black's Formula for caplets satisfied.

Model Setup

- $P_t(T_1, T_2)$: price at time T_1 of a zero coupon bond paying $1 at time T_2.
- Determined at time t.
- $t < T_1 < T_2$.
- FRA for the time interval $[T_1, T_2]$ struck at time t.
- $P_t(T_2)$: $t = T_1$, i.e. spot (rather than forward).

- Forward LIBOR, $L(T_1, T_2)$:

$$P_t(T_1, T_2) = P_t(T_2)(1 + \alpha(T_1, T_2)L_t(T_1, T_2)) \tag{1}$$

- α is the day count convention.
- Solve for L:

$$L_t(T_1, T_2) = \frac{1}{\alpha(T_1, T_2)} \frac{P_t(T_1) - P_t(T_2)}{P_t(T_2)}. \tag{2}$$

- Divide time interval $T_i = i\Delta T, i = 1..N$.
- Get N rates:

$$L_{i,t} \equiv L_t(T_i, T_{i+1}) = \frac{1}{\alpha_i} \frac{P_{i,t} - P_{i+1,t}}{P_{i+1,t}}$$

- Denote $\mathbb{Q}^{i+1}$ the equivalent martingale measure associated with P_{i+1}.
- $L_{i,t}$ is a $\mathbb{Q}^{i+1}$-martingale: difference of two traded assets, deflated by numeraire.

- Assumptions:
  - $L > 0$
  - L continuous in time
  - L follows a lognormal process with deterministic vol.
- Rewrite third assumption:

$$\frac{dL_{i,t}}{L_{i,t}} = \sigma_i(t) dW_t^{i+1}, \tag{3}$$
$$t \in [0, T_i], i = 1...N$$

- W_t^{i+1} is B.M. under $\mathbb{Q}^{i+1}$.

Black's Formula for Caplets

- Price a caplet, payoff: $C_{i,T_{i+1}} \equiv \alpha_i(L_{i,T_i} - K)^+$.
- Settled at time T_i, payoff at time T_{i+1}.
- By definition of Risk Neutral Measure:

$$\frac{C_{i,t}}{P_{i+1,t}} = \mathbb{E}^{Q^{i+1}}\left[\frac{C_{i,T_{i+1}}}{P_{i+1,T_{i+1}}}|\mathcal{F}_{\mathbf{t}}\right]$$

- Denominator in R.H.S. is 1, so

$$C_{i,t} = P_{i+1,t}\mathbb{E}^{Q^{i+1}}\left[\alpha_i(L_{i,T_i} - K)^+|\mathcal{F}_{\mathbf{t}}\right]$$

- Equation 3 implies that under $\mathbb{Q}^{i+1}$, LIBOR satisfies:

$$L_{i,T_i} = L_{i,t}e^{-\frac{1}{2}\int_t^{T_i}\sigma_i^2(u)du+\int_t^{T_i}\sigma_i dW_u^{i+1}},$$

- Get Black's Formula:

$$C_{i,t} = P_{i+1,t}\alpha_i(L_{i,t}N(d_1) - KN(d_2))$$

  –
$$d_1 = \frac{ln(\frac{L_{i,t}}{K}) + \Sigma_i^2/2}{\Sigma_i},$$

  –
$$d_2 = \frac{ln(\frac{L_{i,t}}{K}) - \Sigma_i^2/2}{\Sigma_i}$$

  –
$$\Sigma_i^2 = \int_t^{T_i} \sigma_i^2(u)du$$

Changing the Measure

- Objective: find SDE's for all rates under a measure, $\mathbb{Q}^{N+1}$ (corresponding to the latest maturity).

- All the measures are equivalent, just need to find drift of each rate $L_{i,t}$.

