# Interest Rate Models: Introduction

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# Basic Terminology

- Time-value of money is expressed by the *discount factor*:

$$P(t,T) = \text{value at time } t \text{ of a dollar received at time } T.$$

- Interest rates stochastic $\Rightarrow P(t,T)$ not known until time $t$
- $P(t,T)$ is a function of *two* variables: initiation time $t$ and maturity time $T$.
- Dependence on $T$ reflects *term structure* of interest rates
- $P(t,T)$ fairly smooth as function of $T$ at each $t$, because of averaging.
- Convention: Present time is $t = 0 \Rightarrow$ initial observable is $P(0,T)$ for all $T > 0$.

# Representations of the time-value of money

- The (continuously compounded annualized) *yield-to-maturity* (or just yield) $R(t,T)$ is defined implicitly by:
$$P(t,T) = e^{-R(t,T)(T-t)}.$$

- It is the unique (continuously compounded annualized) *constant* short term interest rate implied by the market price $P(t,T)$.

- Evidently:
$$R(t,T) = -\frac{\log P(t,T)}{T-t}.$$

- The (continuously compounded annualized) *instantaneous forward rate* $f(t,T)$ is defined by:

$$P(t,T) = e^{-\int_t^T f(t,\tau)\,d\tau}.$$

- It is the deterministic time-varying interest rate describing all loans starting at $t$ with various maturities.

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}.$$

- The (continuously compounded annualized) *instantaneous short term interest rate* $r(t)$, (a.k.a. the short rate), is

$$r(t) = f(t, t);$$

- It is the rate earned on the shortest-term loans starting at time $t$.

- Yields $R(t, T)$ and instantaneous forward rates $f(t, T)$ carry the same information as $P(t, T)$.

- The short rate $r(t)$ contains less information: it is a function of just one variable.

# Why is $f(t,T)$ called the instantaneous forward rate?

The ratio $P(0,T)/P(0,t)$ is the time-$t$ borrowing, time-$T$ maturing discount factor locked in at time $0$:

- Consider the portfolio:
  (a) at time 0, go long a zero-coupon bond paying out one dollar at time $T$ (p.v. $=$ $P(0,T)$), and
  (b) at time 0, go short a zero-coupon bond paying out $P(0,T)/P(0,t)$ dollars at time $t$ (present value $-P(0,T)$).
- Has p.v. $0$; holder pays $P(0,T)/P(0,t)$ dollars at time $t$, receives one dollar at time $T$.
- Portfolio "locks in" $P(0,T)/P(0,t)$ as the discount factor from time $T$ to $t$.
- This ratio is called the *forward term rate* at time $0$, for borrowing at time $t$ with maturity $T$.

- Similarly, $P(t,T_2)/P(t,T_1)$ is the forward term rate at time $t$, for borrowing at time $T_1$ with maturity $T_2$.

- The associated yield (locked in at $t$ and applying to $(T_1,T_2)$ is:

$$-\frac{\log P(t,T_2)-\log P(t,T_1)}{T_2-T_1}.$$

- In the limit $T_2 - T_1 \downarrow 0$, we get $-\frac{\partial \log P(t,T)}{\partial T} = f(t,T)$.

# Goal: no-arb framework for pricing and hedging options

- Example 1: put option on a zero-coupon bond.
- Payoff at time $t$: $(K - P(t,T))^+$.
- Example 2: caplet places a cap on the term interest rate $\mathcal{R}$ for lending between times $T_1$ and $T_2$ at the fixed rate $R_0$.

- For a loan with a principal of one dollar, the caplet's payoff at time $T_2$ is:

$$\Delta t\{\mathcal{R} - R_0\}^+ = \{(\mathcal{R} - R_0)\Delta t\}^+,$$

where $\Delta t = T_2 - T_1$ and $\mathcal{R}$ is the actual term interest rate in the market at time $T_1$ (defined by $P(T_1, T_2) = 1/(1 + \mathcal{R}\Delta t)$).

