Interest Rate Models: Hull White

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The Model

- Described by the SDE for the short rate:

$$dr = (\theta(t) - ar)\,dt + \sigma\,dw \tag{1}$$

  - Orignial Article: Rev. Fin. Stud. 3, no. 4 (1990) 573-592
  - See also Sections 23.11-23.12 of Hull(5th edition).
  - Our version simplified : a and σ constant.
  - AKA Extended Vasicek.
  - θ determined uniquely by term structure.

Solving for r(t)

- $$d(e^{at}r) = e^{at}\,dr + ae^{at}r\,dt = \theta(t)e^{at}\,dt + e^{at}\sigma\,dw,$$

- $$e^{at}r(t) = r(0) + \int_0^t \theta(s)e^{as}\,ds + \sigma\int_0^t e^{as}dw(s).$$

- Simplify:

$$r(t) = r(0)e^{-at} + \int_0^t \theta(s)e^{-a(t-s)}\,ds + \sigma\int_0^t e^{-a(t-s)}\,dw(s). \tag{2}$$

- Since the starting time is arbitrary:

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(\tau)e^{-a(t-\tau)}\,ds + \sigma\int_s^t e^{-a(t-\tau)}\,dw(\tau).$$

- Note: $r(t)$ is Gaussian.

Solving for $P(t,T)$

- $P(t,T) = V(t, r(t))$ where V solves the PDE

$$V_t + (\theta(t) - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} - rV = 0$$

- Final-time condition $V(T,r) = 1$ for all r at $t = T$.

- Ansatz:

$$V = A(t,T)e^{-B(t,T)r(t)}. \tag{3}$$

- A and B must satisfy:

$$A_t - \theta(t)AB + \frac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

- Final-time conditions

$$A(T,T) = 1 \quad \text{and} \quad B(T,T) = 0.$$

- B independent of θ, so solution is same as in Vasicek:

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right). \tag{4}$$

- Solving for A requires integration of θ:

$$A(t,T) = \exp\left[-\int_t^T \theta(s)B(s,T)\,ds - \frac{\sigma^2}{2a^2}(B(t,T) - T + t) - \frac{\sigma^2}{4a}B(t,T)^2\right]. \tag{5}$$

Determining θ from the term structure at time 0

- Goal: demonstrate the relation
$$\theta(t) = \frac{\partial f}{\partial T}(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{6}$$

- Note: HJM gives a simple proof of this relation.

- For now, use explicit representation of $P(t,T)$ given by (3)-(5).

- Recall
$$f(t,T) = -\partial \log P(t,T)/\partial T$$
.

- We have

$$-\log P(0,T) = \int_0^T \theta(s)B(s,T)\,ds + \frac{\sigma^2}{2a^2}(B(0,T) - T) + \frac{\sigma^2}{4a}B(0,T)^2 + B(0,T)r_0.$$

- Differentiating and using that $B(T,T) = 0$ and $\partial_T B - 1 = -aB$:

$$f(0,T) = \int_0^T \theta(s)\partial_T B(s,T)\,ds - \frac{\sigma^2}{2a}B(0,T) + \frac{\sigma^2}{2a}B(0,T)\partial_T B(0,T) + \partial_T B(0,T)r_0.$$

- Differentiating again, get:

$$\begin{aligned}\partial_T f(0,T) &= \theta(T) + \int_0^T \theta(s)\partial_{TT} B(s,T)\,ds - \frac{\sigma^2}{2a}\partial_T B(0,T) \\ &\quad + \frac{\sigma^2}{2a}[(\partial_T B(0,T))^2 + B(0,T)\partial_{TT} B(0,T)] + \partial_{TT} B(0,T)r_0.\end{aligned}$$

- Combine these equations, and use $a\partial_T B + \partial_{TT} B = 0$

- Get:

$$af(0,T) + \partial_T f(0,T) = \theta(T) - \frac{\sigma^2}{2a}(aB + \partial_T B) + \frac{\sigma^2}{2a}[aB\partial_T B + (\partial_T B)^2 + B\partial_{TT} B].$$

- Substitute formula for B and simplify, to get

$$af(0,T) + \partial_T f(0,T) = \theta(T) - \frac{\sigma^2}{2a}(1 - e^{-2aT}),$$

- This is equivalent to (6).

A convenient representation

- (6)seems to imply need for *differentiated* term structure $\partial_T f(0,T)$ for calibration.
- Problem: differentiation amplifies effect of observation-error.
- Actually, need only f.
- Try a representation of the form

$$r(t) = \alpha(t) + x(t) \tag{7}$$

- $\alpha(t)$ deterministic, $x(t)$ solves

$$dx = -ax\,dt + \sigma\,dw \quad \text{with } x(0) = 0.$$

- Calculation gives

$$\alpha' + a\alpha = \theta \quad \text{and } \alpha(0) = r_0$$

- $\alpha(t) + x(t)$ solves the SDE for $r(t)$ with initial condition.

- Uniqueness $\Rightarrow$ equals $r(t)$.

- The ODE for α: $(e^{at}\alpha)' = e^{at}\theta$

- Solution:

$$\alpha(t) = r_0 e^{-at} + \int_0^t e^{-a(t-s)}\theta(s)\, ds.$$

- Substituting (6), get

$$\alpha(t) = r_0 e^{-at} + \int_0^t \partial_s[e^{-a(t-s)} f(0,s)] + \frac{\sigma^2}{2a} e^{-a(t-s)}(1 - e^{-2as})\, ds.$$

- Simplifies to

$$\alpha(t) = f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

- Decomposition (7) expresses r as sum of:
  - deterministic $\alpha(t)$ reflecting the term structure at time 0
  - random process $x(t)$ entirely independent of market data. *Validity of Black's formula.* The situation is exactly the same as for Vasicek.

Validity of Black's formula

- SDE for the interest rate under the forward-risk-neutral measure is

$$dr = [\theta(t) - ar - \sigma^2 B(t,T)]dt + \sigma d\overline{w}$$

- $d\overline{w}$ is a Brownian motion under this measure.

- This is a version of Hull-White with a different choice of θ.

- Get bond prices lognormal $\Rightarrow$ Black's formula is valid.