Interest Rate Models: Vasicek

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The Short Rate Dynamics

- The Vasicek model describes the short rate's $\mathbb{Q}$ dynamics by the following SDE:

$$dr_t = (\theta - ar_t)\, dt + \sigma\, dw_t \tag{1}$$

  where θ, $a > 0$, and σ are constants.

- *An explicit formula for r_t:* Start with:

$$d(e^{at} r_t) = e^{at}\, dr_t + ae^{at} r_t\, dt = \theta e^{at}\, dt + e^{at}\sigma\, dw_t,$$

- so:

$$e^{at} r_t = r_0 + \theta \int_0^t e^{as}\, ds + \sigma \int_0^t e^{as} dw_s.$$

- Simplifying:

$$r_t = r_0 e^{-at} + \frac{\theta}{a}(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)}\, dw_s. \tag{2}$$

- Recall
$$r_t = r_0 e^{-at} + \frac{\theta}{a}(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)}\, dw_s. \tag{3}$$
- As the starting time is arbitrary:
$$r_t = r_s e^{-a(t-s)} + \frac{\theta}{a}(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-\tau)}\, dw_\tau. \tag{4}$$
- (??) implies that r_t is Gaussian at each t, with
  - expectation:
$$E^{\mathbb{Q}}[r_t] = r_0 e^{-at} + \frac{\theta}{a}(1 - e^{-at}), \text{ and}$$
  - variance:
$$\text{Var}^{\mathbb{Q}}[r_t] = \sigma^2 E\left[\left(\int_0^t e^{-a(t-s)}\, dw(s)\right)^2\right] = \sigma^2 \int_0^t e^{-2a(t-s)}\, ds = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Dynamics of Bond Price

- We now show that the bond price is lognormally distributed in the Vasicek model:
  - By definition of the risk-neutral measure $\mathbb{Q}$, the zero coupon bond price is:
$$P_t(T) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} \mid \mathcal{F}_t\right]. \tag{5}$$
  - ¿From (??), (interchanging t, s)
$$P_t(T) = A(t,T)e^{-B(t,T)r_t} \tag{6}$$
  -
$$B(t,T) = \int_t^T e^{-a(s-t)}\,ds, \quad \text{and}$$
  -
$$A(t,T) = E\left[e^{-\int_t^T \left\{\frac{\theta}{a}(1-e^{-a(s-t)})+\sigma\int_t^s e^{-a(s-\tau)}\,dw(\tau)\right\}ds}\right].$$
  - $A(t,T)$, $B(t,T)$ deterministic, r_t Gaussian $\Rightarrow$ $P_t(T)$ lognormal.

Explicit Bond Pricing Formula

- Can evaluate $A(t,T)$, $B(t,T)$ - see Lamberton & Lapeyre, pages 128-129.

- Alternative approach: use $P_t(T) = V(t, r_t)$, where $V(t,r)$ solves BVP consisting of PDE:
$$V_t + (\theta - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} = rV$$
subject to the final-time condition $V(T,r) = 1$ for all r.

- Guess a solution of the form:
$$V(t,r;T) = A(t,T)e^{-B(t,T)r}.$$

- Considered as functions of t, $A(t,T)$ and $B(t,T)$ solve the ODE's:

  –
$$A_t - \theta AB + \tfrac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

  subject to:
$$A(T,T) = 1 \quad \text{and} \quad B(T,T) = 0.$$

- Get:
$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

- and:
$$A(t,T) = \exp\left[\left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4a}B^2(t,T)\right].$$

Term Structure and Volatility

- Only three parameters $\Rightarrow$ special term structure.
- By definition, the initial instantaneous forward rate curve $f_0(T) = -\frac{\partial \ln P_0(T)}{\partial T}$.
- After some calculations, in the Vasicek model, one has:

$$f_0(T) = \frac{\theta}{a} + e^{-aT}\left(r_0 - \frac{\theta}{a}\right) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2.$$

- Volatility of f, $\sigma(t,T)$ is defined by

$$df_t(T) = (\text{stuff})\, dt + \sigma(t,T)\, dw_t.$$

- $\ln P_t(T) = \ln A(t,T) - B(t,T)r_t$, implies

$$f_t(T) = -\partial_T \ln A(t,T) + \partial_T B(t,T) r_t,$$

- Itô's formula gives:

$$\sigma(t,T) = \sigma \partial_T B(t,T) = \sigma e^{-a(T-t)}.$$

Validity of Black's Formula

- We now show that $P_t(T)$ is lognormal under the forward-risk-neutral measure $\mathbb{Q}_T$. (The measure under which tradeables normalized by $P(t,T)$ are martingales.)

- Already know that $P_t(T)$ is lognormal under the risk-neutral measure $\mathbb{Q}$, but here we're interested in a different numeraire.

- Change-of-numeraire in the one-factor setting:
  - The risk-neutral measure is associated with the risk-free money-market account β as numeraire (by definition $d\beta_t = r_t\beta_t\,dt$ with $\beta_0 = 1$).
  - Say N is another numeraire, and $\overline{Q}$ is the associated equivalent martingale measure.
  - Only positive tradeables can be numeraires, so the risk-neutral process for N is

$$dN_t = r_t N_t\,dt + \sigma_t^N N_t\,dw_t$$

  where σ_t^N is in general stochastic and w is a $\mathbb{Q}$ standard Brownian motion.

- Itô's formula gives:

$$d\left(\frac{\beta_t}{N_t}\right) = \beta_t\, d(N_t^{-1}) + N_t^{-1}\, d\beta_t$$

- After some algebra:

$$d\left(\frac{\beta_t}{N_t}\right) = \frac{\beta_t}{N_t}(\sigma_t^N)^2\, dt - \frac{\beta_t}{N_t}\sigma_t^N\, dw_t.$$

- $\frac{\beta_t}{N_t}$ is a $\overline{Q}$-martingale, i.e.

$$d\left(\frac{\beta_t}{N_t}\right) = -\frac{\beta_t}{N_t}\sigma_t^N\, d\overline{w}_t$$

  where $\overline{w}$ is a $\overline{Q}$-Brownian motion.

- Therefore:

$$d\overline{w}_t = -\sigma_t^N\, dt + dw_t.$$

- What is the SDE for the short rate in the Vasicek model under the forward-risk-neutral measure $\mathbb{Q}_T$?
  - Numeraire is $P_t(T) = A(t,T)e^{-B(t,T)r_t}$
  - Ito $\Rightarrow$ the (uusal lognormal) volatility of $P_t(T)$ is $-B(t,T)\sigma$.
  - The preceding calculation gives:

$$d\overline{w}_t = \sigma B(t,T)\,dt + dw_t.$$

  - Conclusion:

$$dr_t = (\theta - ar_t)\,dt + \sigma\,dw_t = [\theta - ar_t - \sigma^2 B(t,T)]\,dt + \sigma\,d\overline{w}_t,$$

  where $\overline{w}$ is a $\mathbb{Q}_T$ standard Brownian motion.

- This SDE shows that short rates are normal and bond prices are lognormal, as under the risk-neutral measure $\mathbb{Q}$.