- For any asset A and time $t > 0$,

$$\frac{A_0}{P_{n,0}} = \mathbb{E}^{Q^n}\left[\frac{A_t}{P_{n,t}}|\mathcal{F}_{\mathbf{0}}\right]$$

- Multiplying and dividing by the factors $P_{n+1,0}$ and $P_{n+1,t}$:

$$\frac{A_0}{P_{n+1,0}} = \mathbb{E}^{Q^n}\left[\frac{P_{n,0}}{P_{n+1,0}}\frac{P_{n+1,t}}{P_{n,t}}\frac{A_t}{P_{n+1,t}}|\mathcal{F}_{\mathbf{0}}\right] = \mathbb{E}^{Q^{n+1}}\left[\frac{A_t}{P_{n+1,t}}|\mathcal{F}_{\mathbf{0}}\right]$$

- Implying:

$$\frac{d\mathbb{Q}^{n+1}}{d\mathbb{Q}^n} = \frac{P_{n,0}}{P_{n+1,0}}\frac{P_{n+1,t}}{P_{n,t}}$$

- Define

$$D_t = \mathbb{E}^{Q^n}\left[\frac{d\mathbb{Q}^{n+1}}{d\mathbb{Q}^n}|\mathcal{F}_{\mathbf{0}}\right]$$

- D_t has mean 1, is a positive $\mathbb{Q}^n-$martingale.

- Assume a process X_t defined by

$$dX_t = \mu_t dt + a_t dW_t^n$$

- Satisfies, under $\mathbb{Q}^{n+1}$:

$$dX_t = \mu_t dt + \frac{dD_t}{D_t} dX_t + a_t dW_t^{n+1}.$$

- Take $X_t = L_{n-1,t}$, i.e. $\mu = 0$, $a_t = \sigma_{n-1}(t) L_{n-1,t}$.

- Get:

$$dL_{n-1,t} = \sigma_{n-1}(t) L_{n-1,t} dW_t^{n+1} + \frac{dD_t}{D_t} dL_{n-1,t}$$

.

- "Trick" substitution: $R_t = \frac{1}{D_t}$.
- Preserves differential:

$$\frac{dD_t}{D_t} dL_{n-1,t} = -\frac{dR_t}{R_t} dL_{n-1,t},$$

- Since

$$D_t = \frac{P_{n,0}}{P_{n+1,0}} \frac{P_{n+1,t}}{P_{n,t}},$$

- Get:

$$R_t = \frac{P_{n+1,0}}{P_{n,0}} \frac{P_{n,t}}{P_{n+1,t}} = \frac{P_{n+1,0}}{P_{n,0}} (1 + \alpha_n L_{n,t})$$

- Differentiate:

$$dR_t = \frac{P_{n+1,0}}{P_{n,0}} \alpha_n dL_{n,t}$$

- Substituting the expressions above:

$$\frac{dR_t}{R_t} = \frac{\alpha_n dL_{n,t}}{1 + \alpha_n L_{n,t}} = \frac{\alpha_n}{1 + \alpha_n L_{n,t}} \sigma_n(t) L_{n,t} dW_t^{n+1}$$

- So we finally get:

$$\frac{dD_t}{D_t} dL_{n-1,t} = -\frac{dR_t}{R_t} dL_{n-1,t} = -\sigma_{n-1}(t) L_{n-1,t} \frac{\alpha_n L_{n,t}}{1 + \alpha_n L_{n,t}} \sigma_n(t) dt.$$

- Under $\mathbb{Q}^{n+1}$, the process for $L_{n-1,t}$ thus satisfies the SDE:

$$\frac{dL_{n-1,t}}{L_{n-1,t}} = -\sigma_{n-1}(t) \frac{\alpha_n L_{n,t}}{1 + \alpha_n L_{n,t}} \sigma_n(t) dt + \sigma_{n-1}(t) dW_t^{n+1}$$

- Get drifts for previous intervals recursively:

$$\frac{dL_{i,t}}{L_{i,t}} = -\sum_{k=i+1}^{n} \sigma_k(t) \frac{\alpha_k L_{k,t}}{1 + \alpha_k L_{k,t}} \sigma_i(t) dt + \sigma_i(t) dW_t^{n+1},$$