- The discounted value of this payoff at time $T_1$ is:

$$\frac{\{(\mathcal{R} - R_0)\Delta t\}^+}{1 + \mathcal{R}\Delta t} = \left\{\frac{(\mathcal{R} - R_0)\Delta t}{1 + \mathcal{R}\Delta t}\right\}^+ = (1 + R_0\Delta t)\left\{\frac{1}{1 + R_0\Delta t} - \frac{1}{1 + \mathcal{R}\Delta t}\right\}^+.$$

- Thus, a caplet maturing at $T_2$ has the same discounted payoff as $1 + R_0\Delta t$ put options maturing at $T_1$. The put options have strike $\frac{1}{1+R_0\Delta t}$ and are written on a zero-coupon bond paying one dollar at maturity $T_2$.

- A cap is a collection of caplets, equivalent to a portfolio of puts on zero-coupon bonds.

# How are interest rates both risk-free and random?

- For short-rate models, the standard assumption is that the short rate $r$ solves a stochastic differential equation under $\mathbb{Q}$ of the form $dr_t = \alpha(r_t, t)\, dt + \beta(r_t, t)\, dw_t$.

- Then the value at time $t$ of a dollar received at time $T$ is:

$$P(t,T) = E\left[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\right].$$

- $P(t,T)$ can be determined by solving the BVP consisting of the following PDE for $V(t,r)$:

$$V_t + \alpha V_r + \tfrac{1}{2}\beta^2 V_{rr} - rV = 0,$$

- subject to the final-time condition $V(T,r) = 1$ for all $r$. The value of $P(t,T)$ is then $V(t, r(t))$.

# Modeling interest rates

- As in equities (Black-Scholes v Local Vol Model) - there is a tradeoff between simplicity and accuracy.

- Three basic viewpoints:

  (a) *Simple short rate models.*

  (b) *Richer short-rate models.*

  (c) *One-factor Heath-Jarrow-Morton.*

# (a) Simple short rate models

– Example: Vasicek model, assumes that under r-n probability measure $\mathbb{Q}$ the short rate solves:

$$dr_t = (\theta - ar_t)\,dt + \sigma\,dw_t \tag{1}$$

– $\theta$, $a$, and $\sigma$ constant and $a > 0$.

– Advantage of such a model: it leads to explicit formulas.

– For some short-rate models - including Vasicek - Black's formula for a call on a bnd is validated, since $P(t, T)$ is lognormally distributed under the so-called *forward measure*.

– Disadvantage of such a model: has just a few parameters $\Rightarrow$ no hope of calibrating to the entire yield curve $P(0, T)$.

– As a result, Vasicek and similar short rate models are rarely used in practice.

## (b) Richer short-rate models

– Example: Extended Vasicek model, a.k.a. the Hull-White model:

$$dr_t = [\theta(t) - ar_t]\, dt + \sigma\, dw_t. \tag{2}$$

– $a$ and $\sigma$ are still constant but $\theta$ is now a function of $t$.

– Advantage: when $\theta$ satisfies

$$\theta(t) = \frac{\partial f}{\partial T}(0,t) + af(0,T) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{3}$$

the Hull-White model correctly reproduces the entire yield curve at time $0$.

– Hull-White still leads to explicit formulas and is still consistent with Black's formula.

– Can be approximated by a recombining trinomial tree (very convenient for numerical use).

– Disadvantage: gives little freedom in modeling *evolution* of the yield curve.

## (c) One-factor Heath-Jarrow-Morton

– Theory that can be calibrated to time-$0$ yield curve

– Also permits many possible assumptions about the evolution of the yield curve.

– Specifies the evolution of the instantaneous forward rate $f(t,T)$:

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dw_t. \tag{4}$$

– Initial data $f(0,T)$ obtained from market data.

– The volatility $\sigma(t,T)$ in (**??**) must be specified – it is what determines the model.

– Vasicek corresponds to the choice $\sigma(t,T) = \sigma e^{-a(T-t)}$.

– The drift $\alpha(t,T)$ determined by $\sigma$ and the requirements of no arbitrage.

– Disadvantage: no guidance on how to choose $\sigma(t,T)$.

– Difficult numerically: only a few special cases (mainly corresponding to familiar short-rate models such as Hull-White and Black-Derman-Toy) can be modelled using recombining trees.